## Study Guide/Practice Exam 1

This study guide/practice exam is longer and harder than the actual exam.

**Problem 1:** Know the general formula for the $n$th Taylor approximation to a differentiable function $f$ based at $x = a$.

**Problem 2:** Know the formulae for the $n$th MacLaurin polynomials for the functions $\ln(1 + x)$, and $e^x$.

**Problem 3:** Find the formula for the $2n + 1$st MacLaurin polynomial for $\sin(x)$

**Problem 4:** Find the formula for the $2n$th MacLaurin polynomial for $\cos(x)$.

**Problem 5:** What is the $n$th Taylor polynomial $P_n(x)$ for the function $f(x) = \frac{1}{x^2}$ based at $a = -1$?

**Problem 6:** What is the 3rd MacLaurin polynomial $P_3(x)$ for the function $f(x) = \sin(x^2)$?

**Problem 7:** Find an upper bound on the absolute value of the error of $P_4(x)$ for $x \in [1, 2]$ if $f(x) = \frac{1}{x^2}$. (See problem 5.)

**Problem 8:** Find an upperbound on the absolute value of the error of $P_3(x)$ for $x \in [0, 1]$ if $f(x) = \sin(x^2)$. (See problem 6)

**Problem 9:** This problem is designed to test your understanding of Taylor's theorem on the error bound of Taylor polynomials. You may not directly use that result in your answer to this problem.

Suppose that $f(x)$ is an infinitely differentiable function so that $f''(t) \leq 12$ for $t \geq 0$. Show that $f(x) - P_1(x) \leq 6x^2$ where $P_1(x)$ is the tangent line approximation to $f(x)$ based at $x = 0$..

**Problem 10:** Let $c \in [-1, 1]$ be a fixed real number. Show that

$$\lim_{n\to\infty} |\sin(c) - P_{2n+1}(c)| = 0$$

where $P_{2n+1}(x)$ is the $n$th MacLaurin polynomial of $\sin(x)$.

**Problem 11:** A certain infinitely differentiable function satisfies

$$|f^{(n)}(t)| \leq (n - 1)^2|t|$$

for all $t \in \mathbb{R}$ and for all $n \geq 2$. Let $E_n(x)$ denote the error of the $n$th Taylor approximation to $f(x)$ based at $x = 1$. Find a number $n$, so that

$$|E_n(2)| \leq .01.$$

**Problem 11:** Carefully state the $\epsilon$ definition of the limit of a sequence $(a_n)$.

**Problem 12:** Use the $\epsilon$ definition of limit, to prove that

$$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0.$$

**Problem 13:** Let $a_1 = 1$ and for $n \geq 2$, define $a_n = \frac{1}{1+a_{n-1}}$. Prove that $(a_n)$ converges and find its limit.

**Problem 14:** Suppose that for all $i$, $0 \leq a_i \leq b_i$. Prove that if $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

**Problem 15:** Suppose that $|r| \neq 1$ and that $a \neq 0$. Prove that the geometric series

$$\sum_{i=0}^{\infty} ar^i$$

converges if and only if $|r| < 1$.

**Problem 16:** Carefully explain why the sequence $\sum_{i=0}^{\infty} \frac{1}{i!}$ converges to $e$.

**Problem 17:** Find

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} = 1 - \frac{1}{6} + \frac{1}{5!} - \dots$$

Be sure to thoroughly explain your answer.

(Hint: Consider the $2n+1$st MacLaurin polynomial for $\sin(x)$ evaluated at $x = 1$.)

**Problem 18:** For each of the following series, determine if they converge or diverge. Be sure to provide a thorough explanation for each.

(1) $\sum_{i=1}^{\infty} 1/i$
(2) $\sum_{i=1}^{\infty} 1/i^2$
(3) $\sum_{i=0}^{\infty} \frac{3}{5^i}$
(4) $\sum_{i=0}^{\infty} \frac{3^i}{5^i}$
(5) $\sum_{i=1}^{\infty} \frac{3^i}{5^i i!}$
(6) $\sum_{n=1}^{\infty} \frac{3}{n^2-5}$
(7) $\sum_{n=1}^{\infty} \frac{2}{3^i+4^i}$
(8) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
(9) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

**Problem 19:** For what values of $x$ do the following series converge?

(1) $\sum_{n=1}^{\infty}(3x)^n$
(2) $\sum_{n=1}^{\infty}\frac{3}{x^n}$
(3) $\sum_{n=1}^{\infty}n^n(x-3)^n$
(4) $\sum_{n=1}^{\infty}\frac{(-1)^n(x-3)^n}{n}$
(5) $\sum_{n=1}^{\infty}\frac{n}{x^n}$.