**MA 121: Practice Final Partial Solutions** **Name:**______________________

**Problem #1:**
Find the derivatives of the following functions ($k$ is a constant).

(i)
$$f(x) = \sin(kx)\cos(x^3 - 2)$$
**Solution:** $f'(x) = k\cos(kx)\cos(x^3 - 2) - 3x^2\sin(x^3 - 2)\sin(kx)$

(ii)
$$f(x) = \ln(x^4 + e^{-kx})$$
**Solution:**
$$f'(x) = \frac{4x^3 - ke^{-kx}}{x^4 + e^{-kx}}$$

**Problem #2:**
Figure 1 shows the graph of a function $f(x)$.

(i) Sketch the graph of $f'(x)$.
**Solution:** The blue curve.

(ii) Sketch the graph of an antiderivative of $f(x)$.
**Solution:** The red curve.

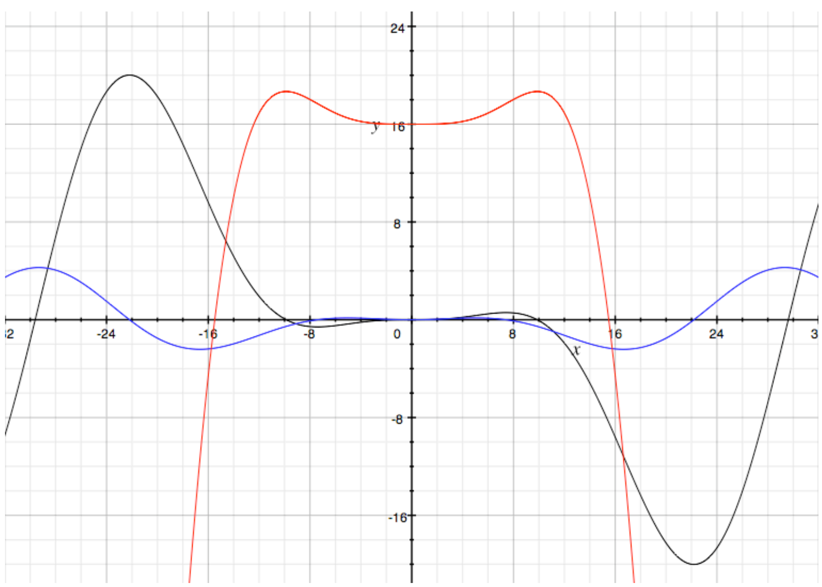


FIGURE 1. The graph of $y = f(x)$, its derivative, and an antiderivative

**Problem #3:**

(i) Find the equation of the line tangent to $y = \arctan(x)$ at $x = 1/\sqrt{3}$.
**Solution:**

$$y = \frac{3}{4}\left(x - \frac{1}{\sqrt{3}}\right) + \frac{\pi}{6}$$

(ii) Use the equation to approximate $\arctan(0.6)$. (Hint: $1/\sqrt{3} \approx 0.577$.)
**Solution:** $\arctan(0.6) \approx (3/4)(.013) + \pi/6$.

(iii) Use Calculus to determine if $y = \arctan(x)$ is increasing or decreasing and concave up or concave down at $x = 1$.
**Solution:** Let $f(x) = \arctan(x)$. Then $f'(x) = 1/(1 + x^2)$. This is always positive, so at $x = 1$, $f(x)$ is increasing. $f''(x) = -(1 + x^2)^{-2}(2x)$. If $x > 0$, $f''(x) < 0$. Thus at $x = 1$, $f(x)$ is concave down.

(iv) Use your answer from (iii) to determine if your answer in (ii) is an underestimate or overestimate of $\arctan(2)$.
**Solution:** Since $f(x)$ is increasing and concave down at $x = 1$, the graph of $f(x)$ is below the graph of the tangent line. Hence, the estimate in (ii) is an overestimate.

