MA 121: Practice Final **Name:**____________________________________

Problem #1:
Find the derivatives of the following functions (k is a constant).

(i)
$$f(x) = \sin(kx)\cos(x^3 - 2)$$
(ii)
$$f(x) = \ln(x^4 + e^{-kx})$$

Problem #2:
Figure 1 shows the graph of a function $f(x)$.

(i) Sketch the graph of $f'(x)$.
(ii) Sketch the graph of an antiderivative of $f(x)$.

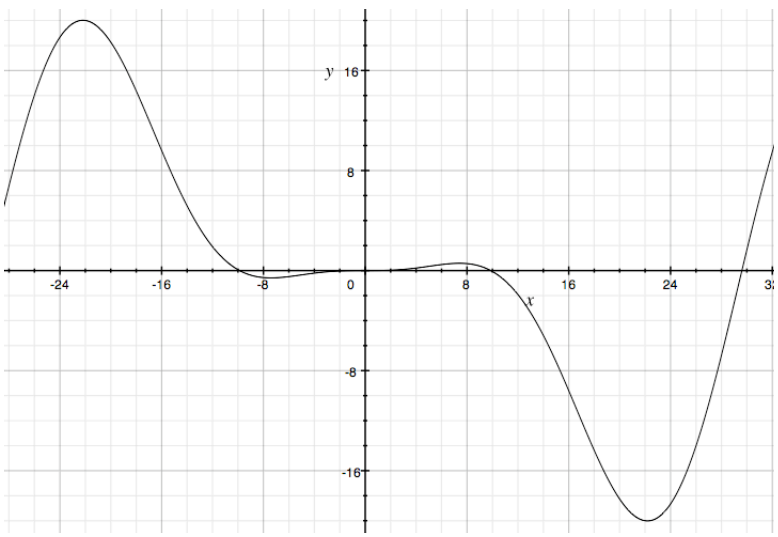

FIGURE 1. The graph of $y = f(x)$

Problem #3:

(i) Find the equation of the line tangent to $y = \arctan(x)$ at $x = 1/\sqrt{3}$.
(ii) Use the equation to approximate $\arctan(0.6)$. (Hint: $1/\sqrt{3} \approx 0.577$.)
(iii) Use Calculus to determine if $y = \arctan(x)$ is increasing or decreasing and concave up or concave down at $x = 1$.
(iv) Use your answer from (iii) to determine if your answer in (ii) is an underestimate or overestimate of $\arctan(2)$.

Problem #4:
Use the limit definition of the derivative to show that $\frac{d}{dx}x^2 = 2x$.

Problem #5:
Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Problem #6:
Let $f(x) = k^x$ for some $k > 0$. Find k so that the tangent line to $f(x)$ through the point $(1, k)$ also passes through the point $(0, 3e^{-2})$.

Problem #7:
Buffalo Bill decides to keep careful count of the population of buffalo in Buffalo, New York. $P(t)$ represents the population of buffalo t years after 1990. Buffalo Bill discovers that $P'(0) = 3$. The population of buffalo in 1990 is 1000.

(i) What are the units for $P'(t)$?
(ii) What does $P'(0) = 3$ signify in terms of buffalo?
(iii) The maximum population of buffalo that Buffalo, NY can sustain is 2000. Use the logistic equation to write down a differential equation and initial value which models the population of buffalo t years after 1990.
(iv) Use separation of variables to solve the initial value problem from (iii). You may wish to notice that

$$\frac{1}{P(2000-P)} = \frac{1}{2000}\left(\frac{1}{P} + \frac{1}{2000-P}\right)$$

Problem #8:
Find and classify all critical points of

$$f(x) = (|x| - 1)^2$$

Problem #9:
Show that the derivative of $f(x) = \arcsin(x)$ is $f'(x) = 1/\sqrt{1-x^2}$.

Problem #10:
Find the equation of the line tangent to the graph of

$$xy^2 = x^2 - y$$

At the point $(1, \frac{-1+\sqrt{5}}{2})$.

Problem #11:
A parabolic bowl of water is being filled at a rate of 2 ft^3 per minute. It is leaking out the bottom at h ft^3 per minute where h is the depth of water (measured at its deepest point.) If the origin is put at the base of the bowl, its (vertical) cross section is given by the equation $y = x^2$. See Figure 2. If the water is h feet deep, then the volume of the water is $\int_0^{\sqrt{h}} \pi x^2 \, dx$.

