Calculus I Practice Exam 2

Instructions: The exam is closed book, closed notes, although you may use a note sheet as in the previous exam. A calculator is allowed, but you must show all of your work. **Your work is your answer.** If you have any questions, please ask immediately! Good luck.

(This practice exam is longer and more difficult than the actual exam.)

Problem #1 : Find the derivatives of the following functions (k is a constant).

a)

$$f(x) = \sqrt{1 + x\ln(7 + kx^2)}$$

Answer:

$$f'(x) = \frac{1}{2}(1 + x\ln(7 + kx^2))^{-1/2}(\frac{2kx^2}{7 + kx^2} + \ln(7 + kx^2))$$

b)

$$g(x) = (x^2 - e^{\sin(x^3 - x)})e^{k\cos(x)}$$

Answer:

$$g'(x) = (x^2 - e^{\sin(x^3-x)})e^{k\cos(x)}(-k\sin(x)) + (2x - \cos(x^3 - x)(3x^2 - 1)e^{\sin(x^3-x)})e^{k\cos(x)}$$

c)

$$h(x) = \frac{1 - \ln(2x)}{3 + e^{2x}}$$

Answer:

$$h'(x) = \frac{(3 + e^{2x})(-1/x) - (1 - \ln(2x))(2e^{2x})}{(3 + e^{2x})^2}$$

Problem #2: Use the tangent line to the graph of $y = \tan(2x)$ at $x = \pi/6$ to approximate $\tan(2\pi/7)$.

Answer: The tangent line has equation

$$y = 8(x - \pi/6) + \sqrt{3}.$$

This means that

$$\tan(2 \cdot \frac{\pi}{7}) \approx 8(\pi/7 - \pi/6) + \sqrt{3}.$$

Problem #3 :

a) Find dy/dx:

$$xy^2 = 2e^x - 2y$$

Answer:

$$\frac{dy}{dx} = \frac{y^2 - 2e^x}{-2xy - 2}$$

b) Find the equation of the line tangent to the above curve at the point $(0, 1)$.

Answer:

$$y = (1/2)x + 1.$$

Problem #4: Calculate the derivative of $y = \arccos(x)$. You must show all of the steps. You may find it helpful to think about triangles.

Answer: We know that $\cos(y) = x$. Taking the derivative of both sides produces $-\sin(y)\frac{dy}{dx} = 1$. That is,

$$\frac{dy}{dx} = -\frac{1}{\sin(y)}.$$

Thinking of y as an angle of a right triangle, we can label the adjacent side x and the hypotenuse 1. This means the opposite side has length $\sqrt{1-x^2}$, by the Pythagorean Theorem. Thus, $\sin(y) = \sqrt{1-x^2}$. Hence,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

Problem #5:

Compute

$$\lim_{x\to 0} \frac{x}{\arctan(4x)}$$

Answer: Since at $x = 0$, both $x = 0$ and $\arctan(4x) = 0$, we can apply L'Hopital's rule.

$$\lim_{x\to 0} \frac{x}{\arctan(4x)} = \lim_{x\to 0} \frac{1}{4/(1+(4x)^2)}.$$

Some algebra shows that this is,

$$\lim_{x\to 0} \frac{1+16x^2}{4}.$$

This last function is continuous at $x = 0$, and so the limit evaluates to $1/4$.

Problem #6: A hanging rope can be modelled by the graph of the function:

$$f(x) = \frac{e^x + e^{-x}}{2}.$$

Find the global maxima and minima of $f(x)$ on the interval $[-2, 2]$.

Answer:

$$f'(x) = \frac{e^x - e^{-x}}{2}.$$

The only critical point is at $x = 0$. The endpoints are $x = -2$ and $x = 2$. Plugging these into the function produces:

$$\begin{aligned} f(0) &= 1 \\ f(-2) &\approx 3.76 \\ f(2) &\approx 3.76 \end{aligned}$$

The the global minimum occurs at $x = 0$ and has a value of 1. The global maximum occurs at the endpoints, $x = \pm 2$, and has an approximate value of 3.76.

Problem #7:

Use calculus to find all of the points where

$$f(x) = x^2 - |x - 1|$$

has local maxima and minima and to classify each as being a maximum or minimum.

Answer: Notice that $f(x)$ does not have a derivative at $x = 1$. Rewrite $f(x)$ as a piecewise defined function:

$$f(x) = \begin{cases} x^2 - (-(x-1)) & \text{if } x \leq 1 \\ x^2 - (+(x-1)) & \text{if } x > 1 \end{cases}$$

Taking the derivative of the pieces provides

$$f'(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

The points $x = -1/2$ and $x = 1/2$ are solutions to $f'(x) = 0$. However, $x = 1/2$ arises when $x > 1$, so this cannot happen.

Thus

$$x = -1/2, x = 1$$

are the critical points of $f(x)$. For $0 < x < 1$, $f'(x) > 0$. For $1 < x < 2$, $f'(x) > 0$. Thus, there is no maximum or minimum at $x = 1$. The second derivative of $f(x)$ is

$$f''(x) = \begin{cases} 2 & \text{if } x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

Hence, at $x = -1/2$, $f(x)$ has a minimum.

Problem #8:

A lighthouse is located on an island 3 miles from a straight shoreline. There is a house on the shore directly opposite the lighthouse. The light on the lighthouse revolves at 4 revolutions per minute. How fast is the light travelling along the shore when it is one mile from the house?

Answer:

Let θ be the angle between the beam of light and the straight path from the light house to the house on the shore. Let s be the distance along the shore from the light to the house on the shore.

