Study Guide/Practice Exam 2 Partial Solutions

This study guide/practice exam is longer and harder than the actual exam.

Problem A: Power Series.

(1) Recall that $P(x) = \sum_{n=0}^{\infty} x^n/n!$. Analyze error terms of the MacLaurin polynomials of e^x to show that $P(x) = e^x$ for all $x \in \mathbb{R}$.

Solution: Let $g(x) = e^x$. Since $g^{(n+1)}(t) = e^t$ for all $t \in \mathbb{R}$, we have:

$$\begin{array}{rcll} |g^{(n+1)}(t)| & \leq & e^x & \text{if } x \geq 0 \text{ and } t \in [0,x] \\ |g^{n+1}(t)| & \leq & e^0 & \text{if } x < 0 \text{ and } t \in [x,0] \end{array}$$

Thus, by Taylor's theorem

$$\begin{array}{rcll} |g(x) - P_n(x)| & \leq & \frac{e^x}{(n+1)!}x^{n+1} & \text{if } x \geq 0 \\ |g(x) - P_n(x)| & \leq & \frac{1}{(n+1)!}|x|^{n+1} & \text{if } x < 0 \end{array}$$

where $P_n(x) = \sum_{i=0}^{n} x^i/i!$ is the nth Taylor approximation to $g(x)$ based at 0. Now by a result proved in class:

$$\lim_{n\to\infty} \frac{e^x x^{n+1}}{(n+1)!} = 0$$

and

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Consequently, by the squeeze theorem,

$$\lim_{n\to\infty} |g(x) - P_n(x)| = 0$$

for all $x \in \mathbb{R}$. This statement is equivalent to the statement that

$$\lim_{n\to\infty} P_n(x) = g(x)$$

for all $x \in \mathbb{R}$. But, by definition, $\lim_{n\to\infty} P_n(x) = P(x)$. $\square$

(2) Find a series representation of $f(x) = e^{x^2}$. Be sure to carefully explain why the series you give converges to e^{x^2} for all values of x.

Solution: Continue to use the notation from the previous part. We know that $P(x) = g(x)$ for all $x \in \mathbb{R}$. If $x \in \mathbb{R}$, then obviously $x^2 \in \mathbb{R}$. Hence, $P(x^2) = g(x^2) = e^{x^2}$. That is,

$$e^{x^2} = P(x^2) = \sum_{i=0}^{\infty} x^{2i}/i!$$

(3) Find a series representation of $\int_0^x e^{t^2}\,dt$. Be sure to carefully explain why the series you give converges to $\int_0^x e^{t^2}\,dt$ for all values of x.

By a theorem discussed in class, integrating (or differentiating) does not change the radius of convergence of a power series. Consequently:

$$\int_0^x e^{t^2}\,dt = \int_0^x \sum_{i=0} t^{2i}/i!\,dt$$

By the same theorem, this equals

$$\sum_{i=0}^{\infty} \int_0^x t^{2i}/i!\,dt = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)i!}.$$

This last series is the one we are looking for.

(4) Find a series representation of

$$\int_0^x \left(t \int_0^t e^{s^2}\,ds\right) dt.$$

Be sure to explain why the series has the right convergence properties.

Solution: The reasons are the same as in the previous problem. Here is the work:

$$\begin{aligned}
\int_0^x \left(t \int_0^t e^{s^2}\,ds\right) dt &= \\
\int_0^x t \sum_{i=0}^{\infty} \tfrac{t^{2i+1}}{(2i+1)i!}\,dt &= \\
\int_0^x \sum_{i=0}^{\infty} \tfrac{t^{2i+2}}{(2i+1)i!}\,dt &= \\
\sum_{i=0}^{\infty} \tfrac{x^{2i+3}}{(2i+3)(2i+1)i!}
\end{aligned}$$

(5) Find a series which represents (on an interval centered at 0) the function

$$f(x) = \frac{4x^2}{1+x^3}.$$

On what interval (centered at 0) does the series represent the function? (Hint: Begin with a series representing $1/(1-x)$.

