Study Guide/Practice Exam 2

This study guide/practice exam is longer and harder than the actual exam.

Problem A: Power Series.

(1) Recall that $P(x) = \sum_{n=0}^{\infty} x^n/n!$. Analyze error terms of the MacLaurin polynomials of e^x to show that $P(x) = e^x$ for all $x \in \mathbb{R}$.
(2) Find a series representation of $f(x) = e^{x^2}$. Be sure to carefully explain why the series you give converges to e^{x^2} for all values of x.
(3) Find a series representation of $\int_0^x e^{t^2}\, dt$. Be sure to carefully explain why the series you give converges to $\int_0^x e^{t^2}\, dt$ for all values of x.
(4) Find a series representation of

$$\int_0^x \left(t \int_0^t e^{s^2}\, ds \right) dt.$$

Be sure to explain why the series has the right convergence properties.
(5) Find a series which represents (on an interval centered at 0) the function

$$f(x) = \frac{4x^2}{1+x^3}.$$

On what interval (centered at 0) does the series represent the function? (Hint: Begin with a series representing $1/(1-x)$.
(6) Find a series solution to the initial value problem:

$$\begin{aligned} f''(x) &= f(x) + f'(x) \\ f(0) &= 1 \\ f'(0) &= 1 \end{aligned}$$

Problem B: Graphing functions $f\colon \mathbb{R}^2 \to \mathbb{R}$.

(1) Let $f(x, y) = x^2 y$. Draw at least 3 x-slices, at least 3 y-slices, and at least 3 level sets. Describe with words or pictures the graph in 3-dimensions of this equation.
(2) For $\mathbf{x} \in \mathbb{R}^2$, define $f(\mathbf{x}) = ||\mathbf{x}||$. Carefully describe, with words and pictures the 3-dimensional graph of this equation.

Problem C: Vector Operations

(1) Use the law of cosines to prove that for two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$,

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos(\theta)$$

where θ is the (interior) angle between the vectors.

(2) Find the equation of the plane in $\mathbb{R}^3$ which contains the point $(7, -19, 21)$ and is perpindicular to the vector $(1, 4, -3)$.
(3) Find the equation of the plane in $\mathbb{R}^3$ which contains the points $(1, 4, -1)$, $(0, 3, 2)$, and $(-1, 1, -1)$.
(4) Find the area of the parallelogram defined by the vectors $(6, -2)$ and $(-1, 8)$ in $\mathbb{R}^2$.
(5) Find the area of the parallelpiped in $\mathbb{R}^3$ defined by the vectors $(1, 4, -1)$, $(0, 3, 2)$, and $(-1, 1, -1)$. Carefully explain why your calculation produces the volume of the parallelpiped.
(6) Find the distance from the point $(1, 4, 7)$ to the plane defined by $3x - 2y + z = 8$. Carefully explain why your calculation produces the correct answer.
(7) A canoe is in the middle of a portion of the Kennebec river which faces north-south. The current is moving at 3 mph southward. In the absence of the current, the wind would blow the canoe at 1 mph northwest. How fast and in what direction must the canoeists paddle to stay in the same position?

Problem D: Limits and Continuity

(1) Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$. Carefully state the formal definition of
$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}).$$
(2) Let
$$f(x, y) = \frac{x^2 y}{x^3 + y^3}.$$
Show that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist
(3) Let
$$f(x, y) = \frac{|x + y|}{x - y}.$$
Show that $\lim_{(x,y)\to\mathbf{0}} f(x, y)$ does not exist (you should assume that we only consider points (x, y) in the domain of f when calculating the limit).
(4) Let
$$f(x, y) = \begin{cases} \frac{xy}{x^2 - xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
Show that f is not continuous at $(0, 0)$.

Problem E: Partial and Directional Derivatives

(1) Calculate $\partial/\partial x$ and $\partial/\partial y$ for the following functions:
   (a) $f(x, y) = xe^x y^2$

  (b) $f(x,y) = \cos(xy)$
  (c) $f(x,y) = (x^2+y^2)e^{x^2+y^2}$

(2) Calculate all second partial derivatives of the following functions
  (a) $f(x,y) = x^3y^2$
  (b) $f(x,y) = x^3 + y^2$.

(3) Carefully describe the geometric meaning of the (first) partial derivatives.

(4) Carefully describe the meaning the second partial derivative $f_{xy}(a,b)$ at a point $(a,b) \in \mathbb{R}^2$.

(5) Let

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Find $f_x(0,0)$.

(6) Let $f(x,y) = xy^2 + x$ and $\mathbf{v} = (1/2, \sqrt{3}/2)$. Find $f_{\mathbf{v}}(0,0)$ using the formal definition of "directional derivative".

(7) Give a careful description of the geometric meaning of the result from the previous problem.

(8) Suppose that $\mathbf{v}$ is a unit vector and that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{a}$. Give a thorough, but not necessarily completely rigorous, explanation of why $f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$.

(9) Explain the meaning of the direction and magnitude of $\nabla f(\mathbf{a})$ (assuming that $\nabla f(\mathbf{a}) \neq 0$ and that f is differentiable at $\mathbf{a}$).

(10) The temperature of a point (x,y) on a metal plate is given by $T(x,y) = 60x/(1+x^2+y^2)$. If an ant is standing at $(1,2)$ in what direction should the ant walk (leaving from $(1,2)$) to get the coolest quickest?

Problem F: Tangent Planes

(1) Carefully explain why the function $f(x,y) = x^2 + y^2$ is differentiable at $(0,0)$.

(2) Find the equation of the plane tangent to the graph of $f(x,y) = x - y^3$ at the point $(2,1)$.

(3) Find the local linearization of the functin $f(x,y) = x/y$ at the point $(1,1)$.

(4) Use a theorem discussed in class to show that the function $f(x,y) = x + y$ is differentiable at $(0,0)$.