Partial Solutions to Study Guide/Practice Exam 1

These solutions are intended to help you check your work. In general, you should provide more detail and explanation for your answers than is provided here.

Problem 1: Know the general formula for the nth Taylor approximation to a differentiable function f based at $x = a$.

Problem 2: Know the formulae for the nth MacLaurin polynomials for the functions $\ln(1 + x)$, and e^x.

Problem 3: Find the formula for the $2n + 1$st MacLaurin polynomial for $\sin(x)$

Solution: Let $f(x) = \sin(x)$. The first few derivatives of f are

$$\begin{array}{rcl} f(x) & = & \sin(x) \\ f'(x) & = & \cos(x) \\ f''(x) & = & -\sin(x) \\ f'''(x) & = & -\cos(x) \\ f^{iv}(x) & = & \sin(x) \end{array}$$

and the other derivatives repeat in a similar fashion. Thus, $f^{(n)}(0) = 0$ whenever n is even. Consequently

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

In summation notation this is:

$$P_{2n+1}(x) = \sum_{i=0}^{2n+1} \frac{(-1)^i x^{2i+1}}{(2i+1)!}.$$

Problem 4: Find the formula for the $2n$th MacLaurin polynomial for $\cos(x)$.

Problem 5: What is the nth Taylor polynomial $P_n(x)$ for the function $f(x) = \frac{1}{x^2}$ based at $a = -1$?

Solution: Here are the first few derivatives of $f(x)$:

$$\begin{array}{rcl} f(x) & = & 1/x^2 \\ f'(x) & = & -2/x^3 \\ f''(x) & = & 6/x^4 \end{array}$$

In general,

$$f^{(n)}(x) = (-1)^{n-1}(n+1)!/x^{n+2}$$

So at $x = -1$, we have:

$$\begin{array}{rcl} f(-1) & = & 1 \\ f'(-1) & = & 2 \\ f''(-1) & = & 6 \\ f^{(n)}(-1) & = & (n+1)! \end{array}$$

Thus,

$$P_n(x) = 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + \ldots + (n+1)(x+1)^n.$$

Problem 6: What is the 3rd MacLaurin polynomial $P_3(x)$ for the function $f(x) = \sin(x^2)$?

Solution: Here are the first 3 derivatives:

$$\begin{array}{rcc} f(x) & = & \sin(x^2) \\ f'(x) & = & 2x\cos(x^2) \\ f''(x) & = & -4x^2\sin(x^2) + 2\cos(x^2) \\ f'''(x) & = & -8x^3\cos(x^2) - 8x\sin(x^2) - 4x\sin(x^2) \end{array}$$

Thus, at $x = 0$:

$$\begin{array}{rcl} f(0) & = & 0 \\ f'(0) & = & 0 \\ f''(0) & = & 2 \\ f'''(0) & = & 0 \end{array}$$

Thus,

$$P_3(x) = x^2.$$

Problem 7: Find an upper bound on the absolute value of the error of $P_4(x)$ for $x \in [1, 2]$ if $f(x) = \frac{1}{x^2}$. (See problem 5.)

Solution: Since we will use Taylor's theorem, we need to find an M, so that

$$|f^{(5)}(t)| \leq M \qquad \text{for all } t \in [1, 2].$$

By our formula from problem 5:

$$|f^{(5)}(t)| = 6!/|t|^7$$

The function $6!/|t|^7$ is a decreasing function for $t \geq 0$, and so its maximum on [1,2] occurs when $t = 1$. Thus,

$$|f^{(5)}(t)| = 6!/|t|^7 \leq 6!$$

Consequently, by the theorem,

$$|f(x) - P_4(x)| \leq \frac{6!}{5!}|x+1|^5 = 6|x+1|^5$$

for $x \in [1, 2]$.

