**MA 253: Final Review and Practice Problems** **Name:**________________________

## 1. PROBLEMS

(1) Let

$$\mathcal{T}_n = \Big\{T : T \text{ is a linear transformation } M_n \to \mathbb{R}\Big\}.$$

You may wish to do this problem assuming that $n = 2$ or $n = 3$.

(a) Prove that $\mathcal{T}_n$ is a vector space.

**Solution:** Let $T, S \in \mathcal{T}_n$ be linear transformations and let $\alpha, \beta$ be real numbers. We begin by showing that $\mathcal{T}_n$ is closed under linear combinations. Define $P = \alpha T + \beta S$. We must show that $P$ is a linear transformation. Let $M, N \in M_n$ and $a, b \in \mathbb{R}$. Then:

$$P(aM + bN) = \alpha T(aM + bN) + \beta S(aM + bN)$$

by the definition of $P$. Since $T$ and $S$ are linear transformations we have

$$\alpha T(aM+bN)+\beta S(aM+bN) = \alpha(aT(M)+bT(N))+\beta(aS(M)+bS(N)).$$

The right hand side consists of real numbers so we have:

$$\alpha(aT(M)+bT(N))+\beta(aS(M)+bS(N)) = a(\alpha T(M)+\beta S(M))+b(\alpha T(N)+\beta S(N))$$

The right hand side is exactly $aP(M)+bP(N)$, so $P$ is a linear transformation.

The fact that addition of elements of $\mathcal{T}_n$ is associative and commutative follows directly from the fact that addition in $\mathbb{R}$ is associative and commutative. The neutral element is the linear transformation $Z\colon M_n \to \mathbb{R}$ defined so that

$$Z(M) = 0$$

for all $n \times n$ matrices $M$. If $T \in \mathcal{T}_n$ then if we define $S\colon M_n \to \mathbb{R}$ by

$$S(M) = -T(M)$$

it is easy to see that $S$ is linear and that $S + T = Z$; thus, elements of $\mathcal{T}_n$ have additive inverses.

The fact that the distributive properties hold follows immediately from the fact that multiplication and addition of real numbers satisfy the distributive properties. It is also clear that for any $T \in \mathcal{T}_n$, $1 \cdot T = T$. Hence, $\mathcal{T}_n$ is a vector space.

(b) Find a basis for $\mathcal{T}_n$.

**Solution:** For a matrix $M$, let $m_{ij}$ denote the entry in the $i$th row and $j$th column. Define

$$T_{ij}(M) = m_{ij}.$$

It is easy to check that $T_{ij}$ is linear. We need to check that $\{T_{ij}\}$ is linearly independent. Suppose that there exist constants $c_{ij}$, not all zero, such that

$$\sum_{ij} c_{ij}T_{ij} = Z.$$

This means that, for all $M \in M_n$

$$\sum_{ij} c_{ij}T_{ij}(M) = 0.$$

Since not all the $c_{ij}$ are zero, there exists at least one $c_{kl} \neq 0$. Let $M_{kl}$ be the matrix having a 1 in the $k$th row and $l$th column and zeroes everywhere else. Then

$$\sum_{ij} c_{ij}T_{ij}(M_{kl}) = c_{kl}T_{kl}(M_{kl}) = c_{kl} \neq 0.$$

Hence, the constants $c_{ij}$ cannot exist. Therefore $\{T_{ij}\}$ is linearly independent.