Solution: We know (by the geometric series test) that for all $x \in (-1,1)$

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

Thus, for all x with $-x^3 \in (-1,1)$ (equivalently, $x \in (-1,1)$)

$$1/(1+x^3) = \sum_{i=0}^{\infty}(-x^3)^i = \sum_{i=0}^{\infty}(-1)^i x^{3i}$$

Thus,

$$4x^2/(1+x^3) = 4x^2\sum_{i=0}^{\infty}(-1)^i x^{3i} = \sum_{i=0}^{\infty}4(-1)^i x^{3i+2}.$$

(6) Find a series solution to the initial value problem:

$$\begin{aligned} f''(x) &= f(x)+f'(x) \\ f(0) &= 1 \\ f'(0) &= 1 \end{aligned}$$

Solution: Let

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

Then,

$$\begin{aligned} f'(x) &= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \dots \\ f''(x) &= 2c_2 + 3\cdot 2c_3x + 4\cdot 3c_4x^2 + 5\cdot 4c_5x^3 + \dots \\ f(x)+f'(x) &= (c_0+c_1) + (c_1+2c_2)x + (c_2+3c_3)x^2 + (c_3+4c_4)x^3 + \dots \end{aligned}$$

Comparing coefficients using the equation $f''(x) = f(x) + f'(x)$ we have

$$\begin{aligned} 2c_2 &= c_0 + c_1 \\ 6c_3 &= c_1 + 2c_2 \\ 12c_4 &= c_2 + 3c_3 \\ 20c_5 &= c_3 + 4c_4 \\ &\dots \end{aligned}$$

Using the initial conditions we obtain $c_0 = 1$ and $c_1 = 1$. Thus,

$$\begin{aligned} c_0 &= 1 \\ c_1 &= 1 \\ c_2 &= 1 \\ c_3 &= 1/2 \\ c_4 &= 5/24 \\ c_5 &= 1/15 \\ &\dots \end{aligned}$$

That is,

$$f(x) = 1 + x + x^2 + x^3/2 + 5x^4/24 + x^5/15 + \dots$$

If you want a general recursive formula for the coefficients, notice that

$$n(n-1)c_n = c_{n-2} + (n-1)c_{n-1}.$$

Problem B: Graphing functions $f\colon \mathbb{R}^2 \to \mathbb{R}$.

(1) Let $f(x,y) = x^2y$. Draw at least 3 x-slices, at least 3 y-slices, and at least 3 level sets. Describe with words or pictures the graph in 3-dimensions of this equation.
(2) For $\mathbf{x} \in \mathbb{R}^2$, define $f(\mathbf{x}) = ||\mathbf{x}||$. Carefully describe, with words and pictures the 3-dimensional graph of this equation.

Problem C: Vector Operations

(1) Use the law of cosines to prove that for two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$,

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos(\theta)$$

where θ is the (interior) angle between the vectors.
(2) Find the equation of the plane in $\mathbb{R}^3$ which contains the point $(7, -19, 21)$ and is perpindicular to the vector $(1, 4, -3)$.
Solution: The solutions to the equation

$$(1, 4, -3) \cdot (x, y, z) = 0$$

consist of all vectors perpindicular to the vector $(1, 4, -3)$ since the dot product of two vectors is zero if and only if the vectors are perpindicular. Every plane parallel to this plane is also perpindicular to $(1, 4, -3)$ and so the plane asked for in the problem has equation

$$(1, 4, -3) \cdot (x - 7, y + 19, z - 21) = 0$$

or equivalently

$$(x - 7) + 4(y + 19) - 3(z - 21) = 0.$$

(3) Find the equation of the plane in $\mathbb{R}^3$ which contains the points $(1, 4, -1)$, $(0, 3, 2)$, and $(-1, 1, -1)$.
Solution: First we find the equation of the plane which contains the points $(0, 0, 0)$, $(-1, -1, 3)$, and $(-2, -3, 0)$. (These points were obtained from the given points by translating them by $-(1, 4, -1)$.) A normal vector to the plane which contains these points can be obtained by using the cross product: $(-1, -1, 3) \times (-2, -3, 0)$. This gives the normal vector $(9, -6, 1)$. Thus the equation of the plane which contains these three points is

$$9x - 6y + z = 0$$

The plane which contains the three given points is obtained by translating this plane by $(1, 4, -1)$. We obtain the equation:

$$9(x - 1) - 6(y - 4) + (z + 1).$$

(4) Find the area of the parallelogram defined by the vectors $(6, -2)$ and $(-1, 8)$ in $\mathbb{R}^2$.

Solution: We put these vectors into the xy plane in $\mathbb{R}^3$ to obtain the vectors $(6, -2, 0)$ and $(-1, 8, 0)$. The magnitude of the cross product is the area of the parallelogram defined by these vectors. That is, the desired area is

$$||(6, -2, 0) \times (-1, 8, 0)|| = ||(0, 0, 46)|| = 46.$$

(5) Find the area of the parallelpiped in $\mathbb{R}^3$ defined by the vectors $(1, 4, -1)$, $(0, 3, 2)$, and $(-1, 1, -1)$. Carefully explain why your calculation produces the volume of the parallelpiped.