Problem 8: Find an upperbound on the absolute value of the error of $P_3(x)$ for $x \in [0,1]$ if $f(x) = \sin(x^2)$. (See problem 6)

Solution: We need the 4th derivative of $f(x)$. In problem 6, we found the third derivative. Thus,

$$\begin{aligned} f^{(4)}(x) &= 16x^4\sin(x^2) - 24x^2\cos(x^2) - 8\sin(x^2 - 16x^2\cos(x^2) - 4\sin(x^2) - 8x^2\cos(x^2) \\ &= 16x^4\sin(x^2) - 12\sin(x^2) - 48x^2\cos(x^2) \end{aligned}$$

The functions sine and cosine have maximum and minimum values of ± 1. Thus (keeping in mind that $\cos(t^2)$ may be negative when $\sin(t^2)$ is positive) we have

$$|f^{(4)}(t)| \le |16t^4 - 12| + 48t^2 \le 16t^4 + 12 + 48t^2$$

If $t \in [0,1]$, then $|f^{(4)}(t)| \le 16 + 12 + 48 = 76$.

Thus,

$$|f(x) - P_3(x)| \le \frac{76}{4!}|x|^4 \le 76/24.$$

Problem 9: This problem is designed to test your understanding of Taylor's theorem on the error bound of Taylor polynomials. You may not directly use that result in your answer to this problem.

Suppose that $f(x)$ is an infinitely differentiable function so that $f''(t) \le 12$ for $t \ge 0$. Show that $f(x) - P_1(x) \le 6x^2$ where $P_1(x)$ is the tangent line approximation to $f(x)$ based at $x = 0$.

Solution: Since f'' and f' are continuous, we may use the fundamental theorem of Calculus. Hence:

$$\begin{array}{ccc} f''(t) & \le & 12 \\ \int_0^x f''(t)\,dt & \le & \int_0^x 12\,dt \\ f'(t)|_0^x & \le & 12t|_0^x \\ f'(x) - f'(0) & \le & 12x \\ \int_0^x f'(t) - f'(0)\,dt & \le & \int_0^x 12t\,dt \\ f(t) - f'(0)t|_0^x & \le & 6t^2|_0^x \\ f(x) - f'(0)x - f(0) & \le & 6x^2 \\ f(x) - (f(0) + f'(0)x) & \le & 6x^2 \\ f(x) - P_1(x) & \le & 6x^2 \end{array}$$

Problem 10: Let $c \in [-1,1]$ be a fixed real number. Show that

$$\lim_{n\to\infty} |\sin(c) - P_{2n+1}(c)| = 0$$

where $P_{2n+1}(x)$ is the nth MacLaurin polynomial of $\sin(x)$.

Solution: Let $f(x) = \sin(x)$. Then since $2n+2$ is an even number, for all $t \in \mathbb{R}$:

$$|f^{2n+2}(t)| = |\sin(t)| \leq 1.$$

Hence, by Taylor's theorem:

$$|f(c) - P_{2n+1}(c)| \leq \frac{1}{(2n+2)!}|c|^{2n+2} \leq \frac{1}{(2n+2)!}.$$

The last inequality is true because $|c| \leq 1$. Since

$$\lim_{n\to\infty} \frac{1}{(2n+2)!} = 0,$$

and since,

$$0 \leq |f(c) - P_{2n+1}(c)| \leq \frac{1}{(2n+2)!}$$

by the squeeze theorem:

$$\lim_{n\to\infty} |f(c) - P_{2n+1}(c)| = 0.$$

Problem 11: A certain infinitely differentiable function satisfies

$$|f^{(n)}(t)| \leq (n-1)^2|t|$$

for all $t \in \mathbb{R}$ and for all $n \geq 2$. Let $E_n(x)$ denote the error of the nth Taylor approximation to $f(x)$ based at $x = 1$. Find a number n, so that

$$|E_n(2)| \leq .01.$$

Solution: Notice that

$$|f^{(n+1)}(t)| \leq n^2|x|$$

for all $t \in [1, x]$. By Taylor's theorem

$$|f(x) - P_n(x)| \leq \frac{n^2|x|}{(n+1)!}|x-1|^{n+1}$$

Thus, if $x = 2$,

$$|f(x) - P_n(x)| \leq \frac{2n^2}{(n+1)!}$$

Now, for $n \geq 3$,

$$\frac{2n^2}{(n+1)n(n-1)(n-2)!} \leq \frac{2}{(n-2)!} \leq \frac{2}{n-2}.$$

Thus, if

$$\frac{2}{n-2} \leq .01$$

then

$$|f(x) - P_n(x)| \leq .01$$

Now,

$$\frac{2}{n-2} \leq .01 \qquad \Leftrightarrow \qquad 202 \leq n.$$

Thus, if $n \geq 202$, then $|f(x) - P_n(x)| \leq .01$.

