MA 253: Practice Exam 1 Solutions

You may not use a graphing calculator, computer, textbook, notes, or refer to other people (except the instructor). Show all of your work; **your work is your answer**.

Problem 1: Solve the following system of linear equations using Gauss-Jordan elimination on a matrix.

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 3 \\ & & 2y & - & z & = & -1 \\ 4x & + & y & & & = & 0 \end{array}$$

Solution: $x = -2/3$, $y = 8/3$, and $z = 19/3$.

Problem 2: Which of the following matrices are in reduced row echelon form? For each that is not, circle an entry in the matrix which shows that it is not in reduced row echelon form.

(a.) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ **(b.)** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c.) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ **(d.)** $\begin{bmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e.) $\begin{bmatrix} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & -2 & \sqrt{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **(f.)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Solution: Matrices (c.) and (e.) are in reduced row echelon form. For matrix (a.), the 2,1 entry violates the requirements for being in reduced row echelon form. For matrix (b.), the 1,2 entry violates the requirements. For matrix (d.), the 1,1 entry violates the requirements and for matrix (f.) the 4,2 entry violates the requirements.

Problem 3: Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Prove that there is a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Solution: Let $\mathbf{e_i}$ be the ith standard basis vector of $\mathbb{R}^n$. Define the matrix:

$$A = \begin{bmatrix} \\ T(\mathbf{e_1}) & T(\mathbf{e_2}) & \dots & T(\mathbf{e_n}) \\ \\ \end{bmatrix}$$

where the ith column of A is $T(\mathbf{e_i})$. We now need to show that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Let $\mathbf{x} \in \mathbb{R}^n$ be an arbitrary vector. Then, since $\{\mathbf{e_i}\}$ is a basis for $\mathbb{R}^n$, there are scalars, $x_1, \dots, x_n$ so that

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_n\mathbf{e_n}.$$

Hence,

$$T(\mathbf{x}) = T(x_1\mathbf{e_1} + \dots + x_n\mathbf{e_n}).$$

Since T is linear this equivalent to:

$$T(\mathbf{x}) = x_1 T(\mathbf{e_1}) + \dots + x_n T(\mathbf{e_n}).$$

By the construction of A,

$$A\mathbf{e_i} = T(\mathbf{e_i}).$$

Hence,

$$T(\mathbf{x}) = x_1 A\mathbf{e_1} + \dots + x_n A\mathbf{e_n}.$$

By the properties of matrix multiplication,

$$T(\mathbf{x}) = A(x_1\mathbf{e_1} + \dots + x_n\mathbf{e_n}).$$

Consequently,

$$T(\mathbf{x}) = A\mathbf{x}$$

as desired. $\square$

Problem 4: Define what it means for a set of vectors to be linearly independent and describe a method whereby you can determine if a given collection of vectors is linearly independent.

Solution: A set of vectors is linearly independent if no nontrivial linear combination (that is, a linear combination with at least one non-zero coefficient) is equal to the zero vector. To test if k vectors are linearly independent, make a matrix whose columns are those vectors. If the matrix has rank k if and only if the vectors are linearly independent.

Problem 5: Suppose that A is an $n \times m$ matrix. Let $T(\mathbf{x}) = A\mathbf{x}$.

(a.) Define $\mathrm{rank}(A)$.
Solution: The number of leading ones in $ref(A)$. Equivalently, the dimension of the image of A.

(b.) If $m > n$, what can you say about $\mathrm{rank}(A)$?
Solution: Since there are more columns than rows, the rank of A is no more than the number of rows, which is n.

(c.) What is the domain of T?
Solution: $\mathbb{R}^m$.

(d.) What is the codomain of T? (In other words, what is k so that $T(\mathbf{x}) \in \mathbb{R}^k$? Your answer should be either $\mathbb{R}^n$ or $\mathbb{R}^m$.
Solution: $\mathbb{R}^n$.