**Problem #4:**
Use the limit definition of the derivative to show that $\frac{d}{dx}\, x^2 = 2x$.

**Solution:**

$$\begin{aligned} \tfrac{d}{dx}\, x^2 &= \lim_{h\to 0} \tfrac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \tfrac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x. \end{aligned}$$

**Problem #5:**
Show that $f(x) = |x|$ is not differentiable at $x = 0$.

**Solution:**

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

Notice that $\Big(f(x+h) - f(x)\Big)/h = |h|/h$ at $x = 0$. By the formula above, if $h > 0$ this is $+1$ and if $h < 0$ it is $-1$. Hence,

$$\lim_{h\to 0^+} \Big(f(x+h) - f(x)\Big)/h = \lim_{h\to 0^+} |h|/h = \lim_{h\to 0^+} +1 = +1.$$

Similarly,

$$\lim_{h\to 0^-} \Big(f(x+h) - f(x)\Big)/h = \lim_{h\to 0^+} |h|/h = \lim_{h\to 0^+} -1 = -1.$$

Since these do not agree

$$\lim_{h\to 0} \Big(f(0+h) - f(0)\Big)/h$$

does not exist. Thus, the derivative of $|x|$ does not exist at $x = 0$.

**Problem #6:**
Let $f(x) = k^x$ for some $k > 0$. Find $k$ so that the tangent line to $f(x)$ through the point $(1, k)$ also passes through the point $(0, 3e^{-2})$.

**Solution:** The tangent line to the graph of $f(x)$ at $(1, k)$ has the equation

$$y = k\ln(k)(x-1) + k$$

If $x = 0$ then

$$y = -k\ln(k) + k = k(1 - \ln(k)).$$

If the line is to pass through $(0, 3e^{-2})$ we must have

$$3e^{-2} = k(1 - \ln(k)).$$

Notice that $k = e^{-2}$ works.

**Problem #7:**
Buffalo Bill decides to keep careful count of the population of buffalo in Buffalo, New York. $P(t)$ represents the population of buffalo $t$ years after 1990. Buffalo Bill discovers that $P'(0) = 3$. The population of buffalo in 1990 is 1000.

(i) What are the units for $P'(t)$?
**Solution:** Buffaloes per year. Or is it, buffalos per year? or, maybe, buffalo per year?

(ii) What does $P'(0) = 3$ signify in terms of buffalo?
**Solution:** In 1990, the population of buffalo is increasing at 3 per year.

(iii) The maximum population of buffalo that Buffalo, NY can sustain is 2000. Use the logistic equation to write down a differential equation and initial value which models the population of buffalo $t$ years after 1990.
**Solution:** The logistic equation says

$$P' = kP(2000 - P)$$

for some $k$. In 1990, $P = 1000$ and $P' = 3$. Thus, $k = .000003$. Thus our IVP is

$$\begin{array}{rcl} P' & = & .000003P(2000-P) \\ P(0) & = & 1000 \end{array}.$$

(iv) Use separation of variables to solve the initial value problem from (iii). You may wish to notice that

$$\frac{1}{P(2000-P)} = \frac{1}{2000}\Big(\frac{1}{P} + \frac{1}{2000-P}\Big)$$

**Solution:** Separating variables produces

$$\int \frac{1}{P(2000-P)}\,dP = \int k\,dt.$$

Using the hint and integrating we obtain:

$$\frac{1}{2000}\Big(\ln|P| - \ln|2000-P|\Big) = kt + C.$$

The properties of logs tells us

$$\ln\Big(\frac{P}{2000-P}\Big) = 2000kt + C.$$

Thus,

$$\frac{P}{2000-P} = Ae^{2000kt}.$$

Solving for $P$ gives us

$$P = \frac{2000Ae^{2000kt}}{1+Ae^{2000kt}}.$$

**Problem #8:**

Find and classify all critical points of

$$f(x) = (|x|-1)^2$$

**Solution:** The critical points are $x = -1, 0, 1$. They are a minimum, a maximum, and a minimum in that order.

**Problem #9:**

Show that the derivative of $f(x) = \arcsin(x)$ is $f'(x) = 1/\sqrt{1-x^2}$.