(i) How fast is the depth of the water (at the deepest point) changing when the water is 3 feet deep (at its deepest point)?
(ii) If the bowl starts out with water that is 1 feet deep, will the water ever be 3 feet deep?

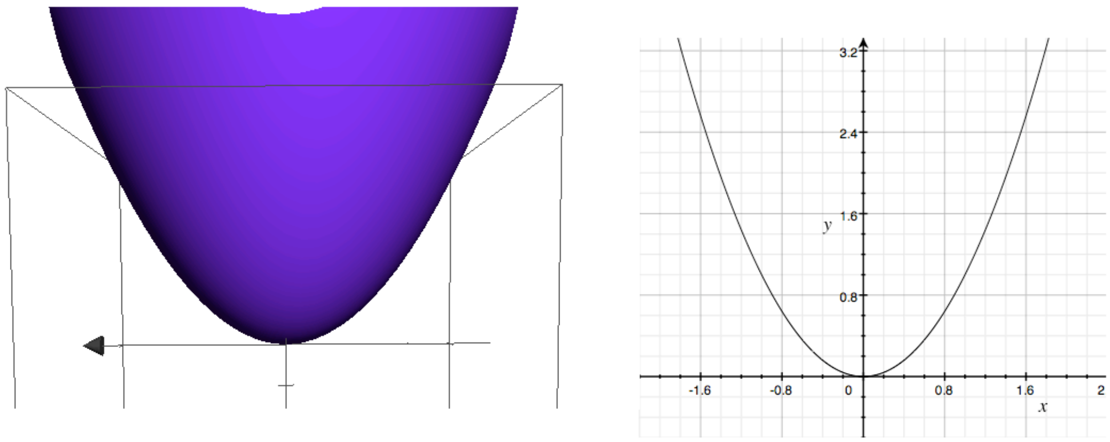

FIGURE 2. Parabolic bowl and cross section

Problem #12:
A man six feet tall walks at a rate of 5 feet per second toward a streetlight that is 16 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light?

Problem #13:
A rectangle has its base on the x–axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions? Explain how you know that you have found the largest possible area.

Problem #14:
The trough in Figure 3 is going to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume? Explain how you know you have found the largest possible volume.

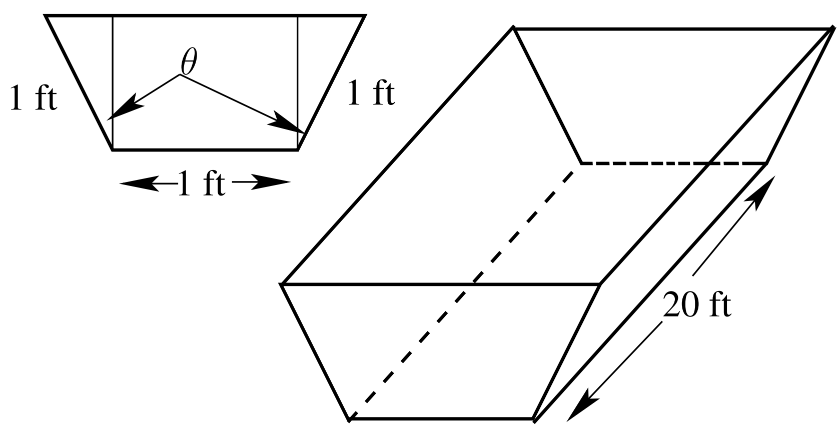

FIGURE 3. The trough

Problem #15:
Calculate $\int_{-7}^{7} f(x)\,dx$ for the function $f(x)$ shown in Figure 4.

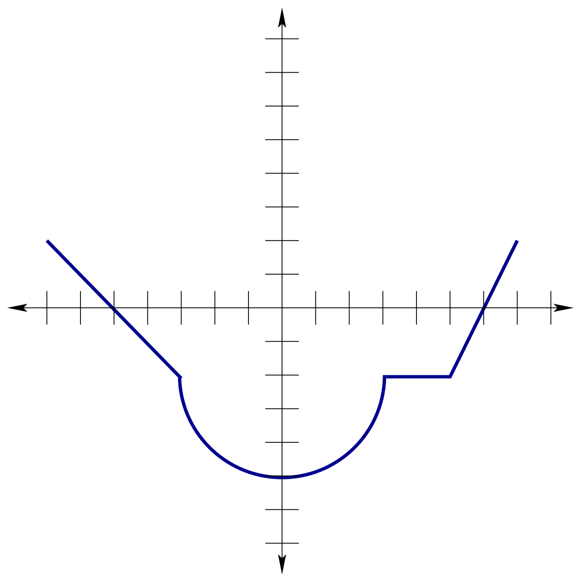

FIGURE 4. The graph of $y = f(x)$.