Solution: Let $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$ be the three vectors (in the given order). The area of the parallelogram defined by $\mathbf{v}$ and $\mathbf{w}$ is $||\mathbf{v} \times \mathbf{w}||$. Some trigonometry shows that the height of the parallelogram in the $\mathbf{u}$ direction is $||\mathbf{u}|||\cos\theta||$ where θ is the angle between $\mathbf{u}$ and a normal vector to the parallelogram defined by $\mathbf{v}$ and $\mathbf{w}$. The cross product $\mathbf{v} \times \mathbf{w}$ is a normal vector to the plane containing $\mathbf{v}$ and $\mathbf{w}$ and so the volume of the parallelpiped is

$$||\mathbf{v} \times \mathbf{w}||||\mathbf{u}||| \cos(\theta)| = |(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}| = 16.$$

(6) Find the distance from the point $(1, 4, 7)$ to the plane defined by $3x - 2y + z = 8$. Carefully explain why your calculation produces the correct answer.

Solution: This is best done with a picture which I won't include here. The given plane can be written as

$$3x - 2y + (z - 8) = 0.$$

It contains the point $(0, 0, 8)$.

We begin by translating the problem so that $\mathbf{0}$ is contained in the plane. Translation does not change distances and so the problem is equivalent to the problem of finding the distance from the point $(1, 4, -1)$ to the plane P defined by

$$3x - 2y + z = 0.$$

Let l be a line segment from the point $(1, 4, -1)$ to P which is at right angles to P. Let θ be the angle between l and the line segment joining $(1, 4, -1)$ to $\mathbf{0}$. The angle θ is the same as the angle between $(1, 4, -1)$ and a normal vector to P which is on the same side of P as $(1, 4, -1)$. The vector $(3, -2, 1)$ is a normal vector to the plane and so

$$\cos\theta = |(1, 4, -1) \cdot (3, -2, 1)|/(||(1, 4, -1)||||(3, -2, 1)|| = \frac{6}{\sqrt{18}\sqrt{14}}.$$

The length of l is $||(1,4,-1)|||\cos\theta| = 6/\sqrt{14}$.

(7) A canoe is in the middle of a portion of the Kennebec river which faces north-south. The current is moving at 3 mph southward. In the absence of the current, the wind would blow the canoe at 1 mph northwest. How fast and in what direction must the canoeists paddle to stay in the same position?

Solution: Choose a coordinate system so that the canoe is at $(0,0)$ and so that north is in the direction $(1,0)$. Then the current $\mathbf{c}$ velocity vector is $(0,3)$. The wind velocity vector $\mathbf{w}$ has magnitude 1 and is at an angle of $135°$ from the positive x axis. Thus $\mathbf{w} = (-\sqrt{2}/2, \sqrt{2}/2)$. The canoeists must paddle with a velocity of $-(\mathbf{w}+\mathbf{c}) = (\sqrt{2}/2, 3+\sqrt{2}/2)$. The magnitude of this vector is approx 3.77 mph.

Problem D: Limits and Continuity

(1) Suppose that $f\colon\ \mathbb{R}^2 \to \mathbb{R}$. Carefully state the formal definition of

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}).$$

Solution:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$$

if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $||\mathbf{x} - \mathbf{a}|| < \delta$ then $|f(\mathbf{x}) - L| < \epsilon$.

(2) Let

$$f(x,y) = \frac{x^2 y}{x^3 + y^3}.$$

Show that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist.

Solution: Notice that

$$\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{x\to 0} 0/x^3 = 0.$$

On the other hand,

$$\lim_{(x,x)\to(0,0)} f(x,y) = \lim_{x\to 0} x^3/2x^3 = 1/2.$$

Since these limits are different, the limit $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist.

(3) Let

$$f(x,y) = \frac{|x+y|}{x-y}.$$

Show that $\lim_{(x,y)\to\mathbf{0}} f(x,y)$ does not exist (you should assume that we only consider points (x,y) in the domain of f when calculating the limit).

Solution: Consider

$$\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{|x|}{x}.$$

This limit does not exist (it is ± 1 depending on the sign of x) thus the requested limit does not exist.