**Solution:** This is in the notes.

**Problem #10:**
Find the equation of the line tangent to the graph of

$$xy^2 = x^2 - y$$

At the point $(1, \frac{-1+\sqrt{5}}{2})$.

**Solution:** Start by finding $dy/dx$. You should discover that it is

$$\frac{dy}{dx} = \frac{2x - y^2}{2yx + 1}$$

Now plug in the point $(1, \frac{-5+\sqrt{5}}{2})$ to find the slope of the tangent line. Call it $m$. Then

$$y = m(x-1) + \frac{-1+\sqrt{5}}{2}$$

is the equation of the tangent line.

**Problem #11:**
A parabolic bowl of water is being filled at a rate of 2 $\text{ft}^3$ per minute. It is leaking out the bottom at h $\text{ft}^3$ per minute where $h$ is the depth of water (measured at its deepest point.) If the origin is put at the base of the bowl, its (vertical) cross section is given by the equation $y = x^2$. See Figure 2. If the water is $h$ feet deep, then the volume of the water is $\int_0^{\sqrt{h}} \pi x^2\, dx$.

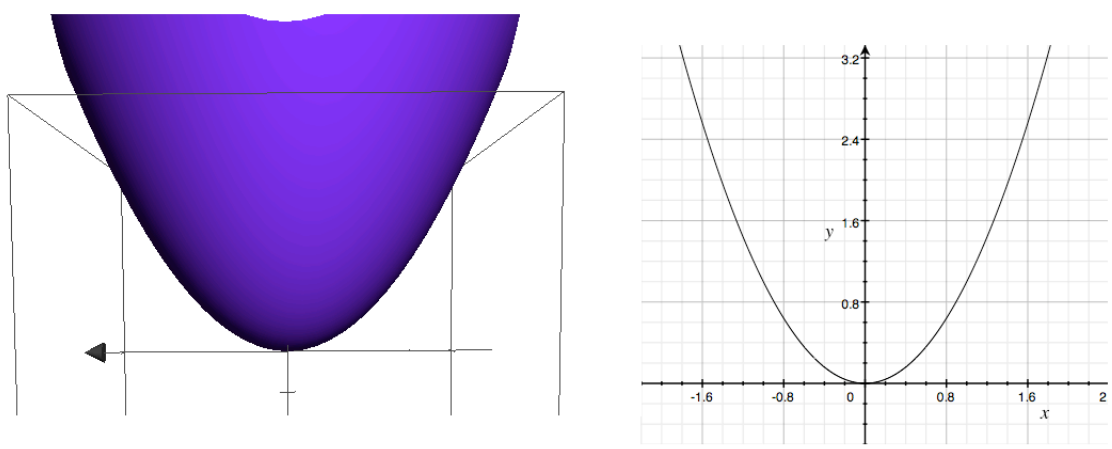


FIGURE 2. Parabolic bowl and cross section

(i) How fast is the depth of the water (at the deepest point) changing when the water is 3 feet deep (at its deepest point)?
**Solution:** Use the integral to find that $V = (1/3)h^{3/2}$. Thus

$$\frac{dV}{dt} = (1/2)h^{1/2}\frac{dh}{dt}.$$

We know that

$$\frac{dV}{dt} = 2 - h = (1/2)h^{1/2}(dh/dt).$$

If $h = 3$, this produces

$$-1 = \sqrt{3}2(dh/dt).$$

Consequently, if $h = 3$

$$dh/dt = -2/\sqrt{3}.$$

(ii) If the bowl starts out with water that is 1 feet deep, will the water ever be 3 feet deep?

