Notes on Determinants

1. Determinants and Row Operations

Theorem 1. Suppose that an \( n \times n \) matrix \( A' \) is obtained from \( A \) by a single row operation. Then the following are true:

1. If the row operation switches two rows, then \( \det(A') = -\det(A) \).
2. If the row operation scales a row by \( k \neq 0 \), then \( \det(A') = k \det(A) \).
3. If the row operation replaces a row \( i \) by the sum of row \( i \) and row \( j \), then \( \det(A') = \det(A) \).

We prove only the third possibility. Our method of proof is by induction.

Base Case: \( n = 2 \)

Suppose that \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( A' = \begin{bmatrix} a & b \\ a + c & b + d \end{bmatrix} \). Then

\[
\det(A) = ad - bc
\]

and

\[
\det(A') = a(b + d) - b(a + c) = ab + ad - ba - bc = ad - bc,
\]

so the theorem holds. If \( A' = \begin{bmatrix} a + c & b + d \\ c & d \end{bmatrix} \), then \( \det(A') = d(a + c) - c(b + d) = ad - bc \), as desired.

Inductive Case: Suppose that the theorem is true for \( n \times n \) matrices and that we want to prove it for \( (n + 1) \times (n + 1) \) matrices with \( n \geq 2 \).

Our row operation replaces row \( i \) with the sum of row \( i \) and row \( j \). By an important theorem, we can calculate \( \det(A') \) by LaPlace expansion across row \( k \), where \( k \neq i, j \) (using the fact that \( n + 1 \geq 3 \). The formula for LaPlace expansion tells us that

\[
\det(A') = \sum_{c=1}^{n+1} a'_{kc} \det(A'_{kc})
\]

where \( a'_{kc} \) is the entry in row \( k \) and column \( c \) of \( A' \). Since row \( k \) of \( A' \) and row \( k \) of \( A \) are exactly the same \( a'_{kc} = a_{kc} \) for all \( c \). The matrix \( A'_{kc} \) is an \( n \times n \) matrix which is obtained from the matrix \( A_{kc} \) by a row replacement operation. Since we are assuming the theorem is true for \( n \times n \) matrices, \( \det(A'_{kc}) = \det(A_{kc}) \). Hence,

\[
\det(A') = \sum_{c=1}^{n+1} a'_{kc} \det(A'_{kc}) = \sum_{c=1}^{n+1} a_{kc} \det(A_{kc}) = \det(A).
\]

This result has an important corollary:

Corollary 2. If \( A \) is an \( n \times n \) matrix and if \( E \) is an elementary matrix, then \( \det(EA) = \det(E) \det(A) \).
Proof. Notice, first, that $\det(I_n) = 1$. Thus, if $E$ is obtained from $I_n$ by a row swap, $\det(E) = -1$. If $E$ is obtained from $I_n$ by scaling a row of $I_n$ by $k \neq 0$, $\det(E) = k$. And if $E$ is obtained from $I_n$ by a row replacement operation, $\det(E) = 1$. The matrix $EA$ is obtained by performing the row operation corresponding to $E$ on $A$, thus, if the row operation is a row swap, $\det(EA) = -\det(A)$; if the row operation scales a row of $A$ by $k \neq 0$, then $\det(EA) = k \det(A)$; and if the row operation is a row replacement operation, $\det(EA) = \det(A)$. The result follows immediately. □

We can now prove the most important theorem about determinants:

**Theorem 3.** If $A$ and $B$ are $n \times n$ matrices, $\det(AB) = \det(A) \det(B)$.

Proof. Let $E_1, \ldots, E_p$ be the elementary matrices so that $E_pE_{p-1}\ldots E_1A = \text{rref}(A)$.

**Case 1:** $A$ is invertible. In this case $\text{rref}(A) = I_n$ and $E_p\ldots E_1A = I_n$. That is, $A = E_1^{-1}\ldots E_p^{-1}$. Then taking the determinant of $AB = E_1^{-1}\ldots E_p^{-1}B$ gives us:

$$\det(AB) = \det(E_1^{-1}\ldots E_p^{-1}B).$$

By Corollary 2, since the inverse of an elementary matrix is an elementary matrix:

$$\det(AB) = \det(E_1^{-1})\ldots \det(E_p^{-1}) \det(B).$$

Applying Corollary 2 again,

$$\det(AB) = \det(E_1^{-1}\ldots E_1^{-1}) \det(B).$$

And this is just,

$$\det(AB) = \det(A) \det(B)$$

as desired.

**Case 2:** $A$ is not invertible. In this case, $\text{rref}(A)$ is not $I_n$ and so must have a row of all zeroes. Performing LaPlace expansion across this row shows that $\det(\text{rref}(A)) = 0$. We have, (using Corollary 2),

$$(*) 0 = \det(\text{rref}(A)) = \det(E_p\ldots E_1A) = \det(E_p)\ldots \det(E_1) \det(A).$$

If $E$ is an elementary matrix, then $E$ is invertible. Hence, by Corollary 2

$$1 = \det(I_n) = \det(E) \det(E^{-1}).$$

This implies that the determinant of an elementary matrix is not zero. Hence, Equation (*) shows us that $\det(A) = 0$. In other words, the determinant of a non-invertible matrix is zero.

If $A$ is not invertible, neither is $AB$ by Exercise 34 of Section 2.4. Hence, $\det(AB) = 0$. Thus, $\det(AB) = \det(A) \det(B) = 0$. □