BAND TAUT SUTURED MANIFOLDS

SCOTT A TAYLOR

ABSTRACT. Attaching a 2–handle to a genus two or greater boundary component of a 3–manifold is a natural generalization of Dehn filling a torus boundary component. We prove that there is an interesting relationship between an essential surface in a sutured 3–manifold, the number of intersections between the boundary of the surface and one of the sutures, and the cocore of the 2–handle in the manifold after attaching a 2–handle along the suture. We use this result to show that tunnels for tunnel number one knots or links in any 3–manifold can be isotoped to lie on a branched surface corresponding to a certain taut sutured manifold hierarchy of the knot or link exterior. In a subsequent paper, we use the theorem to prove that band sums satisfy the cabling conjecture, and to give new proofs that unknotting number one knots are prime and that genus is superadditive under band sum. To prove the theorem, we introduce band taut sutured manifolds and prove the existence of band taut sutured manifold hierarchies.

1. INTRODUCTION

Gabai's sutured manifold theory [G1, G2, G3] is central to a number of stunning results concerning Dehn surgery on knots in 3–manifolds. Many of these insights make use of a famous theorem of Gabai [G2, Corollary 2.4]: with certain mild hypotheses, there is at most one way to fill a torus boundary component of a 3–manifold so that Thurston norm decreases. Lackenby [L1], building on this work, proved a theorem relating Dehn surgery properties of a knot to the intersection between the knot and essential surfaces in the 3–manifold. Lackenby used his results to study the effect of twisting the unknot along a knot having linking number zero with the unknot, and to study [L2] the uniqueness properties of Dehn surgery on certain knots in certain 3-manifolds. Lackenby [L3] and Kalfagianni [K] also used Lackenby's theorem to study the unknotting properties of certain knots.

In this paper, we prove a version of Lackenby's theorem for attaching a 2–handle to a sutured 3–manifold along a suture. Like Lackenby, we use

Scharlemann's combinatorial version of sutured manifold theory [S1]. Although our method is inspired by the proofs of Gabai's and Lackenby's theorems, the proof is very different.

For the statement of the main result, let (N,γ) be a sutured manifold and let $F \subset \partial N$ be a component of genus at least 2. Let $b \subset \gamma \cap F$ be a component. Let $N[b]$ be the 3–manifold obtained by attaching a 2–handle to N along b and let $\beta \subset N[b]$ be the cocore of the 2–handle. For a 3-manifold M with T the union of two torus boundary components, we say that a non-zero homology class $y \in H_2(M,\partial M)$ is **Seifert-like** for T if the projection of y to the first homology of each component of T is non-zero. The main result of this paper is:

Theorem 10.7 (rephrased). *Suppose that (N,γ) is taut and that the components of $\partial N - A(\gamma)$ adjacent to b are both thrice punctured spheres or are both once-punctured tori. Let $Q \subset N$ be a surface having no component a sphere or disc disjoint from γ. Assume that ∂Q intersects γ minimally and that $|\partial Q \cap b| \geq 1$. Then one of the following is true:*

(1) Q has a compressing or b-boundary compressing disc.
(2) $(N[b],\beta) = (M_0',\beta_0')\#(M_1',\beta_1')$ where M_1' is a lens space and β_1' is a core of a genus one Heegaard splitting of M_1'.
(3) The sutured manifold $(N[b],\gamma - b)$ is taut. The arc β can be properly isotoped to be embedded on a branched surface $B(\mathscr{H})$ associated to a taut sutured manifold hierarchy $\mathscr{H}$ for $N[b]$. There is also a proper isotopy of β in $N[b]$ to an arc disjoint from the first decomposing surface in $\mathscr{H}$. If b is adjacent to thrice-punctured spheres, that first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(N[b],\partial N[b])$. If b is adjacent to once-punctured tori, the first decomposing surface can be taken to represent any Seifert-like homology class for the corresponding unpunctured torus components of $\partial N[b]$.
(4)
$$-2\chi(Q) + |\partial Q \cap \gamma| \geq 2|\partial Q \cap b|.$$

A b-**boundary compressing disc** for a properly embedded surface $Q \subset N$ transverse to b is a disc with boundary consisting of an arc on Q and a subarc of b and with interior disjoint from $Q \cup \partial N$. See Figure 1.

In [L2], Lackenby shows how to add sutures to the (non-empty) boundary of a compact, orientable, irreducible and boundary-irreducible manifold (other than a 3–ball) to create a taut sutured manifold. In his construction all components of $R(\gamma)$ are thrice-punctured spheres or tori, so the hypothesis

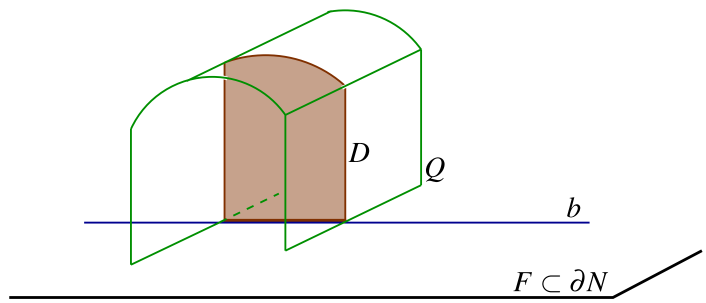

FIGURE 1. D is an b-boundary compressing disc for the surface Q outlined in green.

in Theorem 10.7 that b be adjacent to thrice-punctured sphere components of $R(\gamma)$ is reasonable.