We now need to show that $\{T_{ij}\}$ spans $\mathcal{T}_n$. Let $T \in \mathcal{T}_n$ be an arbitrary linear transformation. Let $M_{ij}$ be the $n \times n$ matrix with a 1 in the $ij$ location and zeroes everywhere else. Let $\alpha_{ij} = T(M_{ij})$. Then for an arbitrary matrix $M$,

$$T(M) = T(\sum_{ij} m_{ij}M_{ij}) = \sum_{ij} m_{ij}\alpha_{ij}$$

since $T$ is linear. On the other hand,

$$\sum_{ij} \alpha_{ij}T_{ij}(M) = \sum_{ij} \alpha_{ij}T_{ij}(\sum_{kl} m_{kl}M_{kl}).$$

Since each $T_{ij}$ is linear and since $T_{ij}(M_{kl}) = 0$ unless $i = k$ and $j = l$ (in which case $T_{ij}(M_{kl}) = 1$) we have,

$$\sum_{ij} \alpha_{ij}T_{ij}(\sum_{kl} m_{kl}M_{kl}) = \sum_{ij} \alpha_{ij}m_{ij}.$$

Thus,

$$T = \sum_{ij} \alpha_{ij}T_{ij}.$$

Thus, $\{T_{ij}\}$ is both linearly independent and spans $\mathcal{T}_n$. It is, therefore, a basis.

(c) Construct an isomorphism from $M_n$ to $\mathcal{T}_n$.
**Solution:** Use the notation from the previous part and define $\tau\colon M_n \to \mathcal{T}_n$ by

$$\tau(M) = T_M$$

where $T_M$ is the linear transformation

$$T_M(N) = \sum_{ij} m_{ij} n_{ij}$$

($n_{ij}$ is the $ij$th entry of $N$.) To see that $\tau$ is an isomorphism, we need only prove that $\ker \tau = \{0\}$ since both $\mathcal{T}_n$ and $M_n$ are finite dimensional. Suppose that $M \in \ker \tau$. Then for all $N \in M_n$:

$$T_M(N) = 0.$$

Notice that

$$T_M(M_{ij}) = m_{ij}$$

Thus, if $M \in \ker \tau$, $m_{ij} = 0$ for all entries of $M$. That is $M$ is the zero matrix. Hence, $\tau$ is an isomorphism.

(2) Let $V$ be the set of all $3 \times 3$ matrices that commute with $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Show that $V$ is a subspace of $M_3$.
**Solution:** Let $A$ be the given matrix and let $M$ and $N$ be arbitrary matrices. We know that $MA = AM$ and $NA = AN$. Let $a, b \in \mathbb{R}$. Then,

$$(aM+bN)A = a(MA)+b(NA) = a(AM)+b(AN) = A(aM)+A(bN) = A(aM+bN)$$

by properties of matrix multiplication. Hence $V$ is closed under linear combinations. It is, therefore, a subspace.

(b) Find a basis for $V$. What is its dimension?
**Solution:** Consider:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Comparing the entries shows that if a matrix $B$ commutes with $A$ then

$$B = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

Thus, a basis for $V$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

Thus, $V$ is 3–dimensional.

(3) Let $A$ be an $n \times n$ matrix such that $A^4 = I$. Show that each real eigenvalue of $A$ is $\pm 1$.

**Solution:** If $\lambda$ is an eigenvalue of $A$, then $\lambda^4$ is an eigenvalue of $A^4$. Since $A^4 = I$, $\lambda^4 = 1$. Thus, if $\lambda$ is real, $\lambda = \pm 1$.

(4) Let $A$ be an $n \times n$ matrix such that $A^4$ is the zero matrix. Find all eigenvalues of $A$.

**Solution:** As in the previous problem, if $\lambda$ is an eigenvalue of $A$, $\lambda^4$ is an eigenvalue of $A^4 = 0$. The only complex number which when raised to the fourth power produces zero is 0. Hence, $\lambda = 0$ and all eigenvalues of $A$ are zero.

(5) Let $V$ be the vector space of all infinite sequences of real numbers having only finitely many nonzero terms. Let $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_m, 0, 0, \ldots)$ be a fixed vector of $V$. Define a function $T\colon\ V \to \mathbb{R}$ by

$$T(\mathbf{b}) = \sum a_i b_i$$

where $\mathbf{b} = (b_0, b_1, b_2, \ldots)$.