**Solution:** No. As long as $h < 2$, $dV/dt > 0$, implying that volume is increasing (and therefore that depth is increasing). If $h > 2$ then $dV/dt < 0$ and volume (and there fore depth) is decreasing. Thus, if we start with 1 foot of water, the depth of water will never exceed 2 feet.

**Problem #12:**

A man six feet tall walks at a rate of 5 feet per second toward a streetlight that is 16 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light?

**Solution:** Let $s$ denote the distance from the tip of the guy's shadow to to the lamppost. Let $p$ denote the length of the shadow and $x$ the distance from the man to the lamppost. Thus $s = x + p$. We know $dx/dt = -5$. Using trigonometry or similar triangles you can deduce

$$16/s = 6/p.$$

This implies that $p = 3/8s$ and $x = 5/8s$. Thus, $ds/dt = 8/5dx/dt = -8$. So the tip of the shadow is moving at 8 feet per second toward the lamppost. Thus, $dp/dt = 3/8ds/dt = -3$. So the shadow is shrinking at 3 feet per second.

**Problem #13:**

A rectangle has its base on the $x$–axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions? Explain how you know that you have found the largest possible area.

**Solution:** Let $(x, y)$ denote the coordinates of the upper right corner of the rectangle and let $A$ denote the rectangle's area. Then

$$A = 2xy = 24x - 2x^3.$$

Hence,

$$dA/dx = 24 - 4x^2.$$

Thus, $A$ has a critical point at

$$x = \sqrt{6}.$$

(We need only the positive square root, since $x$ is the $x$–coordinate of the upper **right** corner.) The second derivative of $A$ is

$$d^2A/dx^2 = -8x.$$

Since $x \geq 0$, $A$ is always concave down. Thus, $x = \sqrt{6}$ is a local maximum. Since $0 \leq x \leq \sqrt{12}$, by the extreme value theorem this must also be our global maximum. You should now solve for $y$ and $A$.

**Problem #14:**

The trough in Figure 3 is going to be made to the dimensions shown. Only the angle $\theta$ can be varied. What value of $\theta$ will maximize the trough's volume? Explain how you know you have found the largest possible volume.

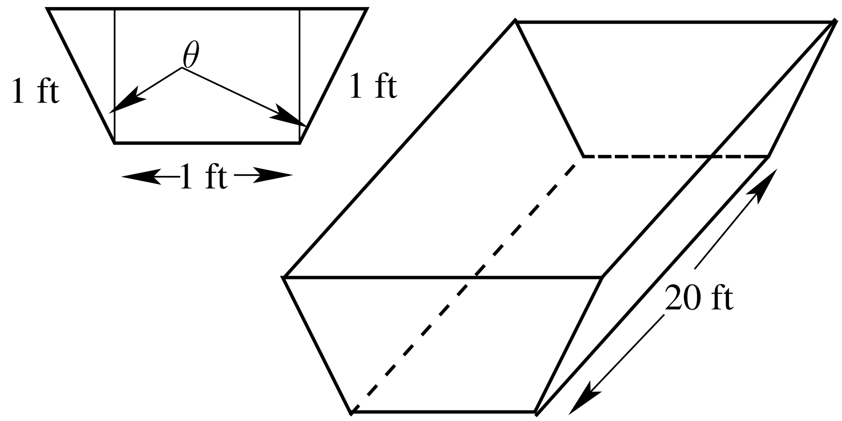


FIGURE 3. The trough

**Solution:** The height of the trapezoid front is $\cos\theta$ and the area of the trapezoid is $(1/2)(2 + 2\sin\theta)\cos\theta$. The volume of the trough is

$$V = 20(1 + \sin\theta)\cos\theta$$

Taking the derivative, setting equal to zero, and a little algebra produces

$$-\sin^2\theta + \cos^2\theta - \sin\theta = 0.$$

Use the trig identity $\cos^2\theta = 1 - \sin^2\theta$ and divide by $-1$ to achieve:

$$2\sin^2\theta + \sin\theta - 1 = 0.$$

Factor:

