Group Project 1 Answers: Visualizing Vectors

Names:

Problem 1: Describe how to represent vector addition graphically. In other words, given vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ draw a picture representing $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}+\mathbf{y}$ and describe in words what is going on. You work a specific example in $\mathbb{R}^2$ and a specific example in $\mathbb{R}^3$.

Answer: Slide $\mathbf{x}$ along $\mathbf{y}$ so that the tail of $\mathbf{x}$ is at the head of $\mathbf{y}$. The head of $\mathbf{x}$ points to $\mathbf{x}+\mathbf{y}$.

Problem 2: Suppose that $\lambda \in \mathbb{R}$ is a constant and that $\mathbf{v}$ is a vector. Compare the arrow representing $\lambda\mathbf{v}$ to the arrow representing $\mathbf{v}$. There are several cases to consider. For each case you should draw picture to demonstrate your point.

Answer: You should have 3 cases: $\lambda > 0$, $\lambda < 0$, and $\lambda = 0$. In the first case, $\lambda\mathbf{x}$ points in the same direction as $\mathbf{x}$ but might be of a different length. In the second case, $\lambda\mathbf{x}$ points in the opposite direction as $\mathbf{x}$. In the third case, $\lambda\mathbf{x} = 0$.

Problem 3: Let $\mathbf{v} \in \mathbb{R}^2$ be a fixed non-zero vector and onsider the set of vectors

$$L_{\mathbf{v}} = \{t\mathbf{v} : t \in \mathbb{R}\}.$$

What do you get if you plot each point in this set? Draw a picture which gives a specific example.

Answer: You get the line passing through $\mathbf{v}$ and the origin.

Problem 4: Let $\mathbf{v} \in \mathbb{R}^2$ be a fixed non-zero vector and let $\mathbf{w} \in \mathbb{R}^2$ be another vector such that $\mathbf{v} \neq \mathbf{w}$. Consider the sets:

$$\begin{aligned} L_{\mathbf{v},\mathbf{w}} &= \{t\mathbf{v} + (1-t)\mathbf{w} : t \in \mathbb{R}\} \\ l_{\mathbf{v},\mathbf{w}} &= \{t\mathbf{v} + (1-t)\mathbf{w} : 0 \leq t \leq 1\}. \end{aligned}$$

Describe each of these sets and use pictures and specific examples to illustrate your answers. You should consider the case when $\mathbf{v}$ and $\mathbf{w}$ are linearly independent and the case when $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent separately.

Answer: $L_{\mathbf{v},\mathbf{w}}$ is the line passing through $\mathbf{v}$ and $\mathbf{w}$. If you want a proof, here is one:

Proof. Notice that:

$$t\mathbf{v}+(1-t)\mathbf{w}-\mathbf{w}=t(\mathbf{v}-\mathbf{w}).$$

Subtracting $\mathbf{w}$ from every point on the line through $\mathbf{v}$ and $\mathbf{w}$ produces the line going through the origin and the vector $\mathbf{v}-\mathbf{w}$. Subtracting $\mathbf{w}$ from every point in $L_{\mathbf{v},\mathbf{w}}$ produces the set

$$L_{\mathbf{v}-\mathbf{w},\mathbf{0}}=\{t(\mathbf{v}-\mathbf{w}) : t\in\mathbb{R}\}.$$

By Problem 3, these are the same. Thus, adding $\mathbf{w}$ to both still produces the same thing. □

$l_{\mathbf{v},\mathbf{w}}$ is the line segment between $\mathbf{v}$ and $\mathbf{w}$.

Problem 5: Let $\mathbf{v}$ and $\mathbf{w}$ be as in problem 4. Describe and illustrate the set

$$M_{\mathbf{v},\mathbf{w}}=\{\mathbf{v}+t\mathbf{w} : t\in\mathbb{R}\}$$

Answer: This is the line through $\mathbf{v}$ in the direction of $\mathbf{w}$.

Problem 6: Consider the vectors $\mathbf{u}=(2,3,5)$, $\mathbf{v}=(0,5,5)$, and $\mathbf{w}=(1,0,1)$ in $\mathbb{R}^3$. Describe and sketch the set

$$P=\{r\mathbf{u}+s\mathbf{v}+t\mathbf{w} : r,s,t\in\mathbb{R}\}.$$

Answer: This is all of $\mathbb{R}^3$.

Problem 7: In Problem 6, what happens if we require $r+s+t=1$?

Answer: You get the plane through $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$.

Problem 8: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$. What are the possibilities for the set:

$$P=\{r\mathbf{u}+s\mathbf{v}+t\mathbf{w} : r,s,t\in\mathbb{R}\}?$$

Hint: Think about the case when the vectors are linearly independent separately from when they are linearly dependent.

Answer: If the vectors are linearly independent, you get all of $\mathbb{R}^3$. If they are linearly dependent you get either a plane through the origin, a line through the origin, or the zero vector.

Problem 9: Do Problem 8 again, but require that $r+s+t=1$.

Answer: In all cases you get a plane containing $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. If the vectors are linearly independent, the plane is unique. If they are linearly dependent, the plane is not unique.

Problem 10: (**Bonus!**) Suppose that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are four linearly indendent vectors in $\mathbb{R}^4$. Describe the set (but don't sketch a picture!)

$$S = \{q\mathbf{a} + r\mathbf{b} + s\mathbf{c} + t\mathbf{d} : q + r + s + t = 1\}.$$

You don't need to prove that your answer is correct, but you should be reasonably confident in it.

Answer: You get a "3–space" in $\mathbb{R}^4$ containing the four vectors.