(a) Show that $T$ is linear.

**Solution:** Let $\mathbf{b} = (b_0, b_1, \ldots)$ and $\mathbf{c} = (c_0, c_1, \ldots) \in V$. Let $a, b \in \mathbb{R}$. Then:

$$T(a\mathbf{b} + b\mathbf{c}) = \sum a_i(ab_i + bc_i) = \sum(aa_ib_i + ba_ic_i).$$

This can be rewritten as

$$a \sum a_i b_i + b \sum a_i c_i = aT(\mathbf{b}) + bT(\mathbf{c}).$$

Hence, $T$ is linear.

(b) Suppose that $\mathbf{a} = (1, -1, 0, 0, \ldots)$. Find $\ker T$.

**Solution:** Suppose $\sum a_i b_i = 0$. Then, because of the choice of $\mathbf{a}$, we have $b_0 - b_1 = 0$. That is, $b_0 = b_1$. Thus, $\ker T$ consists of all sequences of real numbers having only finitely many nonzero terms whose first two entries are equal.

(6) Let $\mathcal{B}$ be the following set in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^4$.
**Solution:** There are several ways to do this. Probably the easiest is to put the four vectors into a matrix (in the order given). Call the matrix $B$. Then the determinant of $B$ is one, so the columns of $B$ are linearly independent. Since we have four linearly independent vectors in $\mathbb{R}^4$, they must be a basis for $\mathbb{R}^4$.

(b) Write the vector $(2, 2, 2, 1)$ in this basis.
**Solution:** $(3, 7, -5, 6)$.

(c) Let $T$ be the following linear transformation. Find a matrix which represents $T$ in the basis $\mathcal{B}$.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

**Solution:**

$$\begin{pmatrix} 2 & -1 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ -1 & 0 & -3 & -4 \\ 0 & 1 & 4 & 4 \end{pmatrix}_{\mathcal{B}}$$

(d) Apply the Gram-Schmidt process to $\mathcal{B}$ to obtain an orthonormal basis for $\mathbb{R}^4$.
**Solution:**

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \\ 0 \end{bmatrix} \right\}.$$

(7) Consider the plane in $\mathbb{R}^3$ defined by $2x - y + z = 0$. Find a $3 \times 3$ matrix which represents orthogonal projection onto this plane.

**Solution:** Notice that $\{(1, 0, -2), (0, 1, 1), (2, -1, 1)\}$ forms a basis $\mathcal{B}$ for $\mathbb{R}^3$ where the first two vectors are in the plane and third is orthogonal to the plane. With respect to $\mathcal{B}$ the matrix for the projection is:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathcal{B}}.$$

Thus, the matrix with respect to the standard basis is $B^{-1}PB$ where

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}.$$

(8) Suppose that $A$ is a $7 \times 7$ matrix with characteristic equation

$$f_A(\lambda) = (-\lambda)^3(2-\lambda)(3-\lambda)(4-\lambda)(-5-\lambda)$$

(a) What are the eigenvalues of $A$?
(b) What is the determinant of $A$?
(c) What is the trace of $A$?
(d) What is the characteristic equation of $A^T$?
(e) Is $A$ invertible?
(f) What can you say about the dimension of $\ker A$?
(g) What can you say about the dimension of $\operatorname{im} A$?

**Solution:** The eigenvalues are $0$ (with mult. 3), $2$, $3$, $4$, and $-5$. The determinant is the product of these and so is 0. This means the matrix is not invertible. The trace is the sum of the eigenvalues and so is 4. The characteristic equation of $A^T$ is the same since $(A^T - \lambda I) = (A - \lambda I)^T$. The eigenspaces of all the non-zero eigenvalues of $A$ are 1–dimensional, this accounts for 4 out of the 7 dimensions. Thus, the dimension of $\ker A$ is not more than 3. By the rank-nullity theorem, the dimension of $\operatorname{im} A$ is at least 4.

