MA 302: Practice Exam 2 Partial Solutions **Name:**________________________

(1) Give an example of a vector field $\mathbf{F}$ having $\operatorname{curl}\mathbf{F} = \mathbf{0}$, but where $\mathbf{F}$ is not a gradient field.

(2) Know the formal definitions of the following terms or the complete precise statements of the following theorems:

(a) Green's theorem

(b) planar divergence theorem

(c) conservative vector field

(d) gradient field

(e) potential function

(f) parameterized surface

(g) orientable surface

(h) one-sided surface

(3) Give an example of a one-sided surface in $\mathbb{R}^3$.

(4) Give an example of an orientable surface in $\mathbb{R}^3$.

(5) Prove the following:

(a) Suppose that $D \subset \mathbb{R}^2$ is a type III region and that $\mathbf{F}\colon D \to \mathbb{R}^2$ satisfies the hypotheses of Green's theorem. Prove that the conclusion of Green's theorem holds.

(b) Suppose that $D \subset \mathbb{R}^2$ is the union of two type III regions along a portion of their boundaries. Suppose also that $\mathbf{F}\colon D \to \mathbb{R}^2$ satisfies the hypotheses of Green's theorem. Prove that the conclusion of Green's theorem holds.

(c) Suppose that $D \subset \mathbb{R}^2$ is simply connected and that $\mathbf{F}\colon D \to \mathbb{R}^2$ has $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Prove that $\mathbf{F}$ has path independent line integrals in D.

(d) Suppose that $\mathbf{F}\colon D \to \mathbb{R}^n$ has path independent line integrals. Describe the creation of a potential function for $\mathbf{F}$ and prove that the gradient of this function is $\mathbf{F}$.

(e) Prove that if $\mathbf{F}\colon D \to \mathbb{R}^n$ is a gradient field and if C is a simple closed curve in D, then $\int_C \mathbf{F}\cdot d\mathbf{s} = 0$.

(6) Let $D \subset \mathbb{R}^2$ be the region bounded by the graphs of the equations $y = x^3$ and $y = x$ and with $x \geq 0$. Suppose that $\mathbf{F}(x,y) = (xy + y, y^2 x)$.

(a) Is D a type I, II, or III region or none of the above?

Solution: It is a type III region, since it can be expressed as both

$$\begin{array}{ll} \{(x,y) : 0 \leq x \leq 1, x^3 \leq y \leq x\} & \text{and} \\ \{(x,y) : 0 \leq y \leq 1, y \leq x \leq \sqrt[3]{y}\} & \end{array}$$

(b) Orient ∂D so that D is always on the left. Calculate $\int\limits_{\partial D} \mathbf{F}\cdot d\mathbf{s}$ directly.

Solution: Parameterize the graph of $y = x$ as $(1-t, 1-t)$ and the graph of $y = x^3$ as (t, t^3) both with $0 \leq t \leq 1$. Notice that this gives ∂D_1 the "correct" orientation for Green's theorem.. Let C_1 and C_2 be the pieces of ∂D_1 corresponding to $y = x^3$ and $y = x$ respectively. Then:

$$\begin{array}{rl} \int_{\partial D} \mathbf{F}\cdot d\mathbf{s} & = \\ \int_0^1 \begin{pmatrix} (1-t)^2 + (1-t) \\ (1-t)^3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} t^4 + t^3 \\ t^7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} dt & = \\ \int_0^1 -(1-t)^2 - (1-t) - (1-t)^3 + (t^4 + t^3) + 3t^9\, dt & = \\ (1-t)^3/3 + (1-t)^2/2 + (1-t)^4/4 + t^5/5 + t^4/4 + 3t^{10}/10\Big|_0^1 & = \\ 1/5 + 1/4 + 3/10 - 1/3 - 1/2 - 1/4 & = \\ -1/3 & \end{array}$$

(c) Calculate $\iint\limits_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\, dA$ directly.

Solution:

$$\begin{array}{rl} \int_0^1 \int_{x^3}^x \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\, dA & = \\ \int_0^1 \int_{x^3}^x y^2 - x - 1\, dy\, dx & = \\ \int_0^1 x^3/3 - x^2 - x - x^9/3 + x^4 + x^3\, dx & = \\ 1/12 - 1/3 - 1/2 - 1/30 + 1/5 + 1/4 & = \\ -1/3 & \end{array}$$

(d) What is the relevance of Green's theorem to the preceding problems?

Solution: Since $\mathbf{F}$ is defined on D and since ∂D is piecewise C^1, Green's theorem asserts the previous two calculations should be equal. Which they are.

(e) Is the vector field $\mathbf{F}$ conservative?

Solution: No. If it were conservative the integral $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ would be 0. (There are other possible reasons.)

(7) Let $\mathbf{F}(x,y,z) = \frac{-1}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Suppose that a particle is located at the point $(1,1,0)$ and moves via the path $\mathbf{x}(t) = (t, \cos t, t\sin t)$ to the point $(\pi/2, 0, \pi/2)$. How much work is done?