$$(2\sin\theta - 1)(\sin\theta + 1) = 0.$$

Since $0 \le \theta \le \pi/2$, $\sin\theta + 1 \ne 0$. Hence, $\theta = \pi/6$. Since we are working on a closed, bounded interval, we need only evaluate $V$ at $\theta = \pi/6$ and at our endpoints $\theta = 0, \pi/2$. At $\theta = \pi/2$ our volume is zero. At $\theta = 0$, our volume is $V = 20$. At $\theta = \pi/6$ our volume is $V = 15\sqrt{3}$. This is slightly greater that $V = 26$. Thus, the maximum volume occurs when $\theta = \pi/6$.

**Problem #15:**

Calculate $\int_{-7}^{7} f(x)\,dx$ for the function $f(x)$ shown in Figure 4.

**Solution:** $-16 - 9\pi/2$

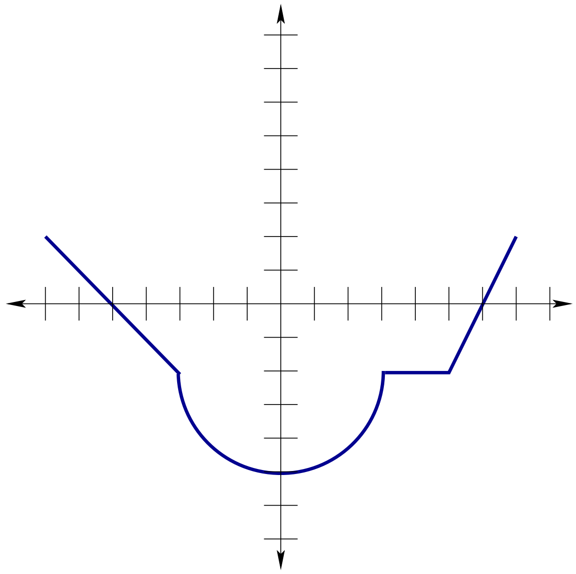

FIGURE 4. The graph of $y = f(x)$.

**Problem #16:**
Approximate $\int_1^2 \sqrt{4-x^2}\,dx$ using a left-hand Riemann sum with $n = 4$ boxes. You do not need to perform the necessary arithmetic.

**Solution:**

$$\frac{1}{4}\left[\sqrt{4-1^2} + \sqrt{4-(5/4)^2} + \sqrt{4-(3/2)^2} + \sqrt{4-(7/4)^2}\right]$$

**Problem #17:**
Let $f(x) = \arccos(x)(x^{16}-1)^{89}$. Compute $\int f'(x)\,dx$.

**Solution:** The problem is asking for all antiderivatives to $f'(x)$. The function $f(x)$ is one such. By a corollary of the Mean Value Theorem, any other antiderivative is of the form $f(x) + C$ for some constant $C$.

**Problem #18:**
Compute $\int_1^3 1 - 2x$ using the limit definition of the Riemann integral. You will need to use the identity

$$\sum_{i=0}^{n-1} i = \frac{n^2-n}{2}.$$

**Solution:** Notice that $\Delta x = 2/n$ and $x_i = 1 + 2i/n$ and $f(x_i) = 1 - 2(1 + 2i/n) = -1 - 4i/n$. The definition of the definite integral tells us that we want to find

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (-1 - 4i/n)(2/n).$$

This can be rewritten as

$$\lim_{n\to\infty}(2/n)\sum_{i=0}^{n-1}-1-(8/n^2)\sum_{i=0}^{n-1}i$$

Using the given formula and common sense, this equals

$$\lim_{n\to\infty}(2/n)(-n)-(8/n^2)(n^2-n)/2$$

Some algebra reduces this to

$$\lim_{n\to\infty}-2-4+4/n.$$

This equals $-6$.

**Problem #19:**
Figure 5 shows the graph of a function $f(x)$. Let $A_f(t)=\int_0^t f(x)\,dx$. Find and classify the critical points of $A_f(t)$.

