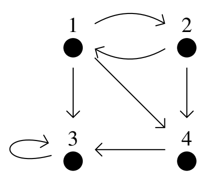
MA 253 Exam 2 - Study Guide Solutions

1. SAMPLE PROBLEM SOLUTIONS

(1) Consider the network below, and let A be its transition matrix.



(a) What is the transition matrix for this system?
Solution:
$$\begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1 & 1 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

(b) What are its eigenvalues?
Solution: $0, 1, -1/\sqrt{6}, 1/\sqrt{6}$

(c) Is there a single equilibrium vector or more than one?
Solution: Just one - since the eigenvectors form a basis, the eigenspace for the eigenvalue 1 is 1-dimensional and there is just one transition vector in that span.

(d) Do the eigenvectors form a basis for $\mathbb{R}^4$?
Solution: Yes, there are 4 distinct eigenvalues.

(2) Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with the eigenvalue $\lambda = 1$ having multiplicity n.

(a) Might T be invertible?
Solution: Yes, 0 is not an eigenvalue since T can have at most n eigenvalues and there already n of them (all equal to 1).

(b) Must T be diagonalizable?
Solution: No. It will be diagonalizable only if there is a basis of eigenvectors.

(3) Show that if A and B are $n \times n$ matrices, and if k is a real number, then

(a) $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$
Solution: When we add two $n \times n$ matrices, the entries along the diagonal are just the sum of the entries of the originals. That is, the sum of thier traces.

(b) $\mathrm{tr}(kA) = k\,\mathrm{tr}(A)$.
Solution: When we multiply A by k each number on the diagonal gets multiplied by k.

(4) Show that if λ is an eigenvalue for a matrix A, then λ^k is an eigenvalue for A^k. What can you say about the associated eigenvectors?

Solution: Let λ be an eigenvalue for A and let $\mathbf{v}$ be an associated eigenvector. Then

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A^2\mathbf{v} &= \lambda A\mathbf{v} \\ &= \lambda^2\mathbf{v} \end{aligned}$$

and so forth.

(5) Explain why if n is odd, then every $n \times n$ matrix A must have at least one real eigenvalue.

Solution: Let $f(\lambda)$ be the characteristic polynomial. As $\lambda \to -\infty$, the polynomial $f(\lambda) \to -\infty$ since it is of odd degree. Similarly, as $\lambda \to \infty$, we also have $f(\lambda) \to \infty$. Thus by the intermediate value theorem, it must cross the x-axis somewhere. That number is an eigenvalue.

(6) Is it possible for an $n \times n$ matrix with entries in $\mathbb{R}$ to have exactly one eigenvalue which has a non-zero imaginary part?

Solution: No. The determinant of a matrix is the product of its eigenvalues. Such a matrix would have a determinant with non-zero imaginary part. However, the entries of the the matrix are in $\mathbb{R}$ and cofactor expansion (or the definition of determinant) show that the determinant of a matrix with real entries is real.

(7) Explain why a matrix whose columns are a basis for $\mathbb{R}^n$ must be the matrix for a one-to-one linear transformation. Must such a matrix be invertible? What if it is square?

Solution: Yes. Yes. It must be square. To see this, let A be a matrix whose columns are a basis for $\mathbb{R}^n$. Since they form a basis, the columns are linearly independent, so when we row reduce rref$A$ has a leading 1 in every column. Since the span of the columns is $\mathbb{R}^n$, the rref$A$ has a leading 1 in every row. Thus, rref is an $n \times n$ identity matrix. So A is square and invertible.

(8) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be reflection across the subspace $x_1 + x_2 + \ldots + x_n = 0$.

(a) Find all the eigenvalues of T and their algebraic multiplicities without doing any matrix computations.

Solution: Let V be the subspace we are reflecting across. It is $n-1$ dimensional, so there are $n-1$ basis vectors in it. Each of those is a basis vector for the eigenspace of the eigenvalue 1, since they are unchanged by the transformation. There is a unique 1-dimensional subspace perpendicular to V (spanned by the vector $(1,1,\dots,1)$. So that vector is a basis for the eigenspace of the eigenvalue -1, since it is reflected. Thus, 1 has algebraic and geometric multiplicity $n-1$ and -1 has algebraic and geometric multiplicity 1.

(b) Find a basis $\mathscr{B}$ for $\mathbb{R}^n$ in which $[T]_{\mathscr{B}}$ is diagonal.

Solution: The basis of eigenvectors makes $[T]_{\mathscr{B}}$ diagonal

(9) If A is an $n\times n$ matrix such that there is an invertible matrix S and an upper triangular matrix U such that

$$A = SUS^{-1}$$

what is the relationship, if any between the eigenvalues of A and those of U? Are the eigenvalues of A easy to find? Why or why not?

Solution: The eigenvalues are the same, since eigenvalues are independent of basis. They are easy to find since they are the diagonal entries of U.

(10) Suppose that $A = XY$ where A,X,Y are $n\times n$ matrices and X and Y are an upper triangular and lower triangular matrices. Explain why 0 is not an eigenvalue of A if and only if neither X nor Y has a 0 on the diagonal.

Solution: Recall that an $n\times n$ matrix as 0 as an eigenvalue if and only if it is not invertible.

Suppose first that neither X nor Y has a 0 on the diagonal. Since they are triangular, neither has 0 as an eigenvalue. Thus, they are both invertible. We have

$$(XY)(Y^{-1}X^{-1}) = I = (Y^{-1}X^{-1})(XY)$$

so A is invertible with inverse $Y^{-1}X^{-1}$. This means that 0 is not an eigenvalue of A.

Now suppose that A is invertible. If Y had a 0 on the diagonal, it would have an eigenvalue of 0. Suppose that $\mathbf{v}$ is an associated eigenvector, then $Y\mathbf{v} = \mathbf{0}$ so

$$A\mathbf{v} = XY\mathbf{v} = X\mathbf{0} = \mathbf{0},$$

so $\mathbf{v}$ is also an eigenvector for A with eigenvalue 0. This implies that A is not invertible, contrary to our hypothesis. Thus, Y does not have a 0 on the diagonal. Then$Y$ is invertible,

$$X = AY^{-1}.$$

Since A is invertible, A inverse exists so

$$X^{-1} = YA^{-1}.$$

Thus, X is invertible. This means that 0 is not an eigenvalue of X. The diagonal entries of X are the eigenvalues, so X does not have a 0 on the diagonal.