MA 302: More Practice Problems **Name:**________________________

(1) Let $S \subset \mathbb{R}^3$ be an ellipsoid enclosing the origin, oriented outward. Let $P \subset \mathbb{R}^3$ be a cube enclosing the origin and enclosed by S. Orient P outward. Let $\mathbf{F}$ be an incompressible vector field defined on $\mathbb{R}^3 - \{\mathbf{0}\}$. Prove that the flux of $\mathbf{F}$ through P is the same as the flux of $\mathbf{F}$ through S.

Solution: Let V be the region between S and P. Orient ∂V with a unit normal that points out of V. Then by the divergence theorem:

$$\begin{aligned}
\textstyle\iint_S \mathbf{F}\cdot d\mathbf{S} - \iint_P \mathbf{F}\cdot d\mathbf{S} &= \textstyle\iint_{\partial V} \mathbf{F}\cdot d\mathbf{S} \\
&= \textstyle\iiint_V \operatorname{div}\mathbf{F}\, dV \\
&= \textstyle\iiint_V 0\, dV \\
&= 0.
\end{aligned}$$

Consequently, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ equals $\iint_P \mathbf{F}\cdot d\mathbf{S}$.

(2) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$. Let $\mathbf{a}$ be a point in $\mathbb{R}^3$. For each $n \in \mathbb{N}$, let V_n be a compact 3-dimensional region containing $\mathbf{a}$, such that the regions V_n limit to $\mathbf{a}$. Orient the boundary of V_n outwards. Use the divergence theorem to prove that

$$\operatorname{div}\mathbf{F}(\mathbf{a}) = \lim_{n\to\infty} \frac{1}{\operatorname{vol} V_n} \iint_{\partial V_n} \mathbf{F}\cdot d\mathbf{S}.$$

Solution: Suppose that n is large enough so that $\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{a})$ for all $\mathbf{x} \in V_n$. Then, by the divergence theorem:

$$\begin{aligned}
\textstyle\iint_{\partial V_n} \mathbf{F}\cdot d\mathbf{S} &= \textstyle\iiint_{V_n} \operatorname{div}\mathbf{F}\, dV \\
&\approx \textstyle\iiint_{V_n} \operatorname{div}\mathbf{F}(\mathbf{a})\, dV \\
&= \operatorname{div}\mathbf{F}(\mathbf{a}) \textstyle\iiint_{V_n} dV \\
&= \operatorname{div}\mathbf{F}(\mathbf{a})(\operatorname{vol} V_n).
\end{aligned}$$

That is,

$$\operatorname{div}\mathbf{F}(\mathbf{a}) \approx \frac{1}{\operatorname{vol} V_n} \iint_{\partial V_n} \mathbf{F}\cdot d\mathbf{S}.$$

As $n \to \infty$ this approximation becomes exact, proving the result.

(Note: This proof is actually non-rigorous. To make it rigorous we would need to use the mean value theorem for integrals.)

(3) Let S be the box with corners $(\pm 1, \pm 1, \pm 1)$, oriented outward. Let $\mathbf{F}(x,y,z) = \begin{pmatrix} xyz \\ xy \\ z \end{pmatrix}$. Find the flux of $\mathbf{F}$ through S.

Solution: Use the divergence theorem. We have $\operatorname{div} \vec{F}(x,y,z) = yz + x + 1$. The divergence says the flux through S is equal to

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} yz + x + 1 \, dx \, dy \, dz = 8.$$

(4) Prove that inside a hollow planet there is no gravity. (You may use Gauss' Law of Gravitation.)

See the online course notes for a solution.

(5) Give a complete, precise statement of the divergence theorem.

See the book or the online notes.