

- (1) Let $F(x, y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F .

Solution:

$$DF(x, y) = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \\ 3 - 2y & -2x \end{pmatrix}$$

- (2) Let $F(x, y) = (x - y, x + y)$ and let $G(x, y) = (x \cos y, x \sin y)$. Find the derivative of $F \circ G$ using the chain rule.

Solution:

$$DF(x, y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$DG(x, y) = \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix}$$

$$\begin{aligned} D(F \circ G)(x, y) &= DF(G(x, y))DG(x, y) \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{pmatrix} \\ &= \begin{pmatrix} \cos y - \sin y & -x \sin y - x \cos y \\ \cos y + \sin y & -x \sin y + x \cos y \end{pmatrix} \end{aligned}$$

- (3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0, 1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P ?

Solution: The rotation of the P relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) can be described by the path $(\cos(6\pi t + \pi/2), \sin(6\pi t + \pi/2))$. Thus, $\mathbf{x}(t) = (\cos(6\pi t + \pi/2) + t, \sin(6\pi t + \pi/2) + \sin t)$.

- (4) A rotating circle of radius 1 follows a helical path in \mathbb{R}^3 so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t , the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let

P be the point on the circle which is at $(1, 0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P .

Solution: Relative to the center of the circle (that is, in $T_{\mathbf{c}(t)}$) the point P follows the path $\cos t\mathbf{T} + \sin t\mathbf{N}$ where \mathbf{T} and \mathbf{N} are the unit tangent and unit normal vectors to $\mathbf{c}(t) = (\sin t, \cos t, t)$ respectively. Those formulae are

$$\begin{aligned}\mathbf{T}(t) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} \\ \mathbf{N}(t) &= \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{c}(t) &= \cos t\mathbf{T} + \sin t\mathbf{N} + \mathbf{c}(t) \\ &= \frac{\cos t}{\sqrt{2}} \begin{pmatrix} \cos t \\ -\sin t \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}.\end{aligned}$$

(5) Explain what it means for curvature to be an intrinsic quantity.

Solution: The curvature of a path $\mathbf{x}(t)$ at t_0 , depends only on the curve itself at t_0 , not on the parameterization \mathbf{x} .

(6) Prove that the curvature at any point of a circle of radius r is $1/r$.

Solution: A circle of radius r can be parameterized as $\mathbf{x}(t) = (r \cos t, r \sin t)$ for $0 \leq t \leq 2\pi$. We have:

$$\begin{aligned}\mathbf{x}'(t) &= (-r \sin t, r \cos t) \\ \|\mathbf{x}'(t)\| &= r \\ \mathbf{T} &= (-\sin t, \cos t) \\ \mathbf{T}' &= (-\cos t, -\sin t) \\ \|\mathbf{T}'\| &= 1 \\ \kappa(t) &= \|\mathbf{T}'\|/\|\mathbf{x}'\| \\ &= 1/r.\end{aligned}$$

(7) Let $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $1 \leq t \leq 2$. Find \mathbf{T} , \mathbf{N} , and \mathbf{B} (that is, the moving frame) for \mathbf{x} as well as κ and τ .

Solution: We have:

$$\begin{aligned}
 \mathbf{x}'(t) &= (-\sin t, \cos t, t) \\
 \|\mathbf{x}'(t)\| &= \sqrt{2} \\
 \mathbf{T}(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) \\
 \mathbf{T}'(t) &= \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
 \mathbf{N}(t) &= (-\cos t, -\sin t, 0) \\
 \mathbf{B}(t) &= \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1) \\
 \kappa(t) &= 1/2 \\
 \mathbf{B}'(t) &= \frac{1}{\sqrt{2}}(\cos t, \sin t, 0) \\
 \frac{\mathbf{B}'(t)}{\|\mathbf{x}'(t)\|} &= \frac{1}{2}(\cos t, \sin t, 0) \\
 \tau(t) &= 1/2.
 \end{aligned}$$

- (8) Suppose that $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^n$ is a C^1 path such that for all t , $\|\mathbf{x}(t)\| = 5$. Prove that at each t , $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are perpendicular.

Solution: Since $5 = \mathbf{x}(t) \cdot \mathbf{x}(t)$, by the product rule, we have $0 = 2\mathbf{x} \cdot \mathbf{x}'$, implying that \mathbf{x} and \mathbf{x}' are perpendicular.

- (9) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \leq t \leq 5$. Find an integral representing the distance travelled by the particle after t seconds.

Solution: The distance travelled after t seconds is

$$\begin{aligned}
 s(t) &= \int_1^t \|\mathbf{x}'(\tau)\| d\tau \\
 &= \int_1^t \sqrt{1 + 4\tau^2 + 9\tau^4} d\tau.
 \end{aligned}$$

- (10) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize \mathbf{x} by arc length.

Solution: We compute,

$$s(t) = \int_1^t \sqrt{4\tau^2 + 36\tau^2} d\tau = \int_1^t 2\tau\sqrt{10} d\tau = \sqrt{10}(t^2 - 1).$$

Then,

$$s^{-1}(t) = \sqrt{t/\sqrt{10} + 1}$$

Consequently,

$$\mathbf{y}(t) = \mathbf{x} \circ s^{-1}(t) = (t/\sqrt{10} + 1, 3t/\sqrt{10} + 3)$$

is the reparameterization of \mathbf{x} by arclength.

- (11) Suppose that $\mathbf{x}(t)$ is a path in \mathbb{R}^n such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, \mathbf{x} is a path joining \mathbf{a} to \mathbf{b} .) Find a path which has the same image as \mathbf{x} but which joins \mathbf{b} to \mathbf{a} .

