**MA 253: Practice Exam 2**

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You may not use a graphing calculator, computer, textbook, notes, or refer to other people (except the instructor). Show all of your work; **your work is your answer**.

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**Problem 1:** Prove or Disprove: The set of $2 \times 2$ matrices with determinant 0 form a vector subspace of $M_2$. ($M_2$ is the vector space of all $2 \times 2$ matrices.

**Answer:** It is not a vector subspace. Consider the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. They each have determinant zero, but adding them produces the identity matrix which has determinant 1. Thus the set is not closed under addition and so cannot be a vector subspace.

**Problem 2:** Prove or Disprove: The set of $2 \times 2$ matrices with trace 0 form a vector subspace of $M_2$.

**Answer:** Suppose that $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ both have trace zero. Let $\alpha$ and $\beta$ be real numbers. Then

$$\mathrm{trace}(\alpha A + \beta B) = \mathrm{trace}\left( \begin{bmatrix} \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 \\ \alpha a_3 + \beta b_3 & \alpha a_4 + \beta b_4 \end{bmatrix} \right) = \alpha a_1 + \beta b_1 + \alpha a_4 + \beta b_4.$$

This can be rewritten as $\alpha(a_1 + a_4) + \beta(b_1 + b_4)$. Since both $A$ and $B$ have trace zero, $a_1 + a_4 = 0$ and $b_1 + b_4 = 0$. Consequently, $\mathrm{trace}(\alpha A + \beta B) = 0$. Hence, the set is closed under linear combinations and so is a subspace.

**Problem 3:** Prove or Disprove: The set of all functions $f \colon \mathbb{N} \to \mathbb{R}$ form a vector space. (That is, functions from the natural numbers to the real numbers.)

**Answer:** We will prove that this set, let's call it $N$ is a vector space. If $f$ and $g$ are two functions from $\mathbb{N}$ to $\mathbb{R}$, then $f + g$ is defined to be the function such that $(f + g)(x) = f(x) + g(x)$ for all $x \in \mathbb{N}$. If $k \in \mathbb{R}$, the function $kf$ is defined to be the function such that $(kf)(x) = kf(x)$. The fact that addition is associative and commutative follows immediately from the fact that addition of real numbers is associative and commutative. The neutral element is the zero function $Z(x) = 0$ for all $x \in \mathbb{N}$. If $f$ is in $N$, then

$-f$ is the function $(-1)f$. Suppose that $f, g \in N$ and that $k \in \mathbb{R}$. Then $k(f+g)(x) = k(f(x)+g(x)) = kf(x)+kg(x) = (kf+kg)(x)$. Hence, $k(f+g) = kf+kg$. Similarly, If $k, l \in \mathbb{R}$ then $(k+l)f(x) = kf(x) + lf(x) = (kf)(x) + (lf)(x) = (kf+lf)(x)$. Hence, $(k+l)f = kf+kl$. Also, $(kl)(f)(x) = klf(x) = (klf)(x)$ and $1f(x) = f(x)$ so $1f = f$. $\square$

**Problem 4:** Let $\mathbf{B}$ denote the set of all biinfinite sequences $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$. Recall that $\mathbf{B}$ is a vector space. For

$$\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

define

$$R(\mathbf{a}) = (\ldots, a_2, a_1, a_0, a_{-1}, a_{-2}, \ldots).$$

(That is, $R\mathbf{a}$) is obtained by "reflecting" $\mathbf{a}$ about the term $a_0$.)

(i) Prove that $R\colon\ \mathbf{B} \to \mathbf{B}$ is a linear transformation. Is it an isomorphism?

**Answer:** Let $\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ and $\mathbf{b} = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$ and let $k, l \in \mathbb{R}$. Then

$$R(k\mathbf{a}+l\mathbf{b}) = R(\ldots, ka_{-1}+lb_{-1}, ka_0+lb_0, ka_1+lb_1, \ldots) = (\ldots, ka_1+lb_1, ka_0+lb_0, ka_{-1}+lb_{-1}, \ldots) =$$

(ii) Prove that

$$W = \{\mathbf{a} \in \mathbf{B} : R(\mathbf{a}) = \mathbf{a}\}$$

is a vector subspace of $\mathbf{B}$. $W$ is the set of "palindromes".

**Answer:** Notice that $(\ldots, 0, 0, 0, \ldots) \in W$ and that if $\mathbf{a}, \mathbf{b} \in W$ then since we add them term by term, $\mathbf{a}+\mathbf{b} \in W$. Similarly, if $k \in R$, then to calculate $k\mathbf{a}$ we take $k$ times each entry in $a$, thus if $\mathbf{a} \in W$ then $k\mathbf{a} \in W$. Thus, $W$ is a subspace.

(iii) Find an eigenvalue and eigenvector for $R$. (i.e. find an $\mathbf{x}$ and a $\lambda \in \mathbb{R}$ so that $R(\mathbf{x}) = \lambda\mathbf{x}$.

