Groups, Linear Algebra, and the Geometry of the Torus

These are some notes and exercises to help you complete the group project. You should try all the exercises but should not feel pressured to complete all, or even many, of them.

Your write-up should include a minimal exposition of the relationship between linear algebra and the geometry of the torus and your solutions to a few of the exercises. Your writing style should be as much like a textbook's writing style as possible. You should definitely include many pictures and examples to demonstrate your point. You may refer the reader to these notes, but your write-up must make it clear that you have internalized these ideas and are able to make use of them.

Your oral presentation should include an overview of the relationship between linear algebra and the geometry of the torus. You should also include a few of your findings, which could be a summary of some of the exercises you've completed or interesting examples or new things you've discovered. The notes and exercises below are intended to be a starting point. You should not feel constrained by them, but you should be learning about interesting mathematics. You are welcome and encouraged to look up these topics in other sources (library or internet), but remember to cite all sources. This includes giving credit for any pictures or diagrams taken or copied from other sources. You are also welcome and encouraged to ask questions and push the boundaries of the topics below.

1. Groups

A **group** is a set G and an operation called either **addition** or **multiplication**, such that the following properties hold (ab denotes a times b or $a+b$.):

- G1. If $a, b \in G$ then $ab \in G$ (closed under multiplication/addition)
- G2. There is an element $I \in G$ such that for every $a \in G$, $aI = Ia = a$ (there is a multiplicative/additive identity)
- G3. For every $a \in G$, there is a $b \in G$ so that $ab = ba = I$. We write $b = a^{-1}$. (Multiplicative/Additive inverses exist)

The most important (for us) examples of groups are the following

(1) The set of pairs of integers $\{(m,n) : m,n \in \mathbb{Z}\}$ with addition as the operation. This set is denoted $\mathbb{Z}^2$.
(2) The set of 2×2 matrices having integer entries and an inverse matrix with integer entries. The operation is multiplication. This set is denoted $GL_2(\mathbb{Z})$.

Exercises.

(1) Show that these two examples are actually groups.

2. THE EASIEST DISTANCE ON THE TORUS

A **metric** or **distance function** is a way of measuring the distance between two points on a surface. As long as the distance function is "nice enough", we can use it to define the length of a path on a surface. To study the geometry of a surface we focus on "geodesics". A path joining points x and y is a **geodesic** if it minimizes length. For example, the geodesics in $\boldsymbol{R}^2$ are just lines and line segments, since the shortest distance between two points is a line segment.

For $(a,b) \in \mathbb{Z}^2$, define $f_{a,b}\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by

$$f_{a,b}(x,y) = (x+a, y+b).$$

Notice that if you consider what happens to the square R with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, under all possible $f_{a,b}$, the square tiles $\boldsymbol{R}^2$.

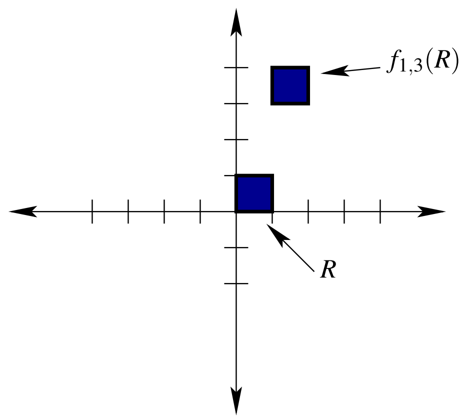


FIGURE 1. The image of R under $f_{a,b}$ tiles the plane.

Given a point $(x,y) \in \boldsymbol{R}^2$, we can figure out which square it is located in. Suppose it is located in the square with lower left corner (a,b). Then $f_{a,b}^{-1}(x,y)$ is a point in the original square R. See Figure 2. The square R is called a **fundamental domain**.

If we glue the top of the fundamental domain to the bottom and the left side to the right side (imagine it made out of fabric), we obtain the torus.

This construction gives us a way of measuring the distance between points on the torus. Suppose that p and q are two points in the torus. By cutting the torus open, we obtain two points $\widetilde{p}$ and $\widetilde{q}$ in the square R.

The shortest path between p and q on the torus, may not correspond to the shortest path between $\widetilde{p}$ and $\widetilde{q}$ in R. See Figure 2.A. To find the shortest path on the torus, let Q be the set of all points in $\boldsymbol{R}^2$ which are translates of $\widetilde{q}$. Find the shortest path from p to a point in Q and move that path back into the fundamental domain using $f_{a,b}^{-1}$ for certain choices of a and b. See Figure 2.B. The length of the path in the torus is just the length of the line segment in $\boldsymbol{R}^2$ joining $\widetilde{p}$ to $\widetilde{q}$.

