MA 302: Practice Exam 1 **Name:**________________________

This practice exam is much longer than the actual exam.

(1) Let $F(x,y) = (x^2y, y^2x, 3x - 2yx)$. Find the derivative of F.

(2) Let $F(x,y) = (x - y, x + y)$ and let $G(x,y) = (x\cos y, x\sin y)$. Find the derivative of $F \circ G$ using the chain rule.

(3) Suppose that a rotating circle of radius 1 is travelling through the plane, so that at time t seconds the center of the circle is at the point $(t, \sin t)$. Let P be the point on the circle which is at $(0,1)$ at time $t = 0$. If the circle makes 3 revolutions per second, what is the path $\mathbf{x}(t)$ taken by the point P?

(4) A rotating circle of radius 1 follows a helical path in $\mathbb{R}^3$ so that at time t the center of the circle is at $(\sin t, \cos t, t)$. At each time t, the circle lies in the osculating plane. (That is, the circle lies in the plane spanned by the unit tangent and the unit normal vectors.) Let P be the point on the circle which is at $(1,0)$ at time $t = 0$. The circle completes one rotation every 2π seconds. Find a formula $\mathbf{x}(t)$ for the path taken by the point P.

(5) Explain what it means for curvature to be an intrinsic quantity.

(6) Prove that the curvature at any point of a circle of radius r is $1/r$.

(7) Let $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $1 \leq t \leq 2$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ (that is, the moving frame) for $\mathbf{x}$ as well as κ and τ (curvature and torsion).

(8) Suppose that $\mathbf{x}\colon\ [a,b] \to \mathbb{R}^n$ is a $\mathbf{C}^1$ path such that for all t, $||\mathbf{x}(t)|| = 5$. Prove that at each t, $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are perpendicular.

(9) A particle is following the path $\mathbf{x}(t) = (t, t^2, t^3)$ for $1 \leq t \leq 5$. Find an integral representing the distance travelled by the particle after t seconds.

(10) Let $\mathbf{x}(t) = (t^2, 3t^2)$ for $t \geq 1$. Reparameterize $\mathbf{x}$ by arc length.

(11) Suppose that $\mathbf{x}(t)$ is a path in $\mathbb{R}^n$ such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$ (that is, $\mathbf{x}$ is a path joining $\mathbf{a}$ to $\mathbf{b}$.) Find a path which has the same image as $\mathbf{x}$ but which joins $\mathbf{b}$ to $\mathbf{a}$.

(12) Let $\mathbf{x}\colon\ [a,b] \to \mathbb{R}^n$ be a path with $\mathbf{x}'(t) \neq \mathbf{0}$ for all t. Let $\mathbf{y} = \mathbf{x} \circ \phi$ be an orientation reversing reparameterization of $\mathbf{x}$. Suppose that $f\colon\ \mathbb{R}^2 \to \mathbb{R}$ is integrable. Prove that $\int_{\mathbf{y}} f\,ds = \int_{\mathbf{x}} f\,ds$.

(13) Let $\mathbf{x}(t) = (t\cos t, t\sin t)$ for $0 \leq t \leq 2\pi$. Let $f(x,y) = y\cos x$. Find a one-variable integral representing $\int_{\mathbf{x}} f\,ds$.

(14) Let $F(x,y) = (ax+by, cx+dy)$ be a transformation of space, where a,b,c,d are constants such that $ad - bc \neq 0$. Suppose that an object is moving in a circle $\mathbf{x}(t) = (\cos t, \sin t)$. Let $\mathbf{y}(t) = F(\mathbf{x}(t))$. If all forces stop acting on an object following the path $\mathbf{y}$ at time $t = \pi$, where will the object be 3 seconds later?

(15) Let $\mathbf{F}(x,y) = (x, -2y)$.

(a) Sketch a portion of the vector field F.

(b) Sketch a flow line for the vector field starting at $(1,1)$.

(c) Find a parameterization for the flow line starting at $(1,1)$.

(d) The vector field F is a gradient field. Find the potential function.

(16) Let $F(x,y) = (2xy, x^2+1)$. Find a potential function for F.

(17) Explain why flow lines for an everywhere non-zero gradient field never close up. Use this to prove that $\mathbf{F}(x,y) = (-y,x)$ is not a gradient field.

(18) Suppose that $\mathbf{F}\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ is a gradient field with potential function $f\colon\ \mathbb{R}^2 \to \mathbb{R}$. Let ϕ be a flow line for $\mathbf{F}$ and let L be an equipotential line (i.e. contour line) for f. Prove that if ϕ and L intersect, they do so at right angles. (That is, they are perpendicular.)

(19) Let $f(x,y) = ye^x$. Find the gradient of f.

(20) Let $F(x,y,0) = (ye^x, xe^{y^2}, 0)$. Find the divergence of F.

(21) Let $F(x,y,z) = (xyz, xe^y\ln(z), x^2+y^2+z^2)$. Find the curl of F.

(22) Find the curl of your answer to problem 16.

(23) Find the divergence of your answer to problem 18.