(e.) Prove that T is a linear transformation. You may assume basic facts about matrix multiplication.
Solution:

$$T(a\mathbf{x} + b\mathbf{y}) = A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y} = aT(\mathbf{x}) + bT(\mathbf{y}).$$

(f.) Define $\ker(T)$ and prove that it is a subspace.
Solution:

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^m : T(\mathbf{x}) = \mathbf{0}\}.$$

Suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are in $\ker(T)$ and that $a, b \in \mathbb{R}$. Then,

$$T(a\mathbf{x_1} + b\mathbf{x_2}) = aT(\mathbf{x_1}) + bT(\mathbf{x_2}) = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$$

since T is a linear tranformation and $\mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^m$. □

(g.) Define $\mathrm{im}(T)$ and prove that it is a subspace.
Solution:

$$\mathrm{im}(T) = \{\mathbf{b} \in \mathbb{R}^n : \text{ there exists } \mathbf{x} \in \mathbb{R}^m \text{ such that } T(\mathbf{x}) = \mathbf{b}\}$$

Suppose that $\mathbf{y_1}, \mathbf{y_2} \in \mathrm{im}(T)$ and that $a, b \in \mathbb{R}$. By the definition of image there are vectors $\mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^m$ such that $T(\mathbf{x_1}) = \mathbf{y_1}$ and $T(\mathbf{x_2}) = \mathbf{y_2}$. Then, since T is a linear transformation

$$T(a\mathbf{x_1} + b\mathbf{x_2}) = aT(\mathbf{x_1}) + bT(\mathbf{x_2}) = a\mathbf{y_1} + b\mathbf{y_2}.$$

Hence, $a\mathbf{y_1} + b\mathbf{y_2} \in \mathrm{im}(T)$ as desired. □

(h.) Carefully state and prove the rank-nullity theorem.

Solution: The statement is: For an $n \times m$ matrix A, $\dim \mathrm{im}(A) + \dim \ker(A) = m$. To prove it, recall that a basis for $\mathrm{im}(A)$ can be formed by taking the non-redundant columns of A. Each redundant column corresponds to a free variable. Thus $\dim \mathrm{im}(A)$ plus the number of free variables of A is equal to the number of columns of A. Each free variable of A gives rise to a vector in a basis for $\ker(A)$. Thus the sum of the dimensions of $\mathrm{im}(A)$ and $\ker(A)$ must be the number of columns of A. □

(i.) Define what it means for T to be injective.

Solution: T is injective if and only if for every two distinct $\mathbf{x_1}$ and $\mathbf{x_2}$ in $\mathbb{R}^m$, $T(\mathbf{x_1}) \neq T(\mathbf{x_2})$.

(j.) Define what it means for T to be surjective.

Solution: T is surjective if and only if for every $\mathbf{b} \in \mathbb{R}^n$, there exists $\mathbf{x} \in \mathbb{R}^m$ such that $T(\mathbf{x}) = \mathbf{b}$.

(k.) Explain why T is injective if and only if the system $T(\mathbf{x}) = \mathbf{b}$ has at most one solution for each $\mathbf{b}$.

Solution: Suppose that there were some $\mathbf{b}$ such that $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to $T(\mathbf{x}) = \mathbf{b}$. Then $T(\mathbf{x_1}) = T(\mathbf{x_2})$ and so by the definition of "injective" given above, $\mathbf{x_1} = \mathbf{x_2}$.

(l.) Explain why T is surjective if and only if the system $T(\mathbf{x}) = \mathbf{b}$ is consistent for each $\mathbf{b}$.

Solution: The system is consistent if and only if there is a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$. But this means that $\mathbf{b} \in \mathrm{im}(T)$ for all $\mathbf{b} \in \mathbb{R}^n$. This is the definition of "surjective".

(m.) Prove that T is injective if and only if $\ker(T) = \{\mathbf{0}\}$.