(9) Suppose that an $n \times n$ matrix $A$ is similar to its inverse $A^{-1}$. What can you say about the determinant of $A$?

**Solution:** Similar matrices have the same determinant and inverse matrices have determinants which are reciprocals. Thus, if $A$ is similar to its inverse it must have determinant $\pm 1$.

(10) Suppose that 30 percent of math students in a given semester take a math course the following semester, while 70 percent of them take a poetry course. Suppose that 40 percent of poetry students in a given semester take a poetry course the following semester while 60 percent of them take a math course. Is there some point in the future where either the mathematics or the poetry department will have to close down because of lack of students? If not, what happens in the long term?

**Solution:** In the long term, the poetry courses will enroll 54 percent of the students and the math courses will enroll 46 percent of the students (approximately).

(11) Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with positive determinant such that $T$ preserves the lengths of all vectors.

(a) Show that there is a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{x}$ (That is, $T$ has a fixed point.)

**Solution:** The characteristic polynomial for $T$ has degree 3. Every degree 3 polynomial factors over the real numbers, and so $T$ has a real eigenvalue. If an eigenvalue is not $\pm 1$, the length of the associated eigenvector is changed. Thus, all eigenvalues must be $\pm 1$. If $T$ has two complex eigenvalues, they are complex conjugates and when multiplied together produce a positive number. Thus, since the determinant of $T$ is positive, in this case $T$ must have $+1$ as an eigenvalue. If $T$ has three real eigenvalues at least one of them must be +1 since the determinant of $T$ is positive. Thus, in all cases $T$ has $+1$ as an eigenvalue. Any associated eigenvector will be a fixed point of $T$.

(b) Must $T$ be an orthogonal transformation? Hint: It is enough to figure out whether $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. Consider the equation

$$(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = (T(\mathbf{x}) + T(\mathbf{y})) \cdot (T(\mathbf{x} + T(\mathbf{y}))$$

**Solution:** Yes. Working out the equation from the hint shows that $T$ preserves the dot product. Thus $T$ preserves the property of being orthonormal. Orthogonal linear transformations are exactly the linear transformations that do this.

(12) Let $W \subset \mathbb{R}^n$ be a subspace. Define

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

(a) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.
**Solution:** Let $\mathbf{u}, \mathbf{v} \in W^\perp$ and $a, b \in \mathbb{R}$. We must show that $a\mathbf{u} + b\mathbf{v} \in W^\perp$. Let $w \in W$. We must show:

$$(a\mathbf{u} + b\mathbf{v}) \cdot w = 0.$$

This follows directly from the properties of dot product and the fact that $\mathbf{u}, \mathbf{v} \in W^\perp$.

(b) Describe a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $W^\perp = \ker T$
**Solution:** Orthogonal projection onto $W$.

(c) What is the dimension of $W^\perp$ in terms of the dimension of $W$?
**Solution:** $n - \dim W$ by the rank-nullity theorem.

(13) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a collection of orthonormal vectors in $\mathbb{R}^n$. Prove that they are linearly independent.
**Solution:** Suppose that $c_i$ are coefficients such that

$$\sum c_i \mathbf{u}_i = \mathbf{0}.$$

If all the $c_i$ are zero we are done. So suppose that $c_j \neq 0$. Then:

$$0 = \mathbf{u}_j \cdot \left(\sum c_i \mathbf{u}_i\right) = \sum c_i (\mathbf{u_j} \cdot \mathbf{u_i}).$$

Since the vectors $\{\mathbf{u}_i\}$ are orthonormal the only term in the sum which is non-zero is $c_j(\mathbf{u_j} \cdot \mathbf{u_j}) = c_j$. Thus we have $c_j \neq 0$ and $c_j = 0$, a contradiction. Hence, all the $c_i = 0$. This implies that the vectors $\{\mathbf{u}_i\}$ are linearly independent.