Problem 11: Carefully state the ϵ definition of the limit of a sequence (a_n).

Solution: The sequence (a_n) converges to a number L if, for each $\epsilon > 0$, there exists a number N such that whenever $n \geq N$, then

$$|a_n - L| < \epsilon.$$

Problem 12: Use the ϵ definition of limit, to prove that

$$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0.$$

Solution: Let ϵ be given. Then

$$\left|\frac{1}{\ln(n)} - 0\right| = \frac{1}{\ln(n)}.$$

Notice,

$$\frac{1}{\ln(n)} \leq \epsilon \qquad \Leftrightarrow \qquad e^{1/\epsilon} \leq n.$$

Thus, if $n \geq e^{1/\epsilon}$ then $|1/\ln(n) - 0| \leq \epsilon$.

Problem 13: Let $a_1 = 1$ and for $n \geq 2$, define $a_n = \ln(1 + a_{n-1})$. Prove that (a_n) converges.

Solution: First we show that (a_n) is an decreasing sequence.

Notice that $a_1 = 1$ and $a_2 = \ln(2) < 1$. So $a_2 < a_1$. Now consider:

$$\begin{array}{cccll} a_{n+1} & < & a_n & & \Leftrightarrow \\ \ln(1+a_n) & < & \ln(1+a_{n-1}) & & \Leftrightarrow \\ 1+a_n & < & 1+a_{n-1} & \text{(since } \ln \text{ is increasing} & \Leftrightarrow \\ a_n & < & a_{n-1} & & \end{array}$$

Continuing in this manner we see that

$$a_{n+1} < a_n \qquad \Leftrightarrow \qquad a_2 < a_1.$$

We know this latter inequality to be true, so $a_{n+1} < a_n$ for all n.

Next we show that (a_n) is bounded. Consider:

$$\begin{array}{ccccc} a_n & < & 1 & & \Leftrightarrow \\ \ln(1+a_{n-1}) & < & 1 & & \Leftrightarrow \\ a_{n-1} & < & e-1 & & \Leftrightarrow \\ \ln(1+a_{n-2}) & < & e^{e-1}-1 & & \Leftrightarrow \\ \ldots & & & & \end{array}$$

Eventually we will end up with

$$a_1 < C$$

where C is some number formed by starting with 1 and repeatedly applying the exponential function and subtracting 1. Notice that $e^x > 1$ if $x > 1$ and so the number C is bigger than 1. Thus, $a_1 < c$ and we conclude that $a_n < 1$ for all n.

Since each a_n is non-negative and less than 1, the sequence (a_n) is an increasing, bounded sequence. Therefore, (a_n) converges.

Problem 14: Suppose that for all i, $0 \le a_i \le b_i$. Prove that if $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

Problem 15: Suppose that $|r| \neq 1$ and that $a \neq 0$. Prove that the geometric series

$$\sum_{i=0}^{\infty} ar^i$$

converges if and only if $|r| < 1$.

Problem 16: Carefully explain why the sequence $\sum_{i=0}^{\infty} \frac{1}{i!}$ converges to e.

Solution:

Let $f(x) = e^x$. Since $f^{(n)}(x) = e^x$ for all n, by Taylor's theorem

$$|f(x) - P_n(x)| \le \frac{e^x}{(n+1)!} x^{n+1}$$

for all $x \ge 0$. If $x = 1$ we have:

$$|f(1) - P_n(1)| \le \frac{e}{(n+1)!}$$

As $n \to \infty$, $\frac{e}{(n+1)!} \to \infty$. Consequently,

$$\lim_{n\to\infty} |f(1) - P_n(1)| = 0.$$

This means that eventually all terms of the sequence $(P_n(1))$ are arbitrarily close to 0. Thus,

$$\lim_{n\to\infty} P_n(1) = f(1) = e$$

By definition,

$$\sum_{i=0}^{\infty} \frac{1}{i!} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!}$$

We also know,

$$P_n(1) = \sum_{i=0}^{n} \frac{1}{i!}.$$

Thus,

$$\sum_{i=0}^{\infty} \frac{1}{i!} = \lim_{n\to infty} P_n(1) = e.$$

Problem 17: Find

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} = 1 - \frac{1}{6} + \frac{1}{5!} - \dots$$

Be sure to thoroughly explain your answer.