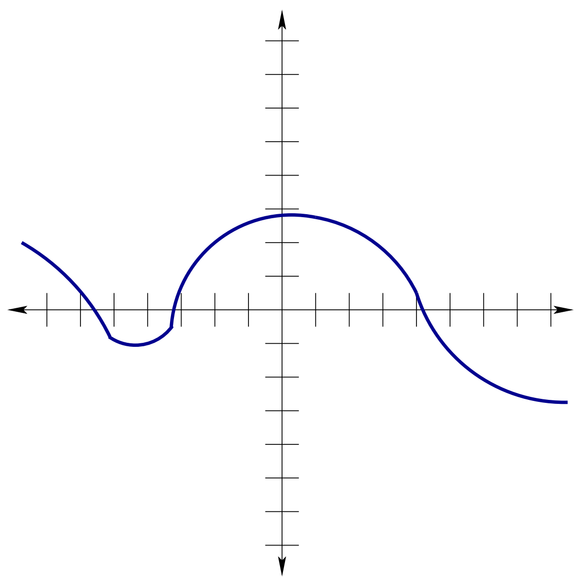

FIGURE 5. The graph of $y=f(x)$.

**Solution:** $t\approx -5.5$ is a maximum, $t\approx -3.2$ is a minimum, and $t\approx 4.1$ is a maximum. To figure this out, use FTC I, which says that $A_f(t)$ is an antiderivative of $f(t)$.

**Problem #20:**
Compute

$$\frac{d}{dt}\int_2^{\cos(t^2)}\sin(x^2)\,dx$$

**Solution:** $\sin\Big(\cos(t^2)^2\Big)\Big(-\sin(t^2)\Big)\Big(2t\Big)$.

**Problem #21:**
Use the first fundamental theorem of Calculus to prove the second fundamental theorem of calculus.

**Solution:** Let $F(x)$ be any antiderivative of $f(x)$. Then there exists a constant $C$ so that

$$F(x) = \int_a^x f(t)\,dt + C$$

This is because $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ by FTC I and any two antiderivatives differ by the addition of a constant by a corollary to the MVT. Then,

$$F(b) - F(a) = \int_a^b f(t)\,dt + C - (\int_a^b f(t)\,dt + C) = \int_a^b f(t)\,dt$$

since the second integral is zero and the $+C$ and $-C$ cancel.

**Problem #22:**
The average value of a function $f(x)$ on an interval $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Give a detailed explanation as to why this is a sensible definition.

**Solution:** If points $x_0 < x_1 < \ldots < x_n$ are chosen in $[a, b]$ so that $x_0 = a$ and $x_n = b$ and so that they are $\Delta x$ units apart, the average of the values of the function for $x = x_0, \ldots, x_{n-1}$ is

$$\frac{1}{n}\sum_{i=0}^{n-1} f(x_i).$$

The average value of the function over the entire interval should be the limit of this as $n \to \infty$. That is,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(x_i).$$

Since the points are evenly spaced $\Delta x = (b - a)/n$. That is, $1/n = \Delta x/(b - a)$. Plugging in we obtain,

$$\lim_{n\to\infty} \frac{1}{(b-a)}\sum_{i=0}^{n-1} f(x_i)\Delta x = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x$$

This, however, is exactly the definition of the definite integral divided by $(b - a)$. So the average value of $f(x)$ on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

**Problem #23:**
Compute the following integrals and antiderivatives.

(1) $\int_0^\pi \sin(x/3)\,dx$
**Solution:** $3/2$.

