MA 302: HW 7 additional problems

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem A: Suppose that $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a C^1 vector field. Let D be a type 3 region in $\mathbb{R}^2$. Orient ∂D so that D is on the left as ∂D is traversed. In class we showed that

$$\iint_D -\frac{\partial M}{\partial y}\, dA = \int_{\partial D} M\, dx.$$

Using similar methods, show the following:

$$\iint_D \frac{\partial N}{\partial x}\, dA = \int_{\partial D} N\, dy.$$

Problem B: Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a C^2 scalar field. Prove that $\mathrm{curl}(\nabla f) = \mathbf{0}$. (This was discussed a while ago, but never proven. You can do it by a simple, somewhat tedious calculation.)

Problem C: Let $\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$. Let D_r be closed disc of radius $r > 0$ centered at the origin.

(1) Compute $\int_{\partial D_r} \mathbf{F} \cdot d\mathbf{s}$. Does your answer depend on r?

(2) Explain why Green's theorem cannot be used to calculate the line integral in the previous part.

(3) Calculate $\mathrm{curl}\,\mathbf{F}(x,y)$.

(4) Suppose that $0 < r_0 < r_1$. Let R be the region between $\partial D_{r_0}$and ∂D_{r_1}. Explain why $\iint_R (\mathrm{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dR$ is defined and calculate it directly.

(5) Explain why the calculation in the previous part, combined with Green's theorem gives another method of showing that the answer from (1) does not depend on r.

Problem D: Suppose that $\mathbf{F}$ is a vector field which is defined and is C^1 on all of $\mathbb{R}^2$ except for n points: $\mathbf{p}_1, \ldots, \mathbf{p}_n$. Suppose that $\mathrm{curl}\,\mathbf{F}(x,y) = \mathbf{0}$ for all $(x,y) \in \mathbb{R}^2 - \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$. Let C_1 and C_2 be two disjoint simple closed curves in $\mathbb{R}^2 - \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$, both oriented counterclockwise or both oriented clockwise. C_2 bounds a closed, bounded region inside $\mathbb{R}^2$; assume that C_1 is contained in that region. Let A be the region between C_1 and C_2.

Prove that if A does not contain any of the points $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}.$$

Problem E: Construct a vector field $\mathbf{F}$ with the following properties:

(1) $\mathbf{F}$ is not defined at the points $\mathbf{p}_1 = (0,0)$ and $\mathbf{p}_2 = (1,0)$. Furthermore, $\lim_{\mathbf{x} \to \mathbf{p}_i} F(\mathbf{x})$ does not exist (for $i = 1,2$). (This last requirement on the limit is basically to keep you from taking some nice well-behaved vector field and simply redefining it at $\mathbf{p}_i$.)

(2) $\operatorname{curl} \mathbf{F}(x,y) = \mathbf{0}$ for $(x,y) \neq \mathbf{p}_1, \mathbf{p}_2$.

(3) For a smooth simple closed curve C_1 (oriented counter-clockwise) which encloses $\mathbf{p}_1$ but not $\mathbf{p}_2$,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \pi.$$

(4) For a smooth simple closed curve C_2 (oriented counter-clockwise) which encloses $\mathbf{p}_2$ but not $\mathbf{p}_1$,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 7.$$

After defining $\mathbf{F}$, answer this question: If C_3 is a simple closed curve (oriented counter-clockwise) which encloses both $\mathbf{p}_1$ and $\mathbf{p}_2$, what must be the value of

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{s}?$$