(Hint: Consider the $2n+1$st MacLaurin polynomial for $\sin(x)$ evaluated at $x = 1$.)

Solution: The $2n+1$st MacLaurin polynomial for $\sin(x)$ evaluated at $x = 1$ is:

$$P_{2n+1}(1) = \sum_{i=0}^{2n+1} \frac{(-1)^i}{(2i+1)!}.$$

In an earlier problem we showed that

$$\lim_{n\to\infty} |\sin(1) - P_{2n+1}(1)| = 0.$$

This is the same as saying that

$$\lim_{n\to\infty} P_n(1) = \sin(1).$$

Then,

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(-1)^i}{(2i+1)!} = \lim_{n\to\infty} P_n(1) = \sin(1).$$

Problem 18: For each of the following series, determine if they converge or diverge. Be sure to provide a thorough explanation for each.

The following are hints: not complete solutions

(1) $\sum_{i=1}^{\infty} 1/i$
Solution: Diverges by the integral test.

(2) $\sum_{i=1}^{\infty} 1/i^2$
Solution: Converges by the integral test.

(3) $\sum_{i=0}^{\infty} \frac{3}{5^i}$
Solution: It is a geometric series and it converges.

(4) $\sum_{i=0}^{\infty} \frac{3^i}{5^i}$
Solution: It is a geometric series and it converges.

(5) $\sum_{i=1}^{\infty} \frac{3^i}{5^i i!}$
Solution: Use a comparison test with the previous series.

(6) $\sum_{n=1}^{\infty} \frac{3}{n^2-5}$
Solution: Use the integral test to see that

$$\sum_{n=3}^{\infty} \frac{3}{n^2 - t}$$

converges. Or use a comparison test using the fact that $n^2 - 5 \geq (n-1)^2$ for $n \geq 3$.

(7) $\sum_{n=1}^{\infty} \frac{2}{3^i+4^i}$
Solution: Use the comparison test and compare to a geometric series.

(8) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
Solution: Use the alternating series test.

(9) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.
Solution: Use the ration test:

$$\frac{(n+1)^{(n+1)}(n+1)!}{n^n n!} = (n+1)\left(\frac{(n+1)}{n}\right)^n \frac{(n+1)!}{n!} = (n+1)^2\left(\frac{(n+1)}{n}\right)^n$$

As $n \to \infty$ this also heads to ∞.

Problem 19: For what values of x do the following series converge?

(1) $\sum_{n=1}^{\infty} (3x)^n$
Solution: Converges if $|3x| < 1$. Hence, converges if $|x| < 1/3$. (Geometric Series)

(2) $\sum_{n=1}^{\infty} \frac{3}{x^n}$
Solution: Use the ratio test to discover that if

$$\frac{1}{|x|} < 1 \qquad \Leftrightarrow \qquad |x| > 1$$

then the series converges and if

$$\frac{1}{|x|} > 1 \qquad \Leftrightarrow \qquad |x| < 1$$

then the series diverges.

If $x = 1$, then

$$\sum_{n=1}^{\infty} \frac{3}{x^n} = \sum_{n=1}^{\infty} 3$$

which diverges.

If $x = -1$ then

$$\sum_{n=1}^{\infty} \frac{3}{(-1)^n} = -3 + 3 - 3 + 3 - \dots$$

which also diverges.

Thus, the series if and only if $|x| > 1$.

(3) $\sum_{n=1}^{\infty} n^n (x-3)^n$

Solution: Use the ratio test. The series diverges if

$$\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{n^n} |x-3| > 1$$

But

$$\frac{(n+1)^{(n+1)}}{n^n}|x-3| = (n+1)\Big(\frac{n+1}{n})^n|x-3|$$

which heads to ∞ as $n \to \infty$ unless $x = 3$. Thus the series converges if and only if $x = 3$.

(4) $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$

Converges if and only if $2 < x \leq 3$.

(5) $\sum_{n=1}^{\infty} \frac{n}{x^n}$.

Solution: Converges if and only if $|x| > 1$.