(4) Let

$$f(x,y) = \begin{cases} \frac{xy}{x^2-xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is not continuous at $(0,0)$.

Solution: Notice that $\lim_{(x,0)\to(0,0)} f(x,y) = 0$ but $\lim_{(x,x)\to(0,0)} f(x,y) = 1$. Since these limits are different, $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist and so the f cannot be continuous at $(0,0)$.

Problem E: Partial and Directional Derivatives

(1) Calculate $\partial/\partial x$ and $\partial/\partial y$ for the following functions:
(a) $f(x,y) = xe^x y^2$
(b) $f(x,y) = \cos(xy)$
(c) $f(x,y) = (x^2+y^2)e^{x^2+y^2}$

Solution:

$$\begin{array}{llcl} (a) & f_x(x,y) & = & e^x y^2 + xe^x y^2 \\ (a) & f_y(x,y) & = & 2yxe^x \\ (b) & f_x(x,y) & = & -y\sin(xy) \\ (c) & f_y(x,y) & = & -x\sin(xy) \\ (c) & f_x(x,y) & = & (x^2+y^2)2xe^{x^2+y^2} + 2xe^{x^2+y^2} \\ (c) & f_y(x,y) & = & (x^2+y^2)2ye^{x^2+y^2} + 2ye^{x^2+y^2} \end{array}$$

(2) Calculate all second partial derivatives of the following functions
(a) $f(x,y) = x^3y^2$
(b) $f(x,y) = x^3 + y^2$.

Solution:

$$\begin{array}{llcl} (a) & f_{xx}(x,y) & = & 6xy^2 \\ (a) & f_{xy}(x,y) & = & 6x^2y \\ (a) & f_{yx}(x,y) & = & 6yx^2 \\ (a) & f_{yy}(x,y) & = & 2x^3 \\ (b) & f_{xx}(x,y) & = & 6x \\ (b) & f_{xy}(x,y) & = & 0 \\ (b) & f_{yx}(x,y) & = & 0 \\ (b) & f_{yy}(x,y) & = & 2 \end{array}$$

(3) Carefully describe the geometric meaning of the (first) partial derivatives.

Solution: The slope of the tangent line in a y–slice.

(4) Carefully describe the meaning the second partial derivative $f_{xy}(a, b)$ at a point $(a, b) \in \mathbb{R}^2$.

Solution: $f_y(a, b)$ is the slope of the tangent line to the graph of f in the slice $x = a$. $f_{xy}(a, b)$ is the rate of change of the slope of the tangent lines in the x–slices as the x–slices pass through $x = a$.

(5) Let

$$f(x, y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Find $f_x(0, 0)$.

Solution: By definition,

$$f_x(0, 0) = \lim_{h\to 0} \frac{(0 + h)^2(0)^2}{(0 + h)^4 + 0^4} = \lim_{h\to 0} 0 = 0.$$

(6) Let $f(x, y) = xy^2 + x$ and $\mathbf{v} = (1/2, \sqrt{3}/2)$. Find $f_{\mathbf{v}}(0, 0)$ using the formal definition of "directional derivative".

Solution: By definition,

$$f_{\mathbf{v}}(0, 0) = \lim_{h\to 0} \frac{1}{h}(f(\mathbf{0} + h\mathbf{v}) - f(\mathbf{v})).$$

Since $\mathbf{0} + h\mathbf{v} = (h/2, h\sqrt{3}/2)$. Thus,

$$f(\mathbf{0} + h\mathbf{v}) = \frac{3h^3}{8} + \frac{h}{2}.$$

By the formula,

$$f_{\mathbf{v}}(\mathbf{0}) = \lim_{h\to 0} \frac{1}{h}(3h^3/8 + h/2 - 0) = 1/2.$$

(7) Give a careful description of the geometric meaning of the result from the previous problem.

Solution: The vector $\mathbf{v}$ is at $60°$ from the x–direction. Thus, if we sliced the graph of $f(x, y)$ by a plane through $(0, 0)$ at an angle of $60°$ and found the slope of the tangent line in that slice passing through the origin, the slope would be 1/2.

(8) Suppose that $\mathbf{v}$ is a unit vector and that $f\colon\ \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{a}$. Give a thorough, but not necessarily completely rigorous, explanation of why $f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$.

