MA 253: Final Review and Practice Problems **Name:**______________________

1. Review

(1) Properties of matrices and linear transformations $\mathbb{R}^n \to \mathbb{R}^m$
  (a) What are all the possible ways to tell if a matrix is invertible?
  (b) What are all the possible ways to tell if a linear transformation is injective?
  (c) What are all the possible ways to tell if a linear transformation is surjective?
  (d) Define and explain "linear independence".
  (e) If $\mathcal{V}$ is a collection of vectors in $\mathbb{R}^n$ and if $T\colon\ \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, what properties of T will guarantee that if $\mathcal{V}$ is linearly independent, so is $T(\mathcal{V})$?
  (f) Show that if T is a linear transformation, then there is a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

(2) Abstract Vector Spaces
  (a) What is the definition of "vector space" (or "linear space")?
  (b) What is the definition of "basis"?
  (c) What are the definitions of "linear transformation" and "isomorphism"?
  (d) Give an example of an infinite dimensional vector space.
  (e) What is the definition of "subspace"?
  (f) Prove that if $T\colon\ V \to W$ is a linear transformation, $\ker T$ and $\operatorname{im} T$ are subspaces.
  (g) Review the rank-nullity theorem. Given an example of a linear transformation $T\colon\ V \to W$ such that V and $\operatorname{im} T$ are infinite dimensional, but $\ker T$ is not 0–dimensional.
  (h) If $\mathcal{B}$ is a basis for $\mathbb{R}^n$ and $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, how do you write a matrix for T with respect to the basis $\mathcal{B}$?
  (i) How do you convert from the standard basis of $\mathbb{R}^n$ to another basis?
  (j) If V is a finite-dimensional vector space, how do you find coordinates for vectors in V? What is the relationship between coordinates and isomorphisms?

(3) Determinants
  (a) How does LaPlace expansion work?

(b) What is an easy way to calculate the determinant of a triangular matrix?
(c) Use induction to prove that your answer to the previous problem is correct.
(d) What are special properties of the determinant?
(e) If A is an $n \times n$ matrix such that $AA^T = I_n$ what is the determinant of A?
(f) How can you use row reduction to calculate a determinant?
(g) Prove that if A and B are similar matrices then they have the same determinant and trace. (Hint: What's the relationship between eigenvalues and determinant and trace?)

(4) Eigenstuff
(a) What are the definitions of "eigenvector" and "eigenvalue" for a matrix? for a linear transformation?
(b) How do you find eigenvectors and values?
(c) How do you show that if an $n \times n$ matrix has n distinct real eigenvalues then it is diagonalizable?
(d) Prove that if A and B are similar matrices then they have the same characteristic equation.
(e) What is the geometric intuition behind both real and complex eigenvalues and eigenvectors?

(5) Orthogonality
(a) What does it mean for a collection of vectors to be orthonormal?
(b) Prove that if A is an $n \times m$ matrix such that $AA^T = I_n$ then the columns of A are orthonormal.
(c) Explain the Gram–Schmidt process.
(d) What is QR–factorization?
(e) What kind of linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^n$ take every orthonormal basis to an orthonormal basis?

(6) Projects
- The potential questions you may be asked are listed separately on the course webpage.

2. Problems

(1) (This problem is very difficult. Don't be discouraged if you can't complete it. I almost didn't include it but then thought, "what the heck...")

Let
$$\mathcal{T}_n = \Big\{ T : T \text{ is a linear transformation } M_n \to \mathbb{R} \Big\}.$$

You may wish to do this problem assuming that $n = 2$ or $n = 3$.

(a) Prove that $\mathcal{T}_n$ is a vector space.

(b) Find a basis for $\mathcal{T}_n$.

(c) Construct an isomorphism from M_n to $\mathcal{T}_n$.

(2) Let V be the set of all 3×3 matrices that commute with $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Show that V is a subspace of M_3.

(b) Find a basis for V. What is its dimension?

(3) Let A be an $n \times n$ matrix such that $A^4 = I$. Show that each real eigenvalue of A is ± 1.

(4) Let A be an $n \times n$ matrix such that A^4 is the zero matrix. Find all eigenvalues of A.

(5) Let V be the vector space of all infinite sequences of real numbers having only finitely many nonzero terms. Let $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_m, 0, 0, \ldots)$ be a fixed vector of V. Define a function $T\colon\ V \to \mathbb{R}$ by

$$T(\mathbf{b}) = \sum a_i b_i$$

where $\mathbf{b} = (b_0, b_1, b_2, \ldots)$.

(a) Show that T is linear.

(b) Suppose that $\mathbf{a} = (1, -1, 0, 0, \ldots)$. Find $\ker T$.

(6) Let $\mathcal{B}$ be the following set in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^4$.

(b) Write the vector $(2, 2, 2, 1)$ in this basis.

(c) Let T be the following linear transformation. Find a matrix which represents T in the basis $\mathcal{B}$.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

(d) Apply the Gram-Schmidt process to $\mathcal{B}$ to obtain an orthonormal basis for $\mathbb{R}^4$.

(7) Consider the plane in $\mathbb{R}^3$ defined by $2x - y + z = 0$. Find a 3×3 matrix which represents orthogonal projection onto this plane.

(8) Suppose that A is a 7×7 matrix with characteristic equation

$$f_A(\lambda) = (-\lambda)^3 (2 - \lambda)(3 - \lambda)(4 - \lambda)(-5 - \lambda)$$

(a) What are the eigenvalues of A?
(b) What is the determinant of A?
(c) What is the trace of A?
(d) What is the characteristic equation of A^T?
(e) Is A invertible?
(f) What can you say about the dimension of $\ker A$?
(g) What can you say the dimension of $\operatorname{im} A$?

(9) Suppose that an $n \times n$ matrix A is similar to its inverse A^{-1}. What can you say about the determinant of A?

(10) Suppose that 30 percent of math students in a given semester take a math course the following semester, while 70 percent of them take a poetry course. Suppose that 40 percent of poetry students in a given semester take a poetry course the following semester while 60 percent of them take a math course. Is there some point in the future where either the mathematics or the poetry department will have to close down because of lack of students? If not, what happens in the long term?

(11) Suppose that $T\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with positive determinant such that T preserves the lengths of all vectors.
(a) Show that there is a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{x}$ (That is, T has a fixed point.)
(b) Must T be an orthogonal transformation? Hint: It is enough to figure out whether $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. Consider the equation
$$(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = (T(\mathbf{x}) + T(\mathbf{y})) \cdot (T(\mathbf{x} + T(\mathbf{y}))$$

(12) Let $W \subset \mathbb{R}^n$ be a subspace. Define
$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$
(a) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.
(b) Describe a linear transformation $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ such that $W^\perp = \ker T$
(c) What is the dimension of $W^\perp$ in terms of the dimension of W?

(13) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a collection of orthonormal vectors in $\mathbb{R}^n$. Prove that they are linearly independent.