MA 253 Exam 3 Review

The final exam is cumulative, although it will not be much longer than the two midterm exams. To prepare, you should study all of the earlier material as well as the material since the 2nd exam. This study guide focuses on the material since Exam 2, see the previous study guides for the earlier material.

1. ORTHOGONALITY

(1) What does it mean for a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to be orthonormal? How can you check if they are orthonormal?

(2) Let $A = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & \cdots & | \end{pmatrix}$ be an orthonormal $n \times k$ matrix.

   (a) Explain why $A^T A = I_k$.

   (b) If $n = k$, explain why also $AA^T = I_n$

   (c) Explain why, if $n \neq k$, then AA^T is the matrix for projecting vectors onto the subspace spanned by the columns of A.

(3) Explain all details of the Gram-Schmidt process and be able to apply to at least 3 vectors.

(4) Explain how the Gram-Schmidt process gives rise to a QR-factorization of a matrix.

(5) Explain how to view QR-factorization as a change of basis.

(6) What is the orthogonal complement $V^\perp$ of a subspace $V \subset \mathbb{R}^n$?

(7) Show that $(V^\perp)^\perp = V$.

(8) Explain how the attempt to fit a curve to "most" data sets gives rise to an inconsistent linear system. Explain why if it gives rise to a consistent linear system you are probably "over-fitting".

(9) Explain the least-squares process in terms of projecting points onto subspaces.

(10) Prove that if A is an $n \times m$ matrix, then $(\mathrm{im} A)^\perp = \ker A^T$.

(11) Prove that if A is an $n \times m$ matrix, then $\ker A^T A = \ker A$.

(12) Suppose that A is an $n \times m$ matrix. Explain why the least squares approximate solution to $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x}^* = (A^T A)^{-1} A^T \mathbf{b} = \mathbf{x}^*.$$

(13) Find a quadratic curve $g(x) = a + bx + cx^2$ which is as close as possible (in the least-squares sense) to a curve which passes through the points

$$(0,0),(4,5),(8,2),(16,0)$$

(14) Define "singular value of an $n \times m$ matrix A".

(15) Let $q(\mathbf{x}) = ||A\mathbf{x}||^2$. Use q to prove that the eigenvalues of $A^T A$ are non-negative.

(16) Prove that if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal eigenvectors for $A^T A$, then the vectors $A\mathbf{v}$ and $A\mathbf{w}$ are orthogonal.

(17) State the spectral theorem.

(18) State the Singular Value Decomposition Theorem and explain the statement (but not why it is true) in terms of change of bases.

(19) Explain how Singular Value Decomposition can be used to efficiently compute powers of a matrix A.

2. Linear Spaces

(1) Know the informal definition of a linear space. (You do not, however, have to know all the axioms for a linear space)

(2) Explain why the following are linear spaces:

  (a) polynomials with real coefficients

  (b) $n \times m$ matrices with real entries

  (c) functions from any given set X to $\mathbb{R}$

  (d) Solutions to a linear homogenous differential equation like $f''(x) - f(x) = 0$.

  (e) Sequences of real numbers

  (f) $n \times n$ matrices with trace equal to 0

  (g) $n \times n$ matrices with entries summing to 0

(3) Know the precise definition of a linear transformation $T: X \to Y$ between two linear spaces.

(4) Know the precise definition of an isomorphism between two linear spaces.

(5) Explain why every finite dimensional linear space is isomorphic to $\mathbb{R}^n$ for some n.

(6) Explain why the following are linear transformations (between what spaces??). What are their images? What are their kernels?

  (a) The derivative

  (b) Taking a limit

  (c) the integral $\int_0^x$.

  (d) the shift operator on sequences of real numbers

$$a_1, a_2, a_3, \ldots \mapsto a_2, a_3, a_4, \ldots$$

(7) Let G be a planar directed graph with finitely many vertices and edges. Let V be the vector space of functions from the vertices of G to $\mathbb{R}$. Let E be the vector space of functions from the edges of G to $\mathbb{R}$. Let F be the vector space of functions from the faces of G to $\mathbb{R}$. For a function $f \in V$, let ∇f be the function which assigns to an edge e having head v and tail w the number $f(v) - f(w)$. For a function $g \in E$, let $c(g)$ be the function which assigns to a face P of G, the sum

$$\pm g(e_1) \pm g(e_2) \pm \ldots \pm g(e_n)$$

where $e_1, \ldots, e_n$ are the edges of P in counterclockwise order and the sign $\pm$ is determined by whether we go with the arrow on the edge or against the arrow on the edge as we move counterclockwise around P.

Prove that $\operatorname{im} \nabla \subset \ker c$.