**Answer:** Consider the vector $\mathbf{a} = (\ldots, 1, 1, 17, 1, 1, \ldots)$ where the 17 is in the $a_0$ spot. Then $a$ is a palindrome, so $R(\mathbf{a}) = \mathbf{a}$. Hence, $\mathbf{a}$ is an eigenvector and is associated to the eigenvalue 1.

(iv) Find a linear transformation $T\colon\ \mathbf{B} \to \mathbf{B}$ such that $W = \ker T$.

**Answer:** Let $T(\mathbf{a}) = R(\mathbf{a}) - \mathbf{a}$. This is the difference of two linear transformations and so is a linear transformation. The kernel of $T$ is those vectors such that $R(\mathbf{a}) = \mathbf{a}$. In other words, $W$.

**Problem 5:** Let $\mathcal{P}_3$ denote the set of polynomials of degree three or less. Let $I(ax^3+bx^2+cx+d) = \int_0^x bx^2+cx+d\,dt$.

(i) Show that $\{x^3-x^2+x-1, x^3-x^2, x^2-x, x+1\}$ is a basis for $\mathcal{P}_3$.

**Answer:** Notice that all the elements, except the first two, of the purported basis have different degrees. Thus, we need only check that the first two are not multiples of each other. This is obvious. Hence, the polynomials are linearly independent. $\mathcal{P}_3$ is a vector space of dimension 4, and so they must also span $\mathcal{P}_3$. Hence, they form a basis.

(ii) Show that $I\colon \mathcal{P}_3 \to \mathcal{P}_3$ is a linear transformation.

**Answer:** Recall that the definite integral is linear. Let $P : \mathcal{P}_3 \to \mathcal{P}_3$ be the projection defined by dropping the $x^3$ term from the polynomial. (So that the image of $P$ is $\mathcal{P}_2 \subset \mathcal{P}_3$. The function $I$ is the composition of the definite integral and the projection $P$. The composition of two linear transformations is linear and so $I$ is linear. Alternatively, you can prove this directly using the formula for $I$.

(iii) Calculate $\ker I$.

**Answer:** Notice that

$$I(ax^3 + bx^2 + cx + d) = (b/3)x^3 + (c/2)x^2 + dx.$$

Thus, $I(f) = 0$ if and only if $f(x) = ax^3$. Thus, $\ker(f) = \{ax^3 : a \in \mathbb{R}\}$.

(iv) Write down a matrix for $I$ with respect to the basis in part (i).

**Answer:** This requires more work and there are at least two ways to do it. Here is a way that requires minimal effort. Consider the basis $\mathcal{B} = \{x^3, x^2, x, 1\}$ for $\mathcal{P}_3$. With respect to $\mathcal{B}$, $I$ has the matrix

$$[I]_{\mathcal{B}} = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The change of basis matrix that takes the basis from part (i) to $\mathcal{B}$ is

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Thus the matrix of $I$ with respect to the basis from part (i) is $S^{-1}[I]_{\mathcal{B}}S$.

**Problem 6:** Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

**Answer:** We prove this by induction on $n$, where the matrix under consideration is size $n \times n$. For $n = 1$, the statement is obvious. Suppose that the statement is true for matrices of size $n \times n$. We will show that it is true for matrics of size $(n+1)\times(n+1)$. Let $A$ be an $(n+1)\times(n+1)$ matrix and denote the entry in row $i$ and column $j$ by $a_{ij}$. We want to show that $\det A = \prod_{1\le i\le n} a_{ii}$. To calculate $\det A$ we use LaPlace expansion down the first column. The only non-zero entry in this column is $a_{11}$. So

$$\det A = a_1 1 \det(A_{11})$$

where $A_{11}$ is the matrix obtained from $A$ by removing the first row and column. $A_{11}$ is an upper triangular matrix of size $n\times n$. So by our induction assumption,

$$\det A_{11} = \prod_{2\le i\le n} a_{ii}.$$

Hence,

$$\det A = a_{11}\prod_{2\le i\le n} a_{ii} = \prod_{1\le i\le n} a_{ii}.$$

$\square$

**Problem 7:** Calculate the following determinant:

$$\det\begin{bmatrix} 2 & -1 & 0 & 3 \\ -1 & 2 & 3 & 0 \\ 3 & 0 & 2 & -1 \\ 0 & 3 & 2 & -1 \end{bmatrix}$$

**Answer:** -72.

**Problem 8:** Find eigenvectors and eigenvalues for the following matrices.

(i)

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

**Answer:** The eigenvalues are $\lambda_1 = \frac{3-\sqrt{5}}{2}$ and $\lambda_2 = \frac{3+\sqrt{5}}{2}$. The eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ corresponding to $\lambda_1$ and $\lambda_2$, respectively are $\mathbf{v_1} = (1, 2-\frac{3-\sqrt{5}}{2})$ and $\mathbf{v_2} = (1, 2-\frac{3+\sqrt{5}}{2})$. (Remember that any multiple of an eigenvector is an eigenvector and so you may have differently looking eigenvectors.)

