Study Guide/Practice Exam 3

This study guide/practice exam covers only the material since exam 2. The final exam, however, is cumulative so you should be sure to thoroughly study earlier material. The distribution of content on this practice exam is not necessarily representative of the distribution of content on the actual exam.

(1) Suppose that f is a function which is the solution to the initial value problem

$$\begin{aligned}\tfrac{\partial}{\partial x} f(x,y) + \tfrac{\partial}{\partial y} f(x,y) &= x f(x,y) \\ f(0,0) &= 1\end{aligned}$$

A theorem from differential equations guarantees that $f \in C^2(\mathbb{R}^2)$.

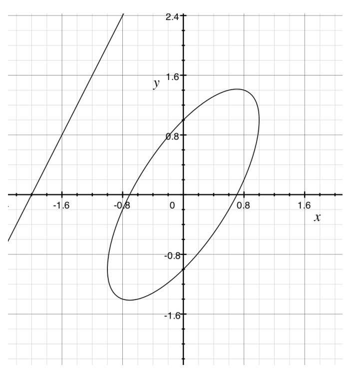
(a) Explain what $f \in C^2(\mathbb{R}^2)$ means.

(b) State a theorem which guarantees that $\frac{\partial}{\partial x \partial y} f(x,y) = \frac{\partial}{\partial y \partial x} f(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

(2) Let $f(x,y) = x^2 - \sin(x) + \cos(y)$. Find the first and second MacLaurin polynomials for f.

(3) Let $\phi(t) = (t^2, t, \sin(t))$. Let $f(x,y,z) = x - y + z$. Use the chain rule to find $\frac{d}{dt} f \circ \phi(t)$.

(4) Let $\phi(t) = (\cos t, \cos t - \sin t)$. Let L be the graph in $\mathbb{R}^2$ of the line $y - 2x = 4$. Find points $t_{\min}, t_{\max} \in [0, 2\pi)$ so that $\phi(t_{\min})$ is the closest point on the ellipse to L and $\phi(t_{\max})$ is the farthest point. You do not need to determine which is which. See below for a picture describing the situation.



(5) Let $f(x,y) = x^3 + x^2 y - y^2$. Find and classify all critical points of f.

(6) Let $f(x,y) = e^{x^2+y^2} - e^{-(x^2+y^2)}$. Find and classify all critical points of f.

(7) Let $g(x,y) = \frac{1}{xy}$. Find the points on the graph of g which are closest to the origin in $\mathbb{R}^3$. (Hint: Let $s(x,y,z) = x^2+y^2+z^2$ be the square of the distance from (x,y,z) to the origin and set $z = g(x,y)$.)

(8) A company operates two plants which manufacture the same item and whose total cost functions are

$$\begin{aligned} C_1 &= 8.5 + 0.03q_1^2 \\ C_2 &= 5.2 + 0.04q_2^2 \end{aligned}$$

where q_1 and q_2 are the quantities produced by each plant. If the item costs p dollars then

$$p = 60 - .04(q_1 + q_2).$$

The goal is to find values for q_1 and q_2 which will maximize the company's profit.

Carefully set up and describe how you would solve this problem using multi-variable derivatives. You need not actually perform the calculations.

(9) Let $f(x,y) = x + y$ and let R be the rectangle in $\mathbb{R}^2$ with corners $(1,1)$, $(1,3)$, $(2,1)$ and $(2,3)$.

(a) Subdivide R into four subrectangles of equal area and write down a sum which approximates $\iint\limits_R f\,dA$.

(b) Use lower left corners of subdivisions of R and the limit definition of the Riemann integral to calculate $\iint\limits_R f\,dA$. You will need the formula;

$$\sum_{k=0}^{n}(k-1) = \frac{(n-1)n}{2}.$$

(c) Write $\iint\limits_R f\,dA$ as an iterated integral and solve.