Problem #16:
Approximate $\int_1^2 \sqrt{4 - x^2}\,dx$ using a left-hand Riemann sum with $n = 4$ boxes. You do not need to perform the necessary arithmetic.

Problem #17:
Let $f(x) = \arccos(x)(x^{16} - 1)^{89}$. Compute $\int f'(x)\,dx$.

Problem #18:
Compute $\int_1^3 1 - 2x$ using the limit definition of the Riemann integral. You will need to use the identity

$$\sum_{i=0}^{n-1} i = \frac{n^2 - n}{2}.$$

Problem #19:
Figure 5 shows the graph of a function $f(x)$. Let $A_f(t) = \int_0^t f(x)\,dx$. Find and classify the critical points of $A_f(t)$.

Problem #20:
Compute

$$\frac{d}{dt} \int_2^{\cos(t^2)} \sin(x^2)\,dx$$

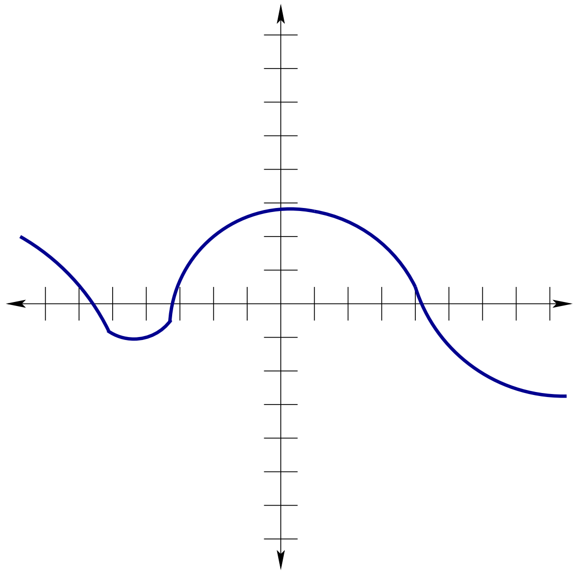

FIGURE 5. The graph of $y = f(x)$.

Problem #21:
Use the first fundamental theorem of Calculus to prove the second fundamental theorem of calculus.

Problem #22:
The average value of a function $f(x)$ on an interval $[a, b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Give a detailed explanation as to why this is a sensible definition.

Problem #23:
Compute the following integrals and antiderivatives.

(1) $\int_0^\pi \sin(x/3)\,dx$
(2) $\int x^3 - \frac{1}{\sqrt{x}}\,dx$
(3) $\int \frac{x}{\sqrt{x-5}}\,dx$
(4) $\int_0^3 te^{t^2-7}\,dt$
(5) $\int \frac{\arctan x}{1+x^2}\,dx$
(6) $\int \frac{18\tan^2(x)\sec^2(x)}{(2+\tan^3(x))^2}\,dx$

Problem #24:
Figure 6 shows a slope field for the differential equation

$$y' = ty - y^2$$

Sketch a solution to the DE $y' = ty - y^2$ which passes through the point $(0, 0.8)$.