Solution: Suppose first that T is injective. Since $T(\mathbf{0}) = \mathbf{0}$, by (k) $\mathbf{0}$ is the only solution to $T(\mathbf{x}) = \mathbf{0}$. Hence $\ker(T) = \{\mathbf{0}\}$. Now suppose that $\ker(T) = \{\mathbf{0}\}$. Suppose that $T(\mathbf{x_1}) = T(\mathbf{x_2})$. Then,

$$T(\mathbf{x_1}) - T(\mathbf{x_2}) = \mathbf{0}.$$

Since T is linear, this is equivalent to

$$T(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0}.$$

Hence, $\mathbf{x_1} - \mathbf{x_2} \in \ker(T)$. Since $\ker(T) = \{\mathbf{0}\}$, $\mathbf{x_1} - \mathbf{x_2} = \mathbf{0}$. But this means that $\mathbf{x_1} = \mathbf{x_2}$. Hence, by the definition of "injective", T is injective. □

(n.) Prove that if T is invertible, then its inverse function T^{-1} is a linear transformation.

Solution: We must show that

$$T^{-1}(a\mathbf{x} + b\mathbf{y}) = aT^{-1}(\mathbf{x}) + bT^{-1}(\mathbf{b}).$$

T is invertible and therefore injective. Thus the above equation is equivalent to

$$T\big(T^{-1}(a\mathbf{x}+b\mathbf{y})\big) = T\big(aT^{-1}(\mathbf{x}) + bT^{-1}(\mathbf{b})\big).$$

Since T is linear and T and T^{-1} are inverses, this is equivalen to

$$a\mathbf{x}+b\mathbf{y} = aT\big(T^{-1}(\mathbf{x})\big) + bT\big(T^{-1}(\mathbf{b})\big).$$

This is equivalen to

$$a\mathbf{x}+b\mathbf{y} = a\mathbf{x}+b\mathbf{y},$$

which is obviously true. □

(o.) Prove that if A is invertible then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

Solution: Suppose that A^{-1} is the inverse of A. Then

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \;\Rightarrow \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

Since matrix multiplication is well-defined, $A^{-1}\mathbf{b}$ is a single vector in $\mathbb{R}^n$. Hence, there is a solution to $A\mathbf{x} = \mathbf{b}$ and it is unique. □

(p.) Prove that A is invertible if and only if $\mathrm{rref}(A)$ is the identity matrix.

Solution: The matrix $ref(A)$ is obtained from A by finitely many row operations. Let $E_1, \dots, E_p$ be the elementary matrices corresponding to these row operations so that

$$\mathrm{rref}(A) = E_p E_{p-1} \dots E_2 E_1 A.$$

Every elementary matrix is invertible and so the above equation can be rewritten as

$$E_1^{-1}E_2^{-1} \dots E_{p-1}^{-1}E_p^{-1}\,\mathrm{rref}(A) = A.$$

Hence, if $\mathrm{rref}(A)$ is the identity matrix, A is the product of elementary matrices. The product of invertible matrices is invertible and so A is also invertible. Now suppose that A is invertible. The first equation shows that $\mathrm{rref}(A)$ is the product of A and elementary matrices. Hence $\mathrm{rref}(A)$ is invertible. This implies that it is square and that the transformation

$$T(\mathbf{x}) = \mathrm{rref}(A)\mathbf{x}$$

is injective and surjective. Since it is injective, the rank of $\mathrm{rref}(A)$ is the number of columns (which is also the number of rows) or A. Hence, every column and every row of $\mathrm{rref}(A)$ has a leading one. Since $\mathrm{rref}(A)$ is in reduced row echelon form, $\mathrm{rref}(A)$ is the identity matrix. □

(q.) Suppose that $\{\mathbf{x_1}, \ldots, \mathbf{x_k}\}$ is a set of vectors such that each vector is a solution to the equation $T(\mathbf{x}) = \mathbf{b}$. Let

$$\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{x_i}$$

where the c_i are scalars such that $\sum_{i=1}^{k} c_i = 1$. Prove that $T(\mathbf{u}) = \mathbf{b}$.

Solution: Since T is linear,

$$T(\mathbf{u}) = T\left(\sum c_i \mathbf{x_i}\right) = \sum c_i T(\mathbf{x_i}) = \sum c_i \mathbf{b} = \mathbf{b} \sum c_i = \mathbf{b}.$$

Problem 6: Let

$$M = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find M^{-1} and show that your answer is correct by performing a matrix multiplication.

Solution: $M = \begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$.

