MA 122: Weekly HW 4

Answer these questions on a separate sheet of paper. Remember that your work must be very neat and complete.

Problem 1: For each of the following power series determine:

(a) all values of x for which it converges
(b) its derivative and the interval of convergence of the derivative
(c) its antiderivative and the interval of convergence of the antiderivative

(1)
$$\sum_{i=0}^{\infty} \frac{(-2)^i x^i}{i!}$$
(2)
$$\sum_{i=1}^{\infty} \frac{x^i}{i^2}$$
(3)
$$\sum_{i=0}^{\infty} (-1)^i i^2 x^i$$

Problem 2: For each of the following functions,

(a) Find (or remember) its MacLaurin series
(b) All values of x for which the MacLaurin series converges
(c) All values of x for which the MacLaurin series converges to the original function.

(1) $f(x) = \frac{1}{1-x}$ (Hint: Use the geometric series test)
(2) $f(x) = \frac{1}{2x^2-1}$ (Hint: Use 2.1 above.)
(3) $f(x) = \sin(x)$.

Problem 3: The standard normal distribution (i.e. with mean 0 and standard deviation 1) is
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$
If the results of repeated measurements follow the standard normal distribution, the probability that a future measurement will give a result within

one standard deviation of the mean is:

$$\int_{-1}^{1} f(x)\,dx.$$

(1) Summarize the following: The Maclaurin polynomial for e^x, why it converges for all x, and why it converges to e^x for all x.
(2) Find a power series representation (based at 0) for $e^{-x^2/2}$. What is its interval of convergence?
(3) Find a series representation for $f(x)$.
(4) Find a series representation for an antiderivative of $f(x)$. What is the interval of convergence?
(5) Express $\int_{-1}^{1} f(x)\,dx$ as a series. (Your answer should not have an x in it. Use Mathematica to evaluate this series to 4 decimal places. (You may need to use the `N[]` function in Mathematica.) Compare your answer to the answer(s) given by a web search of "area under a normal distribution within one standard deviation of the mean".
(6) Let $F(x) = \int_0^x f(t)\,dt$. Write $F(x)$ as a series and graph it using Mathematica.

Problem 4: This problem concerns the function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

This function arises in signal processing when studying "ringing artifacts". "Ringing artifacts" are examples of flaws that arise in JPG images.

(1) Use Mathematica to graph the function $f(x) = \sin(x)/x$ for $x \in [0, 2\pi]$.
(2) Find a series representation for $\sin(x)/x$. (Hint: Since $1/x$ is not a convergent power series, multiplying each term of your answer from 1 is a little fishy. However, in this case the maneuvre does actually produce a correct result.) What it the interval of convergence? If you were forced to define $\sin(0)/0$, what value would the series suggest?
(3) Find a series representation for $\mathrm{Si}(x)$. What is the interval of convergence?
(4) Use your answer from 4.4 to graph $\mathrm{Si}(x)$ in Mathematica.