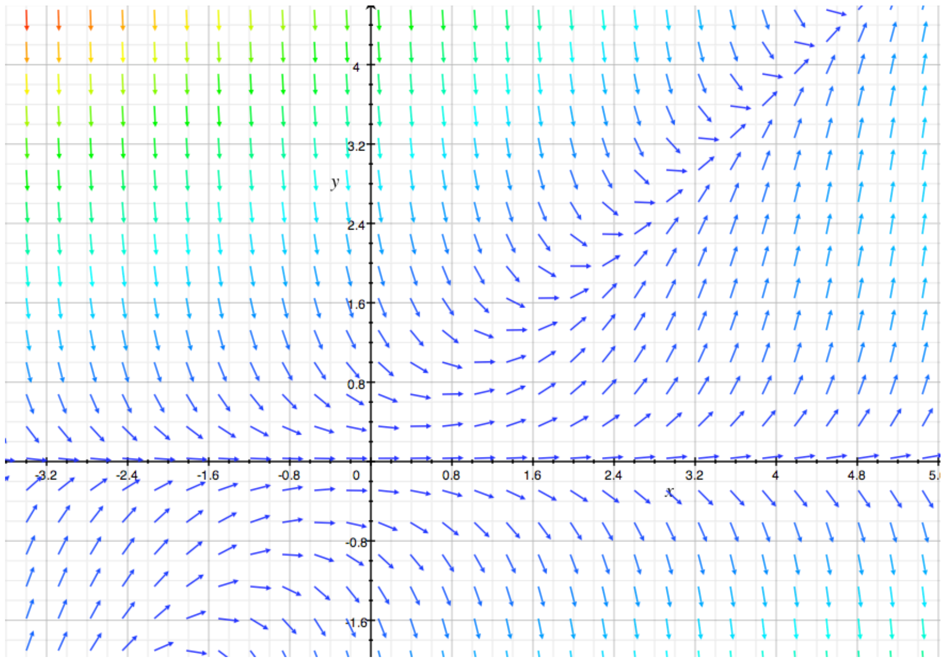

FIGURE 6. The slope field for $y' = ty - y^2$.

Problem #25:
Figure out if

$$y = (-2/3)e^{-t} - (1/3)e^{t/2}$$

is a solution to the DE

$$y' - y/2 = e^{-t}$$

Problem #26:
Find an implicit solution to the DE

$$\frac{dy}{dt} = \frac{t - e^{-t}}{y + e^y}.$$

(This means that you do not need to solve for y in your answer.)

Problem #27:
Consider the DE

$$y' = y - t.$$

Suppose that $y = f(t)$ is a solution to the DE and that $f(0) = 0.5$.

(i) What is the equation for the tangent line to the graph of $f(t)$ at the point $(0, 0.5)$?
(ii) Use your answer from (i) to estimate $f(0.5)$.
(iii) Use the DE and your answer from (ii) to estimate $f'(0.5)$.
(iv) Use your previous work to write down an equation for the tangent line to the graph of $f(t)$ at the point $(0.5, f(0.5))$. (Because your previous work has included approximations, this is also an approximation.)
(v) Use your answer from (iv) to estimate $f(1)$.

(vi) Continuing in this way you can approximate $f(t)$. Figure 7 shows the slope field for the DE. Plot the points $(0, f(0))$, $(.5, f(.5))$, and $(1, f(1))$ and connect them with straight lines. Does this look like an approximation to a solution of the DE with intial condition $f(0) = 0.5$? This method of constructing solutions to a DE is called Euler's method.

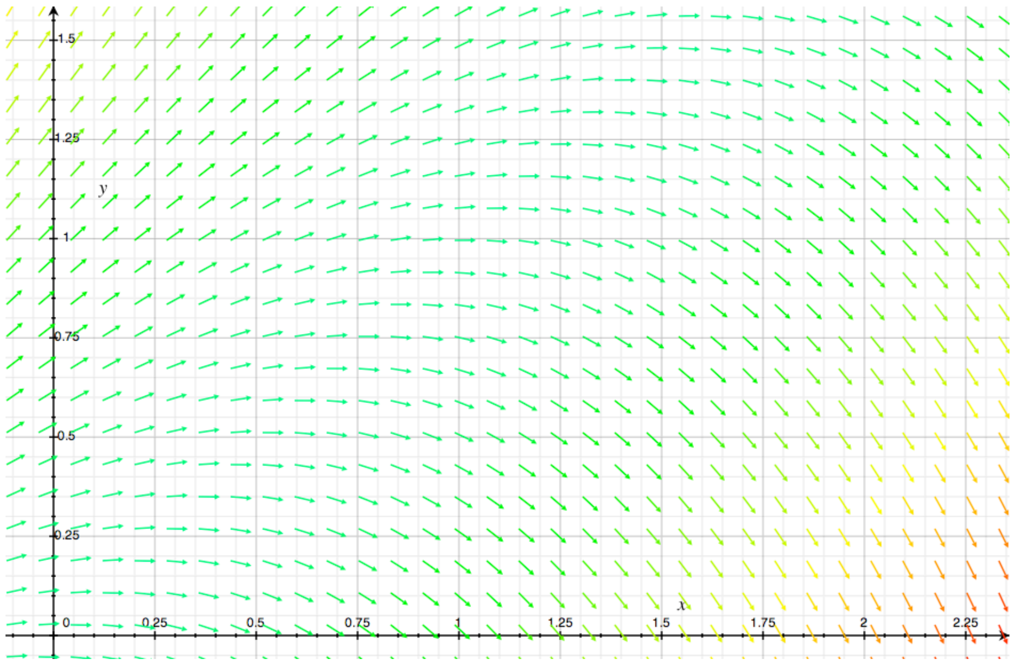

FIGURE 7. The slope field for $y' = y - t$.