Problem 7: Let

$$A = \begin{bmatrix} 1 & 10 & -17 & 38 \\ 2 & 10 & -14 & 36 \\ 3 & 2 & 5 & 2 \\ 4 & 0 & 12 & -8 \\ 5 & 0 & 15 & -10 \\ 6 & 8 & 2 & 20 \end{bmatrix}.$$

The reduced row echelon form of A is

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a.) Find a basis for $\ker(A)$.

Solution: A basis for $\ker(A)$ is

$$\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b.) Find a basis for $\text{im}(A)$.

Solution: A basis for $\text{im}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 10 \\ 10 \\ 2 \\ 0 \\ 0 \\ 8 \end{bmatrix} \right\}$$

Problem 8: Let W be the subset of $\mathbb{R}^3$ consisting of the x axis, the y axis, the z axis, and the line through the origin and the vector $(1, 1, 1)$. Explain why W is not a subspace of $\mathbb{R}^3$.

Solution: W is not closed under addition. For example, $(1, 0, 0)+(0, 1, 0) = (1, 1, 0) \notin W$.

Problem 9: Prove that if $\mathcal{B}$ is a basis for the subspace $V \subset \mathbb{R}^n$ then each vector in V can be uniquely written as a linear combination of vectors in $\mathcal{B}$.

Solution: Since $\text{span}\,\mathcal{B} = V$, each vector in V is a linear combination of vectors in $\mathcal{B}$. Suppose that $\mathbf{v} \in V$ is a vector which can be written as a linear combination of vectors in $\mathcal{B}$ in more than one way. That is, suppose

$$\mathbf{v} = \sum c_i \mathbf{b_i} = \sum d_i \mathbf{b_i}$$

where the (finite) sum is taken over all elements of $\mathcal{B}$. Hence,

$$\mathbf{0} = \sum c_i \mathbf{b_i} - \sum d_i \mathbf{b_i} = \sum (c_i - d_i)\mathbf{b_i}.$$

We have a linear combination of the vectors of $\mathcal{B}$ equalling zero. Since $\mathcal{B}$ is a basis, its vectors are linearly independent. Consequently, the linear combination must have all coefficients equal to zero. That is, for each i, $c_i - d_i = 0$. Therefore, $c_i = d_i$ for all i. Thus, there is a unique way of writing $\mathbf{v}$ as the linear combination of vectors in $\mathcal{B}$. $\square$

Problem 10: Let W be a subspace of $\mathbb{R}^n$ and define

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in \mathbb{R}^n\}.$$

Prove that $W^\perp$ is a subspace.

Solution: Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $W^{\perp}$ and $a, b \in \mathbb{R}$. We must show that $a\mathbf{u} + b\mathbf{v} \in W^{\perp}$. Let $\mathbf{w}$ be an arbitrary vector in W. Then

$$(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} = (a\mathbf{u}) \cdot \mathbf{w} + (b\mathbf{v}) \cdot \mathbf{w} = a\mathbf{u} \cdot \mathbf{w} + b\mathbf{v} \cdot \mathbf{w} = a(0) + b(0) = 0$$

as desired. □

Problem 11: Consider the set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

It is a fact that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(1) Write the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as a linear combination of the vectors in $\mathcal{B}$.

Solution: Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then

$$B^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 1 & -2 \\ -2 & 1 & 3 \\ 2 & -1 & 2 \end{bmatrix}.$$

Let

$$\mathbf{c} = B^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 3/5 \end{bmatrix}.$$

Then $2/5(1, 2, 0) + 2/5(0, 2, 1) + 3/5(1, -1, 1) = (1, 1, 1)$.

(2) In $\mathcal{B}$-coordinates the vector $\mathbf{w}$ is written

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \\ 3 \end{bmatrix}.$$

Write $\mathbf{w}$ using the standard coordinate system for $\mathbb{R}^3$.

Solution:

$$\mathbf{w} = 10 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \\ 1 \end{bmatrix}$$

(3) Consider the linear tranformation $T\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ given (in standard coordinates) by

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

What is the matrix for T in $\mathcal{B}$ coordinates? (You do not need to perform the required calculations. Simply write down an expression which, if computed, will produce the matrix.)

Solution: Continue using the notation from part (a). Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The answer is then $B^{-1}AB$.