Since the light revolves at four revolutions per minute,

$$d\theta/dt = \pm 4\text{rev}/\text{min} = \pm 8\pi\text{radians}/\text{min}.$$

We know that $\tan\theta = s/3$ and so,

$$\sec^2\theta d\theta/dt = 1/3 ds/dt.$$

When the light is one mile from the house, $\theta = \arctan(1/3)$. The length of the beam of light at this moment is $\sqrt{1^2+3^2} = \sqrt{10}$. Thus, $\sec\theta = \sqrt{10}/3$.

Hence,

$$\sqrt{10}(\pm 8\pi) = ds/dt.$$

Notice that we can't tell whether or not the distance between the light and the house is increasing or decreasing.

Problem #9:

A trough, as shown in the diagram, contains eggnog half a foot deep. The organizer of the holiday party, adds eggnog to the trough at a rate of $.5/h$ feet per minute, where h is the height of the . How fast is the eggnog rising when the height of the water is 2 feet? (Hint: The volume of a trapezoidal box (such as the one pictured) is $(b+B)hd/2$ where b is the length of the bottom, B is the length of the top, h is the vertical height of the trapezoid, and d is the depth of the box.)

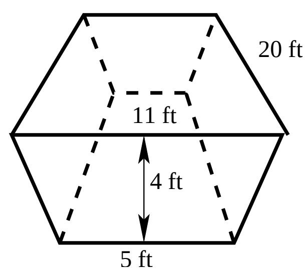

FIGURE 1. The eggnog trough.

Answer: We need a formula for the volume of eggnog in the trough. Let h denote the height of the eggnog and let B denote the top width of the eggnog at time t. Placing the lower right front corner of the trough at the origin, we see that the right slanted side of the trapezoid follows the equation $y = 4/3x$ or $x = 3/4y$. This means that at time t,

$$B = 5 + 2 \cdot (3/4)h.$$

Thus, the volume of eggnog at time t is

$$V = (5 + 5 + (3/2)h)h(20)/2 = 100h + 15h^2.$$

Taking the derivative we get:

$$dV/dt = (100 + 30h)(dh/dt).$$

We are told that $dV/dt = .5/h$. Hence,

$$\frac{.5}{h(100 + 30h)} = dh/dt.$$

Consequently, when $h = 2$, we have

$$\frac{.5}{2(160)} = dh/dt.$$

<u>Problem #10</u>: Harpo Marx is on top of a 6 foot ladder leaning against a vertical wall. Zeppo Marx is pushing the base of the ladder towards the wall at .5 feet per second. How fast is Harpo rising when the angle between the base of the ladder and the floor is $60°$?

Answer: $1/2\sqrt{3}$ feet per second.

<u>Problem #11</u>: A producer of canned tuna is trying to figure out the optimal size of a tuna can. The can must hold 15 in^3 of tuna when initially packed. The can will be a cylinder of radius r and height h. The cost to make the sides of the can is $\$.15$/in^2. The cost to make the top of the can is $\$.10$/in^2. What is the least amount it will cost to make a can for tuna? (Hint: The volume of the cylinder is $V = \pi r^2 h$ and the surface area of the side of the can is $2\pi rh$ and the surface area of the top and bottom of the can (combined) is $2\pi r^2$.)

Answer: We know that

$$15 = \pi r^2 h.$$

The cost to make the can is:

$$C = (.15)2\pi rh + (.1)2\pi r^2.$$

Plugging in $h = 15/(\pi r^2)$ gives us

$$C = 4.5/r + .2\pi r^2.$$

Taking the derivative gives us

$$dC/dr = -4.5/(r^2) + .4\pi r.$$

Setting this equal to zero and solving for r provides a critical point of C at

$$r = \sqrt[3]{11.25/\pi}.$$

The second derivative of C is

$$d^2C/dr^2 = 9/r^3 + .4\pi$$

Since this is always positive, C is always concave up and so $r = \sqrt[3]{11.25/pi}$ is a minimum point. Thus the mininum cost for making the can is

$$C = 4.5/\sqrt[3]{11.25/\pi} + .2\pi(11.25/\pi)^{2/3}.$$

Problem #12:

Consider the function:

$$f(x) = 4 + x - 2x^3$$

on the interval $[0, 1]$.

a) Carefully explain why there is some number c in $[0, 1]$ such that $f(c) = 3.5$.

Answer: $f(x)$ is continuous on $[0, 1]$. $f(0) = 4$ and $f(1) = 3$. The number $y = 3.5$ is between $f(0)$ and $f(3)$ so by the intermediate value theorem, there is a c such that $f(c) = 3.5$ and $0 < c < 1$.

b) Carefully explain what the Mean Value Theorem says about this function on this interval.

Answer: Since $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$, the Mean Value Theorem says that there is a c, with $0 < c < 1$ so that $f'(c) = [f(b) - f(a)]/(b - a)$. That is, $f'(c) = -1$.

Problem #13: Use the Mean Value Theorem to prove that if $f(x)$ is a function which is continuous on $[a, b]$ and differentiable on (a, b) such that $f'(x) = 0$ for all x in (a, b) then there is a constant k such that $f'(x) = k$ for all x in $[a, b]$.

Answer: Let x_1 and x_2 be two numbers in $[a, b]$ with $x_1 < x_2$. Notice that the conditions to apply the mean value theorem to $f(x)$ on $[x_1, x_2]$ are satisfied. Thus, the Mean Value Theorem says that there is a c such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) = 0$. Thus, $f(x_2) - f(x_1) = 0$. In other words, $f(x_2) = f(x_1)$. Since the choice of x_1 and x_2 was arbitrary, $f(x)$ is a constant. $\square$

Problem 14 is on the next page

Problem #14: Sketch part of a slope field for the differential equation:

$$y' = y - t$$

You slope field should include at least four slopes in each quadrant (and not on the axes).

Answer: Here is a computer generated slope field. Your answer, of course, does not need to be this detailed.