(2) $\int x^3 - \frac{1}{\sqrt{x}}\, dx$

**Solution:** $\frac{1}{4}x^4 - 2\sqrt{x} + C$.

(3) $\int \frac{x}{\sqrt{x-5}}\, dx$

**Solution:** $(2/3)(x-5)^{3/2} + \frac{5}{\sqrt{(x-5)}}$

(4) $\int_0^3 te^{t^2-7}\, dt$

**Solution:** The given integral equals $\int_{-7}^{2} (1/2)e^u\, du$. This equals $(1/2)e^2 - (1/2)e^{-7}$.

(5) $\int \frac{\arctan x}{1+x^2}\, dx$

**Solution:** Make the substitution $u = $ arctanx.

(6) $\int \frac{18\tan^2(x)\sec^2(x)}{(2+\tan^3(x))^2}\, dx$

**Solution:** Make the substitution

$$\begin{aligned} u &= 2 + \tan^3(x) \\ du &= 3\tan^2(x)\sec^2 x\, dx \end{aligned}$$

**Problem #24:**

Figure 6 shows a slope field for the differential equation

$$y' = ty - y^2$$

Sketch a solution to the DE $y' = ty - y^2$ which passes through the point $(0, 0.8)$.

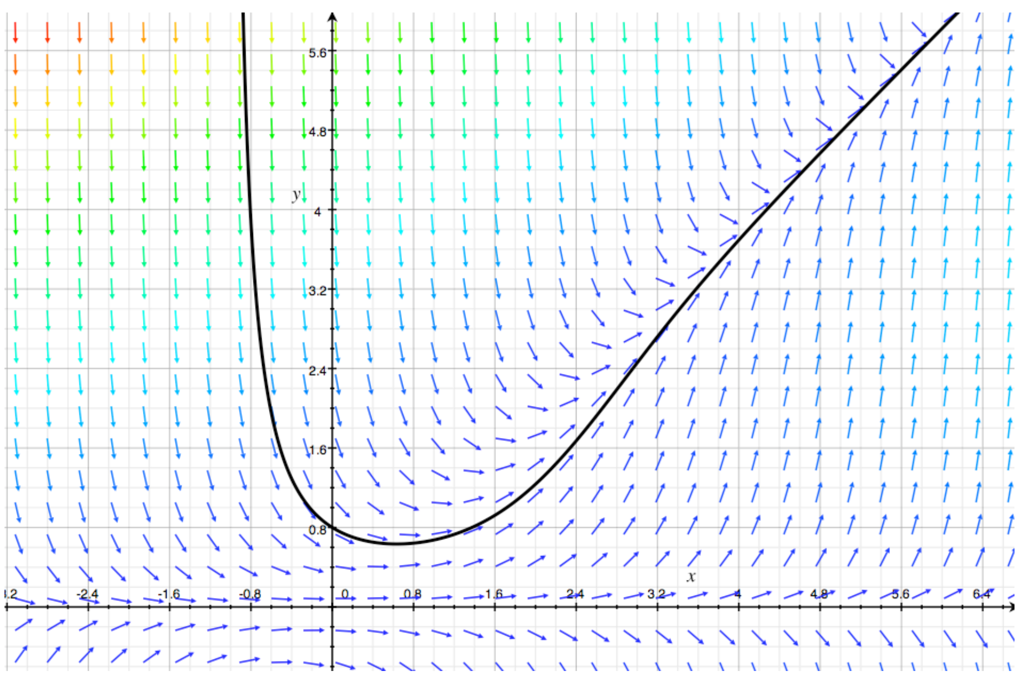


FIGURE 6. The slope field for $y' = ty - y^2$.

**Solution:** The solution has been drawn in.

**Problem #25:**
Figure out if

$$y = (-2/3)e^{-t} - (1/3)e^{t/2}$$

is a solution to the DE

$$y' - y/2 = e^{-t}$$

**Solution:** It is. You need to show the work of finding the derivative and plugging $y'$ and $y$ into the DE to see that it works.

**Problem #26:**
Find an implicit solution to the DE

$$\frac{dy}{dt} = \frac{t - e^{-t}}{y + e^{y}}.$$

(This means that you do not need to solve for $y$ in your answer.)

