MA 253: Practice Exam 2

You may not use a graphing calculator, computer, textbook, notes, or refer to other people (except the instructor). Show all of your work; **your work is your answer**.

Problem 1: Prove or Disprove: The set of 2×2 matrices with determinant 0 form a vector subspace of M_2. (M_2 is the vector space of all 2×2 matrices.

Problem 2: Prove or Disprove: The set of 2×2 matrices with trace 0 form a vector subspace of M_2.

Problem 3: Prove or Disprove: The set of all functions $f\colon \mathbb{N} \to \mathbb{R}$ form a vector space. (That is, functions from the natural numbers to the real numbers.)

Problem 4: Let $\mathbf{B}$ denote the set of all biinfinite sequences $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$. Recall that B is a vector space. For

$$\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

define

$$R(\mathbf{a}) = (\ldots, a_2, a_1, a_0, a_{-1}, a_{-2}, \ldots).$$

(That is, $R\mathbf{a}$) is obtained by "reflecting" $\mathbf{a}$ about the term a_0.)

(i) Prove that $R\colon \mathbf{B} \to \mathbf{B}$ is a linear transformation. Is it an isomorphism?

(ii) Prove that

$$W = \{\mathbf{a} \in \mathbf{B} : R(\mathbf{a}) = \mathbf{a}\}$$

is a vector subspace of $\mathbf{B}$. W is the set of "palindromes".

(iii) Find an eigenvalue and eigenvector for R. (i.e. find an $\mathbf{x}$ and a $\lambda \in \mathbb{R}$ so that $R(\mathbf{x}) = \lambda\mathbf{x}$.

(iv) Find a linear transformation $T\colon \mathbf{B} \to \mathbf{B}$ such that $W = \ker T$.

Problem 5: Let $\mathcal{P}_3$ denote the set of polynomials of degree three or less. Let $I(ax^3 + bx^2 + cx + d) = \int_0^x bx^2 + cx + d\, dt$.

(i) Show that $\{t^3 - t^2 + t - 1, t^3 - t^2, t^2 - t, t + 1\}$ is a basis for $\mathcal{P}_3$.

(ii) Show that $I\colon \mathcal{P}_3 \to \mathcal{P}_3$ is a linear transformation.

(iii) Calculate $\ker I$.

(iv) Write down a matrix for I with respect to the basis in part (i).

Problem 6: Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Problem 7: Calculate the following determinant:

$$\det \begin{bmatrix} 2 & -1 & 0 & 3 \\ -1 & 2 & 3 & 0 \\ 3 & 0 & 2 & -1 \\ 0 & 3 & 2 & -1 \end{bmatrix}$$

Problem 8: Find eigenvectors and eigenvalues for the following matrices.

(i)

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

Problem 9: Prove that similar matrices have the same eigenvalues.

Problem 10: Suppose that A is an $n \times n$ matrix with n real eigenvalues (counted with algebraic multiplicity). Show that the trace of A is the sum of the eigenvalues and that the determinant of A is the product of the eigenvalues.

Problem 11: Suppose that A is an $n \times n$ matrix with n an odd number. Prove that A has at least one real eigenvalue.

Problem 12: Prove that if A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalizable. Give an example of a 3×3 matrix without 3 distinct eigenvalues, which is, nevertheless, diagonalizable.

Problem 13: Let $P\colon\ M_2 \to M_2$ be the linear transformation

$$P(M) = \frac{1}{2}(M + M^T).$$

Find all the eigenvalues of M and their associated eigenvectors.