(ii)

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

**Answer:** The eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 3$. The eigenspace associated to $\lambda = 2$ is one-dimensional. The vector $(-1,-1,1)$ forms a basis for it. The eigenspace associated to $\lambda = 3$ is also one-dimensional and the vector $(-1, 0, 1)$ is a basis for it.

**Problem 9:** Prove that similar matrices have the same eigenvalues.

**Answer:** Suppose that $A$ and $B$ are similar matrices and that $S$ is a matrix so that $A = SBS^{-1}$. We present two proofs.

Here is the first proof. We first show that every eigenvalue for $B$ is also an eigenvalue for $A$. Let $\lambda$ be an eigenvalue for $B$ and let $\mathbf{v}$ be an associated eigenvector. Define $\mathbf{w} = S\mathbf{v}$. Then

$$A\mathbf{w} = SBS^{-1}(S\mathbf{v}) = SB\mathbf{v} = S(\lambda\mathbf{v}) = \lambda S\mathbf{v} = \lambda\mathbf{w}.$$

Thus, $\lambda$ is an eigenvalue for $A$ with associated eigenvector $\mathbf{w}$. We must now show that every eigenvalue for $A$ is also an eigenvalue for $B$. Here's a trick. Let $T = S^{-1}$. Then $B = TAT^{-1}$ and so we can apply our previous work to conclude that every eigenvalue for $A$ is also an eigenvalue for $B$. Hence, they have the same eigenvalues.

Here is the second proof. We show that $A$ and $B$ have the same characteristic polynomials.

$$\det(A-\lambda I) = \det(SBS^{-1}-S\lambda IS^{-1}) = \det(S(B-\lambda I)S^{-1}) = \det(B-\lambda I).$$

**Problem 10:** Suppose that $A$ is an $n \times n$ matrix with $n$ real eigenvalues (counted with algebraic multiplicity). Show that the trace of $A$ is the sum of the eigenvalues and that the determinant of $A$ is the product of the eigenvalues.

**Answer:** We will use the fact (Fact 7.2.5) that the coefficient of $\lambda^{n-1}$ in the characteristic polynomial is $\pm tr(A)$ and the constant term is $\det A$. Since we have $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$ our characteristic polynomial factors as:

$$f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda).$$

Multiplying this out, we see that the constant term is the product $\lambda_1 \ldots \lambda_n$ which by the aforementioned fact is $\det(A)$. The term containing $(-\lambda)^{n-1}$ is obtained by multiplying $(-\lambda)$ times itself $(n-1)$ times and then multiplying by $\lambda_i$ for some $i$. Thus, there are $n$ of these terms, each with coefficient $\lambda_i$ and so the coefficient of $(-\lambda)^{n-1}$ is the sum of the $\lambda_i$.

**Problem 11:** Suppose that $A$ is an $n \times n$ matrix with $n$ an odd number. Prove that $A$ has at least one real eigenvalue.

**Answer:** The characteristic polynomial $f$ has degree $n$ which is odd. Hence $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to-\infty} = \mp\infty$. Thus, by the intermediate value theorem, $f$ must have a root. This root is a real eigenvalue.

**Problem 12:** Prove that if $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues then $A$ is diagonalizable. Give an example of a $3 \times 3$ matrix without 3 distinct eigenvalues, which is, nevertheless, diagonalizable.

**Answer:** Here is one posible answer: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The eigenvalues of a triangular matrix are the diagonal entries and so this matrix has an eigenvalue of 1 with (algebraic) multiplicity 3. It is obviously diagonalizable.

**Problem 13:** Let $P\colon\ M_2 \to M_2$ be the linear transformation

$$P(M) = \frac{1}{2}(M + M^T).$$

Find all the eigenvalues of $M$ and their associated eigenvectors.

**Answer:**

Notice that if $M$ is a symmetric matrix, then $P(M) = M$. The set of symmetric matrices form a subspace of $M_2$ of dimension 3. (They are all of the form

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Thus $\lambda = 1$ is an eigenvalue of multiplicity 3 and has (linearly independent) eigenvectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The kernel of $P$ consists of matrices $M$ such that $M = -M^T$. In other words, matrices of the form

$$\begin{bmatrix} 0 & b \\ -b & 0. \end{bmatrix}$$

Thus $\lambda = 0$ is an eigenvalue of multiplicity 1 and has eigenvector $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Since $M_2$ is four-dimensional, any matrix representing $P$ is $4 \times 4$. Hence $P$ has at most 4 eigenvalues (counted with multiplicity). Therefore, these are all the eigenvalues.