Notes on Determinants

1. Determinants and Row Operations

Theorem 1. *Suppose that an $n \times n$ matrix A' is obtained from A by a single row operation. Then the following are true:*

(1) *If the row operation switches two rows, then* $\det(A') = -\det(A)$.
(2) *If the row operation scales a row by $k \neq 0$, then* $\det(A') = k\det(A)$.
(3) *If the row operation replaces a row i by the sum of row i and row j, then* $\det(A') = \det(A)$.

We prove only the third possibility. Our method of proof is by induction.
Base Case: $n = 2$
Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A' = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$. Then

$$\det(A) = ad - bc$$

and

$$\det(A') = a(b+d) - b(a+c) = ab + ad - ba - bc = ad - bc,$$

so the theorem holds. If $A' = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$, then $\det(A') = d(a+c) - c(b+d) = ad - bc$, as desired.
Inductive Case: Suppose that the theorem is true for $n \times n$ matrices and that we want to prove it for $(n+1) \times (n+1)$ matrices with $n \geq 2$.
Our row operation replaces row i with the sum of row i and row j. By an important theorem, we can calculate $\det(A')$ by LaPlace expansion across row k, where $k \neq i, j$ (using the fact that $n + 1 \geq 3$. The formula for LaPlace expansion tells us that

$$\det(A') = \sum_{c=1}^{n+1} a'_{kc} \det(A'_{kc})$$

where a'_{kc} is the entry in row k and column c of A'. Since row k of A' and row k of A are exactly the same $a'_{kc} = a_{kc}$ for all c. The matrix A'_{kc} is an $n \times n$ matrix which is obtained from the matrix A_{kc} by a row replacement operation. Since we are assuming the theorem is true for $n \times n$ matrices, $\det(A'_{kc}) = \det(A_{kc})$. Hence,

$$\det(A') = \sum_{c=1}^{n+1} a'_{kc} \det(A'_{kc}) = \sum_{c=1}^{n+1} a_{kc} \det(A_{kc}) = \det(A).$$

□

This result has an important corollary:

Corollary 2. *If A is an $n \times n$ matrix and if E is an elementary matrix, then* $\det(EA) = \det(E)\det(A)$.

Proof. Notice, first, that $\det(I_n) = 1$. Thus, if E is obtained from I_n by a row swap, $\det(E) = -1$. If E is obtained from I_n by scaling a row of I_n by $k \neq 0$, $\det(E) = k$. And if E is obtained from I_n by a row replacement operation, $\det(E) = 1$. The matrix EA is obtained by performing the row operation corresponding to E on A, thus, if the row operation is a row swap, $\det(EA) = -\det(A)$; if the row operation scales a row of A by $k \neq 0$, then $\det(EA) = k\det(A)$; and if the row operation is a row replacement operation, $\det(EA) = \det(A)$. The result follows immediately. □

We can now prove the most important theorem about determinants:

Theorem 3. *If A and B are $n \times n$ matrices,* $\det(AB) = \det(A)\det(B)$.

Proof. Let $E_1, \dots, E_p$ be the elementary matrices so that $E_pE_{p-1}\dots E_1A = \mathrm{rref}(A)$.

Case 1: A is invertible.

In this case $\mathrm{rref}(A) = I_n$ and $E_p \dots E_1 A = I_n$. That is, $A = E_1^{-1} \dots E_p^{-1}$. Then taking the determinant of $AB = E_1^{-1} \dots E_p^{-1}B$ gives us:

$$\det(AB) = \det(E_1^{-1} \dots E_p^{-1}B).$$

By Corollary 2, since the inverse of an elementary matrix is an elementary matrix:

$$\det(AB) = \det(E_1^{-1}) \dots \det(E_p^{-1})\det(B).$$

Applying Corollary 2 again,

$$\det(AB) = \det(E_1^{-1} \dots E_1^{-1})\det(B).$$

And this is just,

$$\det(AB) = \det(A)\det(B)$$

as desired.

Case 2: A is not invertible.

In this case, $\mathrm{rref}(A)$ is not I_n and so must have a row of all zeroes. Performing LaPlace expansion across this row shows that $\det(\mathrm{rref}(A)) = 0$. We have, (using Corollary 2),

$$(*) \qquad 0 = \det(\mathrm{rref}(A)) = \det(E_p \dots E_1 A) = \det(E_p)\dots\det(E_1)\det(A).$$

If E is an elementary matrix, then E is invertible. Hence, by Corollary 2

$$1 = \det(I_n) = \det(E)\det(E^{-1}).$$

This implies that the determinant of an elementary matrix is not zero. Hence, Equation (*) shows us that $\det(A) = 0$. In other words, the determinant of a non-invertible matrix is zero.

If A is not invertible, neither is AB by Exercise 34 of Section 2.4. Hence, $\det(AB) = 0$. Thus, $\det(AB) = \det(A)\det(B) = 0$. □