**Solution:** $y^2/2 + e^y = t^2/2 + e^{-t} + C.$

**Problem #27:**
Consider the DE

$$y' = y - t.$$

Suppose that $y = f(t)$ is a solution to the DE and that $f(0) = 0.5$.

(i) What is the equation for the tangent line to the graph of $f(t)$ at the point $(0, 0.5)$?
**Solution:** $y = .5t + .5$

(ii) Use your answer from (i) to estimate $f(0.5)$.
**Solution:** $f(0.5) \approx .75$

(iii) Use the DE and your answer from (ii) to estimate $f'(0.5)$.
**Solution:** $f'(0.5) = f(0.5) - 0.5 \approx .75 - .5 = .25$.

(iv) Use your previous work to write down an equation for the tangent line to the graph of $f(t)$ at the point $(0.5, f(0.5))$. (Because your previous work has included approximations, this is also an approximation.)
**Solution:** $y \approx .25(t - .5) + .75$.

(v) Use your answer from (iv) to estimate $f(1)$.
**Solution:** $f(1) \approx .25(1 - .5) + .75 = .875$.

(vi) Continuing in this way you can approximate $f(t)$. Figure 7 shows the slope field for the DE. Plot the points $(0, f(0))$, $(.5, f(.5))$, and $(1, f(1))$ and connect them with straight lines. Does this look like an approximation to a solution of the DE with intial condition $f(0) = 0.5$? This method of constructing solutions to a DE is called Euler's method.
**Solution:** The points have been drawn on the slope field and a solution curve has also been drawn. The points don't give a great approximation to the real solution.

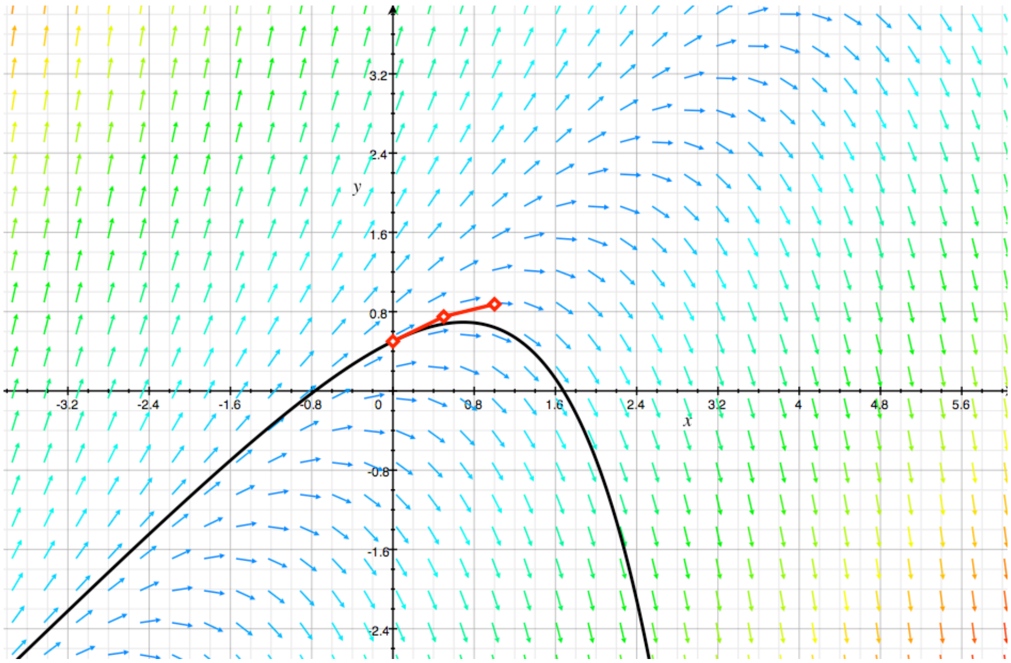


FIGURE 7. The slope field for $y' = y - t$.