Linear Independence of Sine and Cosine

Let $\mathscr{F}$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let

$$f(t) = \sin(t)$$

and

$$g(t) = \cos(t).$$

We claim that f and g are linearly independent.

Proof. Suppose that

$$af + bg = 0$$

with $a, b \in \mathbb{R}$. We must show that $a = 0$ and $b = 0$.

Suppose that $a \neq 0$. Then

$$f = (b/a)g.$$

In other words,

$$\sin(t) = (b/a)\cos(t) \qquad \text{for all } t.$$

In particular,

$$\begin{array}{ccccc} \sin(\pi/2) & = & (b/a)\cos(\pi/2) & & \text{which implies} \\ 1 & = & (b/a)0 & & \text{which implies} \\ 1 & = & 0. & & \end{array}$$

This contradiction implies that $a = 0$.

This means that

$$0\sin(t) + b\cos(t) = 0 \qquad \text{for all } t.$$

In particular,

$$0 = b\cos(0) = b.$$

Hence, $a = 0$ and $b = 0$, showing that $\sin(t)$ and $\cos(t)$ are linearly independent. □