With this method of measuring distance on the torus, each geodesic on the torus is obtained from a line segment in $\boldsymbol{R}^2$. We can measure the length of a geodesic in the torus by measuring the distance

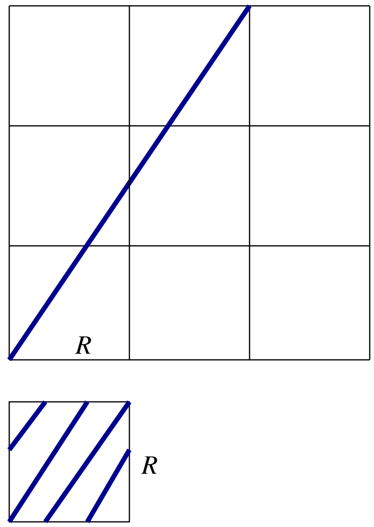


FIGURE 2. The result of translating the line $y = 3x$ back to the fundamental domain.

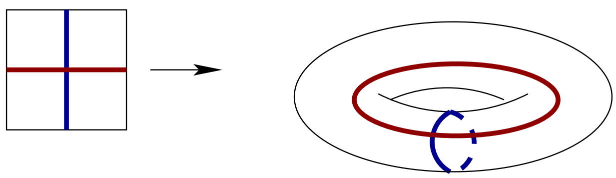


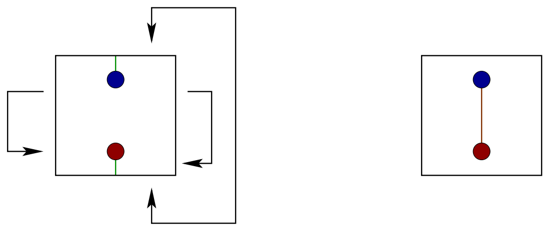
FIGURE 3. The result of gluing the top and bottom of R and the left and right sides. Two lines on the fundamental domain are drawn to show you where they end up on the torus.

of the line segment in $\boldsymbol{R}^2$. For example, the path in the torus (drawn in the fundamental domain and which starts and ends at $(0,0)$) in Figure 2 has length $\sqrt{13}$ because the line segment in $\boldsymbol{R}^2$ joins $(0,0)$ to $(2,3)$.

Exercises.

(1) Draw some geodesics on the torus and measure their length by the above method. Some of your examples should be geodesics which start and end at $(0,0)$.
(2) Let P denote the set of geodesics on the torus which start and end at the same point (say, $(0,0) \in R$) and which don't run over themselves more than once. Let $\mathbb{Q}^*$ denote the rational numbers, together with ∞. Show that there is a one-to-one correspondence between P and $\mathbb{Q}^*$. That is, it is possible to assign a rational number or ∞ to each geodesic in P such that each rational number is assigned to a unique geodesic. (Hint: Think about the slopes of line segments in $\boldsymbol{R}^2$ that go through $(0,0)$.)
(3) Consider L, a line in $\boldsymbol{R}^2$ which goes through $(0,0)$ and which has irrational slope. What can you say about the result of translating L into R? Does it hit every point in R? For a fixed point $p \in R$, what is the distance from p to the translated L? (It is quite difficult to give a rigourous answer to this question; just do your best.)

A.



B.

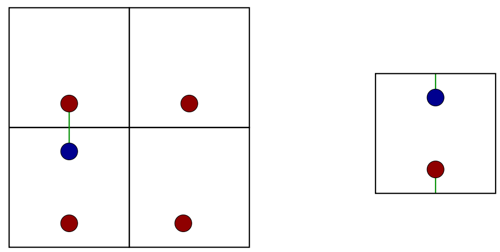


FIGURE 4. A: On the left we have the shortest path on the torus joining p and q. On the right, we have the shortest path in R joining $\widetilde{p}$ and $\widetilde{q}$. B: On the left we have the path in $\boldsymbol{R}^2$ joining $\widetilde{p}$ to Q. On the right we have translated this path into the fundamental domain.

(4) Let T^2 denote the torus. What is the area of the torus with the metric we have constructed? (Hint: What's the area of R?)

