MA 253: Practice Exam 1

You may not use a graphing calculator, computer, textbook, notes, or refer to other people (except the instructor). Show all of your work; **your work is your answer**.

Problem 1: Solve the following system of linear equations using Gauss-Jordan elimination on a matrix.

$$\begin{array}{ccccccc} x & - & y & + & z & = & 3 \\ & & 2y & - & z & = & -1 \\ 4x & + & y & & & = & 0 \end{array}$$

Problem 2: Which of the following matrices are in reduced row echelon form? For each that is not, circle an entry in the matrix which shows that it is not in reduced row echelon form.

(a.) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ **(b.)** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c.) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ **(d.)** $\begin{bmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e.) $\begin{bmatrix} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & -2 & \sqrt{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **(f.)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Problem 3: Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Prove that there is a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Problem 4: Define what it means for a set of vectors to be linearly independent and describe a method whereby you can determine if a given collection of vectors is linearly independent.

Problem 5: Suppose that A is an $n \times m$ matrix. Let $T(\mathbf{x}) = A\mathbf{x}$.

(a.) Define $\operatorname{rank}(A)$.
(b.) If $m > n$, what can you say about $\operatorname{rank}(A)$?
(c.) What is the domain of T?
(d.) What is the codomain of T? (In other words, what is k so that $T(\mathbf{x}) \in \mathbb{R}^k$? Your answer should be either $\mathbb{R}^n$ or $\mathbb{R}^m$.
(e.) Prove that T is a linear transformation. You may assume basic facts about matrix multiplication.
(f.) Define $\ker(T)$ and prove that it is a subspace.
(g.) Define $\operatorname{im}(T)$ and prove that it is a subspace.
(h.) Carefully state and prove the rank-nullity theorem.
(i.) Define what it means for T to be injective.
(j.) Define what it means for T to be surjective.
(k.) Explain why T is injective if and only if the system $T(\mathbf{x}) = \mathbf{b}$ has at most one solution for each $\mathbf{b}$.
(l.) Explain why T is surjective if and only if the system $T(\mathbf{x}) = \mathbf{b}$ is consistent for each $\mathbf{b}$.
(m.) Prove that T is injective if and only if $\ker(T) = \{\mathbf{0}\}$.
(n.) Prove that if T is invertible, then its inverse function T^{-1} is a linear transformation.
(o.) Prove that if A is invertible then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
(p.) Prove that A is invertible if and only if $\operatorname{rref}(A)$ is the identity matrix.
(q.) Suppose that $\{\mathbf{x_1}, \dots, \mathbf{x_k}\}$ is a set of vectors such that each vector is a solution to the equation $T(\mathbf{x}) = \mathbf{b}$. Let

$$\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{x_i}$$

where the c_i are scalars such that $\sum_{i=1}^{k} c_i = 1$. Prove that $T(\mathbf{u}) = \mathbf{b}$.

Problem 6: Let

$$M = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find M^{-1} and show that your answer is correct by performing a matrix multiplication.

Problem 7: Let

$$A = \begin{bmatrix} 1 & 10 & -17 & 38 \\ 2 & 10 & -14 & 36 \\ 3 & 2 & 5 & 2 \\ 4 & 0 & 12 & -8 \\ 5 & 0 & 15 & -10 \\ 6 & 8 & 2 & 20 \end{bmatrix}.$$

The reduced row echelon form of A is

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a.) Find a basis for $\ker(A)$.
(b.) Find a basis for $\mathrm{im}(A)$.

Problem 8: Let W be the subset of $\mathbb{R}^3$ consisting of the x axis, the y axis, the z axis, and the line through the origin and the vector $(1, 1, 1)$. Explain why W is not a subspace of $\mathbb{R}^3$.

Problem 9: Prove that if $\mathcal{B}$ is a basis for the subspace $V \subset \mathbb{R}^n$ then each vector in V can be uniquely written as a linear combination of vectors in $\mathcal{B}$.

Problem 10: Let W be a subspace of $\mathbb{R}^n$ and define

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in \mathbb{R}^n\}.$$

Prove that $W^\perp$ is a subspace.

Problem 11: Consider the set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

It is a fact that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(1) Write the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

as a linear combination of the vectors in $\mathcal{B}$.

(2) In $\mathcal{B}$-coordinates the vector $\mathbf{w}$ is written

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \\ 3 \end{bmatrix}.$$

Write $\mathbf{w}$ using the standard coordinate system for $\mathbb{R}^3$.

(3) Consider the linear tranformation $T\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ given (in standard coordinates) by

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

What is the matrix for T in $\mathcal{B}$ coordinates? (You do not need to perform the required calculations. Simply write down an expression which, if computed, will produce the matrix.)