(10) For the following functions f and regions R set up (but do not solve) an iterated integral equal to $\iint\limits_R f\,dA$. Your answer should be something that can be plugged into *Mathematica* to find the answer.

(a) $f(x,y) = x^3y$ and R is a disc of radius 1 centered at the point $(1,-1)$.

(b) $f(x,y) = \sin(xy)$ and R is the triangular region with corners $(0,0)$, $(2,0)$, and $(1,5)$.

(c) $f(x, y) = x^2 - y^2$ and R is the region bounded by the graphs of $y = x^5$ and $y = x^3$.

(11) Set up iterated triple integrals to find the volumes of the following objects in $\mathbb{R}^3$. You do not need to solve the integrals.

(a) The region trapped between the graphs of $y = -1$, $y = 1$, $y = x^3$, $z = x$, and $z = 5$. See Figure 1.

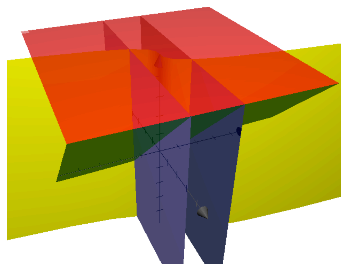


FIGURE 1. The red plane is the graph of $z = 5$. The green plane is $z = x$ and the two blue planes are $y = -1$ and $y = 1$.

(b) The region which is trapped between the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. See Figure 2.

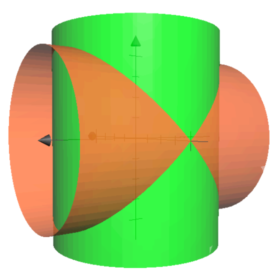


FIGURE 2. The green cylinder is $x^2 + y^2 = 1$ and the orange cylinder is $x^2 + z^2 = 1$.

(12) Carefully state and explain the Change of Variables Theorem for double integrals.

(13) Let $f(x,y) = y\sqrt{4-(x^2+y^2)}$. Let R be the half disc defined by $x^2+y^2 \leq 1$ and $y \geq 0$. Set up an iterated integral in polar coordinates which is equal to $\iint\limits_R f\,dA$.

(14) The following problems each give a function f, a region R, and a change of coordinates. For each, write down an iterated integral in the new coordinate system which equals $\iint\limits_R f\,dA$.

(a) $f(x,y) = x-y$. R is the triangle with corners $(0,0)$, $(3,1)$ and $(0,2)$. The coordinate change is given by $x = s-t$, $y = -s-t$.

(b) $f(x,y) = \sqrt{4}x+y$. R is the region bounded by $x^2-y^2 = 1$, $x^2-y^2 = 4$, $y = 0$, and $y = x/2$. The region appears in Figure 3. The coordinate change is given by $x = r\cosh\theta$, $y = r\sinh\theta$. It may help to remember the following facts:

$$\begin{aligned} d/d\theta\,\cosh\theta &= \sinh\theta \\ d/d\theta\,\sinh\theta &= \cosh\theta \\ \cosh^2\theta - \sinh^2\theta &= 1 \\ \cosh\theta &= (e^\theta + e^{-\theta})/2 \\ \sinh\theta &= (e^\theta - e^{-\theta})/2 \end{aligned}$$

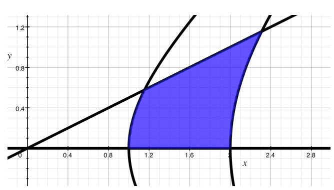


FIGURE 3

(c) Let $f(x,y) = 1$. Let R be the elliptical region

$$Ax^2 + Bxy + Cy^2 \leq 1,$$

where A, B, and C are positive constants such that $C > B^2/(4A^2)$. Use the coordinate change

$$\begin{aligned} s &= \left(x + \tfrac{B}{2A}y\right)\sqrt{A} \\ t &= y\sqrt{C - \tfrac{B^2}{4A}}. \end{aligned}$$

Computing $\iint\limits_R f\,dA$ will give the area of R.