(5) A path in T^2 which consists of a single horizontal line segment in R is called a **longitude**. A path in T^2 which consists of a single vertical line segment in R is a **meridian**. Figure 2 depicts a meridian and longitude on T^2 and in R. What are the lengths of the meridian and longitude?

(6) An **isometry** of T^2 is a map $f\colon T^2 \to T^2$ which doesn't change distance. Let I denote the set of isometries of T^2 which fix a point p. We may assume that p corresponds to $(0,0) \in R$. Show that I consists of the identity isometry and the isometry which reflects R about a diagonal.

(7) Show that if $h\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is a linear map which takes $\mathbb{Z}^2$ to itself, then h defines a function $h'\colon T^2 \to T^2$.

(8) A bijective continuous function $f\colon T^2 \to T^2$ is called a homeomorphism. Each homeomorphism can be continuously deformed to a linear map of $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ which takes $\mathbb{Z}^2$ to itself. The inverse homemorphism can be deformed to an inverse linear transformation which takes $\mathbb{Z}^2$ to itself. Thus, to study homeomorphisms of T^2, it turns out that it suffices to study linear maps $h\colon R^2 \to \boldsymbol{R}^2$ which are bijections of $\mathbb{Z}^2$ and whose inverse linear map also takes $\mathbb{Z}^2$ to itself. Show that the set of all such maps can be naturally identified with

$$GL_2(\mathbb{Z}) = \{2 \times 2 \text{ matrices with integer entries and non-zero determinant}\}.$$

(9) Define a **basic matrix** to be a matrix of the form,

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$$

where $n,m \in \mathbb{Z}$. Notice that basic matrices correspond to certain types of elementary row or column operations.

Show that if $A \in GL_2(\mathbb{Z})$, there are basic matrices $E_1,\ldots,E_p$ and $F_1,\ldots,F_q$ such that $E_p \ldots E_1 A F_1 \ldots F_q$ is one of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

(Hint: Try an easier version of this exercise first, by assuming that the 1,1 entry of A is equal to 1. You may also want to see the instructor for further hints.)

(10) Use the previous exercise to show that if $A \in GL_2(\mathbb{Z})$ then the determinant of A is ± 1.

(11) Suppose that $\det A = 1$. Look up the Cayley-Hamilton theorem and use it to show the following:
  (a) If $\mathrm{tr}(A) = 0$ then $A^4 = I$.
  (b) If $\mathrm{tr}(A) = \pm 1$ then $A^{12} = I$.

  In either case, the homeomorphism of the torus associated to A is said to be **periodic**. In other words, if we repeat it often enough, eventually it's the same as not doing anything.

(12) Suppose that $\det A = 1$ and that $\mathrm{tr}(A) = \pm 2$. Show that A has a single real eigenvector and that the homeomorphism of T^2 associated to A leaves a curve, beginning and ending at $(0,0)$ fixed. In this case, it can be shown that the homeomorphism is a power of what is called a **Dehn Twist** of T^2 along this curve. Look up the definition of "Dehn Twist". (See the picture on the wikipedia page, for example.) The homeomorphism is called "reducible".

(13) Suppose that $\det A = 1$ and that $|\mathrm{tr}(A)| \geq 3$. Show that there is a real number such that $|\lambda| > 1$ and such that both λ and $1/\lambda$ are eigenvalues for A. Explain why this implies that A expands $\boldsymbol{R}^2$ in one direction and shrinks $\boldsymbol{R}^2$ in a different direction. The homeomorphism of T^2 associated to A is called an **Anosov** homeomorphism.

(14) If you can, write a program in Maple, or the language of your choice, to show what happens to a point in the torus under repeated applications of an Anosov map. The way it would work is that you would show a picture of R and we would watch the point move around in R based on the action of the Anosov map. If you have never programmed before, you probably shouldn't try this.

(15) Notice that we constructed a metric on the torus by tiling $\boldsymbol{R}^2$ with squares. You can also do this by tiling $\boldsymbol{R}^2$ with parallelograms. This is most commonly done with parallelograms of area 1. Explain how this works and show that this corresponds to choosing two curves on the torus which intersect exactly once and calling one of them a meridian and the other a longitude. Explain why the meridian and longitude will have the same length if and only if the fundamental domain is a square.

(16) Look up the definition of "Teichmuller Space" and "Moduli Space" for the torus and explain its connection to the previous exercise and to linear algebra.