Solution: $\mathbf{y}: [-1, 0] \rightarrow \mathbb{R}^n$ defined by $\mathbf{y}(t) = \mathbf{x}(-t)$ will do the trick since $\mathbf{y}(0) = \mathbf{a}$ and $\mathbf{y}(-1) = \mathbf{b}$.

- (12) Let $\mathbf{x}: [a, b] \rightarrow \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t . Let $\mathbf{y} = \mathbf{x} \circ \phi$ be an orientation reversing reparameterization of \mathbf{x} . Suppose that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f ds = \int_{\mathbf{x}} f ds$.

Solution: Since ϕ is orientation reversing, $|\phi'(t)| = -\phi'(t)$. Hence, $\|\mathbf{y}'(t)\| = -\|\mathbf{x}'(\phi(t))\|\phi'(t)$. Thus,

$$\int_{\mathbf{y}} f ds = - \int_c^d f(\mathbf{x}(\phi(t))) \|\mathbf{x}'(\phi(t))\| \phi'(t) dt.$$

Substitute $u = \phi(t)$ and $du = \phi'(t)dt$ to get:

$$\int_{\mathbf{y}} f ds = - \int_b^a f(\mathbf{x}(u)) \|\mathbf{x}'(u)\| du.$$

Reversing the limits of integration eliminates the negative sign and so the result follows.

- (13) Let $\mathbf{x}(t) = (t \cos t, t \sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x, y) = y \cos x$. Let $F(x, y) = (-y, x)$. Find one-variable integrals representing $\int_{\mathbf{x}} f ds$ and $\int_{\mathbf{x}} F \cdot ds$.

Solution: Notice that

$$\begin{aligned} \mathbf{x}(t) &= (t \cos t, t \sin t) \\ \mathbf{x}'(t) &= (\cos t - t \sin t, t \cos t + \sin t) \\ \|\mathbf{x}'(t)\| &= \sqrt{(\cos t - t \sin t)^2 + (t \cos t + \sin t)^2} \end{aligned}$$

Thus,

$$\int_{\mathbf{x}} f ds = \int_0^{2\pi} t \sin t \cos(t \cos t) \sqrt{(\cos t - t \sin t)^2 + (t \cos t + \sin t)^2} dt.$$

And,

$$\begin{aligned} \int_{\mathbf{x}} F \cdot ds &= \int_0^{2\pi} \begin{pmatrix} -t \sin t \\ t \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t - t \sin t \\ \sin t + t \cos t \end{pmatrix} dt \\ &= \int_0^{2\pi} t^2 dt \\ &= 8\pi^3/3. \end{aligned}$$

- (14) Let $F(x, y) = (ax+by, cx+dy)$ be a transformation of space, where a, b, c, d are constants such that $ad - bc \neq 0$. Suppose that an object is moving in a circle $\mathbf{x}(t) = (\cos t, \sin t)$. Let $\mathbf{y}(t) = F(\mathbf{x}(t))$. If all forces stop acting on an object following the path \mathbf{y} at time $t = \pi$, where will the object be 3 seconds later?

Solution: We find that $\mathbf{y}(\pi) = F(\mathbf{x}(\pi)) = (-a, -c)$. By the chain rule,

$$\mathbf{y}'(t) = DF D\mathbf{x} = (-a \sin t + b \cos t, -c \sin t + d \cos t).$$

Thus the object will follow the path

$$l(t) = t(-b, -d) + (-a, -c)$$

at time $t + \pi$ seconds. Hence, at time $t = \pi + 3$, the particle is at $l(3) = (-3b - a, -3d - c)$.

(15) Let $\mathbf{F}(x, y) = (x, -2y)$.

- Sketch a portion of the vector field F .
- Sketch a flow line for the vector field starting at $(1, 1)$.
- Find a parameterization for the flow line starting at $(1, 1)$.

Solution: Let $\phi(t) = (x(t), y(t))$ be the flow line. Then we are looking for x and y so that:

$$\begin{aligned} x'(t) &= x(t) \\ y'(t) &= -2y(t) \\ x(0) &= 1 \\ y(0) &= 1 \end{aligned}$$

Using $x(t) = e^t$ and $y(t) = e^{-2t}$ does the trick.

- The vector field F is a gradient field. Find the potential function.

Solution: $f(x, y) = x^2/2 - y^2$.

(16) Let $F(x, y) = (2xy, x^2 + 1)$. Find a potential function for F .

Solution: $f(x, y) = x^2y + y$.

(17) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that $\mathbf{F}(x, y) = (-y, x)$ is not a gradient field.

Solution: See course notes.

(18) Suppose that $\mathbf{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a gradient field with potential function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. Let ϕ be a flow line for \mathbf{F} and let L be an equipotential line (i.e. contour line) for f . Prove that if ϕ and L intersect, they do so at right angles. (That is, they are perpendicular.)

Solution: See course notes.

(19) Let $f(x, y) = ye^x$. Find the gradient of f .

Solution: $\nabla f(x, y) = (ye^x, e^x)$.

(20) Let $F(x, y, 0) = (ye^x, xe^{y^2}, 0)$. Find the divergence of F .

Solution: $\operatorname{div} f(x, y) = ye^x + 2yx e^{y^2}$

(21) Let $F(x, y, z) = (xyz, xe^y \ln(z), x^2 + y^2 + z^2)$. Find the curl of F .

Solution:

$$\begin{pmatrix} 2y - xe^y/z \\ xy - 2x \\ e^y \ln z - xz \end{pmatrix}$$

(22) Find the curl of your answer to problem 16.

Solution:0

(23) Find the divergence of your answer to problem 18.

Solution: 0.