Solution: Since $f(x, y)$ is differentiable at $\mathbf{a}$, there is a function $L(x, y)$ whose graph is the tangent plane to $f(x, y)$ at $\mathbf{a}$. Furthermore, $L(x, y)$ is a good approximation to $f(x, y)$. The tangent plane has the equation

$$L(x, y) = \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + f(\mathbf{a}).$$

Thus, for $\mathbf{x}$ near $\mathbf{a}$:

$$\begin{aligned} f(\mathbf{x}) &\approx \nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})+f(\mathbf{a}) \\ f(\mathbf{x})-f(\mathbf{a}) &\approx \nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a}) \end{aligned}$$

Choosing $\mathbf{x}=\mathbf{a}+h\mathbf{v}$ we obtain:

$$f(\mathbf{a}+h\mathbf{v})-f(\mathbf{a})\approx\nabla f(\mathbf{a})\cdot(h\mathbf{v}).$$

Dividing by h we obtain:

$$\frac{1}{h}\Big(f(\mathbf{a}+\mathbf{hv}-f(\mathbf{a})\Big)\approx\nabla f(\mathbf{a})\cdot\mathbf{v}.$$

As $h\to 0$, this approximation becomes exact, so that

$$\begin{aligned} \lim_{h\to 0}\tfrac{1}{h}\Big(f(\mathbf{a}+h\mathbf{v})-f(\mathbf{a})\Big) &= \nabla f(\mathbf{a})\cdot\mathbf{v} \\ f_v(\mathbf{a}) &= \nabla f(\mathbf{a})\cdot\mathbf{v} \end{aligned}$$

(9) Explain the meaning of the direction and magnitude of $\nabla f(\mathbf{a})$ (assuming that $\nabla f(\mathbf{a})\neq 0$ and that f is differentiable at $\mathbf{a}$).
Solution: The direction is the direction of greatest increase in f. The magnitude is the amount of the greatest increase of f. (Equivalently, the direction is the direction of maximum rate of change and the magnitude is the amount of maximum rate of change of f.)

(10) The temperature of a point (x,y) on a metal plate is given by $T(x,y)=60x/(1+x^2+y^2)$. If an ant is standing at $(1,2)$ in what direction should the ant walk (leaving from $(1,2)$) to get the coolest quickest?

Solution:

$$\begin{aligned} \nabla T(x,y) &= \left(\tfrac{60(1+x^2+y^2)-120x^2}{(1+x^2+y^2)^2},\tfrac{-120xy}{(1+x^2+y^2)^2}\right) \\ \nabla T(1,2) &= \left(\tfrac{240}{36},-\tfrac{240}{36}\right) \end{aligned}$$

Since $\nabla T(1,2)$ is the direction of greatest increase in T, we need to use $-\nabla T(1,2)$, the direction of greatest decrease in T. Thus, the direction of greatest decrease in T is $(-\frac{20}{3},\frac{20}{3})$.

Problem F: Tangent Planes

(1) Carefully explain why the function $f(x,y)=x^2+y^2$ is differentiable at $(0,0)$.
Solution: The function $f(x,y)$ is differentiable at $(0,0)$ (by definition) if and only if both partial derivatives exist and the relative error goes to 0 as $(x,y)\to(0,0)$. It is clear that $f_x(0,0)=0$ and $f_y(0,0)=0$ both exist. Then $L(x,y)=0$ and so the limit of relative

error is

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{||(x,y)-(0,0)||} &= \\ \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} &= \\ \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} &= 0\end{aligned}$$

Thus, $f(x,y)$ is differentiable at $(0,0)$.

(2) Find the equation of the plane tangent to the graph of $f(x,y) = x - y^3$ at the point $(2,1)$.

Solution:

$$\begin{aligned}L(x,y) &= \nabla f(2,1)\cdot(x-2,y-1)+f(2,1)\\ L(x,y) &= (1,-3)\cdot(x-2,y-1)+1\\ L(x,y) &= (x-2)-3(y-1)+1\end{aligned}$$

(3) Find the local linearization of the function $f(x,y) = x/y$ at the point $(1,1)$.

Solution: The local linearization is the same as the function whose graph is the tangent plane $L(x,y)$. Thus,

$$L(x,y) = (x-1)-(y-1)+1.$$

(4) Use a theorem discussed in class to show that the function $f(x,y) = x+y$ is differentiable at $(0,0)$.

Solution: We have $f_x(x,y) = 1$ and $f_y(x,y) = 1$. These functions are clearly continuous on all of $\mathbb{R}^2$. If $f(x,y)$ has continuous partial derivatives on a disc centered at $(0,0)$ it is differentiable at $(0,0)$. Thus, $f(x,y)$ is differentiable at $(0,0)$.