Solution: The vector field $\mathbf{F}$ has potential function:

$$f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$$

Let $\mathbf{a} = (1,1,0)$ and $\mathbf{b} = (\pi/2, 0, \pi/2)$. By the FTC, the work done (which is $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$) is

$$f(\mathbf{b}) - f(\mathbf{a}) = \frac{\sqrt{2}}{\pi} - \frac{1}{\sqrt{2}}.$$

(8) What is the flux of the vector field $\mathbf{F}(x,y) = (-y^2x, x^2y)$ across the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

Solution: Let $\mathbf{x}(t) = (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$. The unit normal pointing outside the region bounded by the circle is $\mathbf{n}(t) = (\cos t, \sin(t))$. Consequently, the flux is

$$\int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} \begin{pmatrix} -8\sin^2 t \cos t \\ 8\cos^2 t \sin t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} (2)\, dt.$$

This is equal to:

$$2\int_0^{2\pi} -8\cos^2 t \sin^2 t + 8\sin^2 t \cos^2 t\, dt = 0.$$

(9) What is the circulation of the vector field $\mathbf{F}(x,y) = (-y^2x, x^2y)$ around the circle of radius 2 centered at the origin? Just set up an appropriate single-variable integral. You do not need to solve it.

Solution: We use the same notation as in the previous problem. The circulation of the vector field is:

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \begin{pmatrix} -8\cos^2 t \sin t \\ \sin^2 t \cos t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt.$$

This is equal to:

$$\int_0^{2\pi} 32\cos^2 t \sin^2 t \, dt.$$

(10) A wire C is bent into the shape of a circle of radius 1 centered at the origin in $\mathbb{R}^2$. It is given a charge of $+1$ and so generates an electric field $\mathbf{F}$. How much work is done in moving a charged particle from $(1/2, 0)$ to $(0, 0)$? Does it matter what path is taken? Why not? (You may leave your answer in integral form.)

Solution: Let q be the charge of the particle. Let C be the wire. By the principle of superposition, we can obtain a potential function for $\mathbf{F}$ by calculation:

$$f(a,b) = \int_C \frac{-q}{\sqrt{(x-a)^2 + (y-b)^2}} ds$$

since $\frac{-q}{\sqrt{a^2+b^2}}$ is a potential function for the electric field generated by a single particle at the origin. Choosing the usual parameterization for C and letting $b = 0$, we obtain:

$$f(a,0) = -q \int_0^{2\pi} \frac{1}{\sqrt{1 - 2a\cos t + a^2}} dt.$$

Since we have a potential function we can simply evaluate f on the endpoints of the path (the path not mattering the slightest) and subtract in order to find the work. So for (a) we obtain:

$$f(1/2,0) - f(0,0) = -q\left(\int_0^{2\pi} \frac{1}{\sqrt{1 - \cos t + 1/4}} dt - 2\pi\right)$$

(11) Explain why $\operatorname{curl} \mathbf{F}(\mathbf{a}) \cdot \mathbf{k} = \lim_{r \to 0^+} \frac{1}{r^2} \int_{S_r} \mathbf{F} \cdot d\mathbf{s}$ where S_r is a square centered at $\mathbf{a}$ with the distance between midpoints of opposite sides equal to r. $\mathbf{F}$ is a planar vector field.

Solution: By Green's theorem,

$$\int_{S_r} \mathbf{F} \cdot d\mathbf{s} = \iint_{D_r} \operatorname{curl} \mathbf{F} \cdot \mathbf{k} \, dA.$$

For small enough r, $\mathbf{F}(\mathbf{x}) \cdot \mathbf{k} \approx \mathbf{F}(\mathbf{a})$ and so the above integral is approximately $r^2 \operatorname{curl} \mathbf{F}(\mathbf{a}) \cdot \mathbf{k}$. Hence,

$$\operatorname{curl} \mathbf{F}(\mathbf{a}) \cdot \mathbf{k} \approx \frac{1}{r^2} \int_{S_r} \mathbf{F} \cdot d\mathbf{s}.$$

The approximation tends to an equality as $r \to 0^+$.

(12) Explain why $\operatorname{div} \mathbf{F}(\mathbf{a}) = \lim_{r \to 0^+} \frac{1}{r^2} \int_{S_r} \mathbf{F} \cdot \mathbf{n}\, ds$ where S_r is a square centered at $\mathbf{a}$ with the distance between midpoints of opposite sides equal to r. $\mathbf{F}$ is a planar vector field.

(13) Find a single variable integral representing the area enclosed by the path $\phi(t) = (2\cos(2t), 3\sin(3t))$ for $-\pi/3 \leq t \leq \pi/3$.

Solution: We note that the orientation of the path ϕ has the bounded region D always on the left. Hence by Green's theorem and the fact that $\operatorname{curl} \begin{pmatrix} 0 \\ x \end{pmatrix} = 1$:

$$\begin{aligned} \iint_D 1\, dA &= \int_{-\pi/3}^{\pi/3} \begin{pmatrix} 0 \\ 2\cos 2t \end{pmatrix} \cdot \begin{pmatrix} -4\sin 2t \\ 9\cos 3t \end{pmatrix} dt \\ &= \int_{-\pi/3}^{\pi/3} 18\cos(2t)\cos(3t)\, dt. \end{aligned}$$