MA 122: Weekly HW 8

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem 1: Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function. Recall that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$$

if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $0 < ||\mathbf{x}-\mathbf{a}|| < \delta$ implies that $|f(\mathbf{x}) - L| < \varepsilon$.

For this problem, all questions concern the function:

$$f(x,y) = x^2 + y^2.$$

(a) Suppose that $\varepsilon = .01$. Find a δ so that if $0 < ||(x,y)-(0,0)|| < \delta$ then $|f(x,y) - 0| < \varepsilon$.
(b) Suppose that $\varepsilon = .001$. Find a δ so that if $0 < ||(x,y)-(0,0)|| < \delta$ then $|f(x,y) - 0| < \varepsilon$.
(c) Suppose that $\varepsilon > 0$ is arbitrary. Find a δ (depending on ε) so that if $0 < ||(x,y)-(0,0)|| < \delta$ then $|f(x,y) - 0| < \varepsilon$.
(d) Explain why your answer to part (c) shows that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$.
(e) Prove that $f(x,y)$ is continuous at $(0,0)$.

Problem 2: The temperature T at a location in the Northern Hemisphere depends on the longitude x, latitude y, and time t, so we can write $T = f(x,y,t)$. Let's measure time in hours from the beginning of January.

(a) What are the meanings of the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, $\partial T/\partial t$?
(b) Honolulu has a longitude of 158° W and a latitude 21° N. Suppose that at 9 AM on January 1 the wind is blowing hot air to the northeast, so the air to the west and south is warm and the air to the north and east is cooler. Would you expect $f_x(-158,21,9)$, $f_y(-158,21,9)$ and $f_t(-158,21,9)$ to be positive or negative? Explain.

Problem 3: Let $f(x,y) = x\sqrt{y}$. Find the equation of the plane tangent to the graph of $z = f(x,y)$ at the point $(1,4,2)$.

Problem 4: Use the formal definition of the partial derivative to show that

$$\frac{\partial}{\partial x}|_{(-1,3)} x^2y^3 = -54.$$

Problem 5: Let $\mathbf{v} = (1/\sqrt{2}, 1/\sqrt{2})$. Let $f(x,y) = xy$. Find $f_{\mathbf{v}}(x,y)$ using the formal definition of the directional derivative.

Problem 6: Find the gradients of the following functions:

(a) $f(x,y) = x/y$
(b) $f(x,y) = (x+y)/(x-y)$
(c) $f(x,y) = x^2 + y^2$.

Problem 7: Explain why the function $f(x,y) = xe^{xy}$ is differentiable at (1,1) and find its linearization.

Problem 8: Consider the function:

$$f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0 \end{cases}$$

(a) Sketch the graph of this function.
(b) Find $f_x(a,b)$ and $f_y(a,b)$ in the following cases:
  (i) $a = 0$ and $b = 0$
  (ii) $a = 0$ and $b \neq 0$
  (iii) $a \neq 0$ and $b = 0$
  (iv) $a \neq 0$ and $b \neq 0$.
(c) Give a thorough explanation as to why $f(x,y)$ is not differentiable at $(0,0)$.
(d) Notice that at (0,0), $f(x,y)$ has both partial derivatives. In class we discussed a theorem which gives conditions that guarantee that a function is differentiable. Relate this example to that theorem (i.e. explain why the conclusions of that theorem do not apply to this example.)