The fourth conclusion is useful, since the inequality can be rearranged to be:

$$-2\chi(Q) + |\partial Q \cap (\gamma - b)| \geq |\partial Q \cap b|.$$

Thus, for example, if ∂Q happens to be disjoint from $\gamma - b$, then twice the negative euler characteristic of the surface is an upper bound for the number of times the boundary of the surface intersects b.

The third conclusion of the theorem is of particular interest in that it is related to several well-known and very useful facts:

- If K is an unknotting number one knot in S^3 and if β is an arc in the knot complement defining a crossing change converting K into the unknot then β is isotopic into a minimal genus Seifert surface for K.
- If K is a tunnel number one knot in S^3 and if β is a tunnel then, if the Scharlemann-Thompson invariant [ST] is not 1, β can be isotoped into a minimal genus Seifert surface for K.

Any minimal genus Seifert surface can be used as the first surface in a taut sutured manifold hierarchy of the knot exterior, and so any minimal genus Seifert surface can be thought of as part of a branched surface associated to a taut sutured manifold hierarchy of the knot exterior. Since these facts have proven to be very useful, the third conclusion of the main theorem of this paper also has the potential to be useful and perhaps points to a connection between the various *ad hoc* methods used to push certain arcs onto minimal genus Seifert surfaces.

Applications of Theorem 10.7 include a proof that knots that are band sums satisfy the cabling conjecture [T2, Theorem 8.1], a partial solution to a conjecture of Scharlemann and Wu [T2, Corollary 5.4], a near complete solution of a conjecture of Scharlemann [T2, Corollary 6.2], and new proofs of three classical facts:

- Knot genus is superadditive under band connect sum [T2, Theorem 7.3].
- Unknotting number one knots are prime [T2, Theorem 7.2].
- Tunnel number one knots in S^3 have minimal genus Seifert surfaces disjoint from a given tunnel (Theorem 11.2 below).

These three facts previously all had proofs which use sutured manifold theory, but the methods were different. The advantage of the new proofs is that they are all nearly identical. Since some effort is required to rephrase the theorems in a way in which the main theorem of this paper can be usefully applied, we defer proofs for all but the last fact to [T2]; the new proof of the last fact is given in this paper. Indeed, we prove the following stronger theorem for tunnel number one knots and 2–component links in any 3–manifold admitting such a knot or link (see Section 11 for the definitions):

Theorem 11.1. *Suppose that L_b is a knot or 2-component link in a closed, orientable 3-manifold M such that L_b has tunnel number one. Let β be a tunnel for L_b. Assume also that $(M - L_b, \beta)$ does not have a (lens space, core) summand. Then there exist (possibly empty) curves $\widehat{\gamma}$ on $\partial(M - \mathring{\eta}(L_b))$ such that $(M - \mathring{\eta}(L_b), \widehat{\gamma})$ is a taut sutured manifold and the arc β can be properly isotoped to lie on the branched surface associated to a taut sutured manifold hierarchy of $(M - \mathring{\eta}(L_b), \widehat{\gamma})$. In particular, if L_b has a (generalized) Seifert surface, then there exists a minimal genus (generalized) Seifert surface for L_b that is disjoint from β.*

2. Motivation and Outline

As motivation for our proof of Theorem 10.7, we briefly review the proofs of Gabai's and Lackenby's theorems. For reference, here are (simplified and weakened) statements of Gabai's and Lackenby's theorems. The sutured manifold terminology will be explained in the next section.

Theorem (Gabai)**.** *Let N be an atoroidal Haken 3–manifold whose boundary is the non-empty union of tori. Let S be a Thurston norm minimizing surface representing an element of $H_2(N, \partial N)$ and let P be a component of ∂M such that $P \cap S = \varnothing$. Then, with at most one exception, S remains norm minimizing in each manifold obtained by Dehn filling N along a slope in P.*

Theorem ([L1, Theorem 1.4])**.** *Suppose that (N, γ) is a taut atoroidal sutured manifold. Let $P \subset \partial N$ be a torus component disjoint from γ.*

Suppose that b is a slope on P such that Dehn filling N with slope b creates a sutured manifold $(N(b),\gamma)$ that is not taut. Let $Q \subset N$ be an essential surface such that ∂Q intersects b minimally and $|\partial Q \cap b| \geq 1$. Then

$$-2\chi(Q) + |\partial Q \cap \gamma| \geq 2|\partial Q \cap b|.$$

Gabai's theorem is proved by taking a taut sutured manifold hierarchy for N such that each decomposing surface in the hierarchy is disjoint from P. The first decomposing surface is the given surface S. The hierarchy ends at a taut sutured manifold (N_n, γ_n) such that $H_2(N_n, \partial N_n - P) = 0$. Our additional assumption that N is atoroidal implies that N_n consists of 3–balls and one additional component that is homeomorphic to $P \times [0,1]$. Dehn filling N_n along a slope $b \subset P$ creates another sutured manifold which we call $(N_n(b), \gamma_n)$. An examination of the sutures $\gamma_n \cap \partial N_n$ shows that for all but at most one choice of b, $(N_n(b), \gamma_n)$ remains taut. One of the fundamental theorems of sutured manifold theory [S1, Corollary 3.9] (see Theorem 7.2 below) implies that, except for the exceptional slope, $(N(b), \gamma)$ and S are taut.

Equivalently, we can begin with the Dehn-filled β-taut sutured manifold $(N(b), \gamma, \beta)$ where β is the core of the surgery solid torus. The hierarchy for N is then a β-taut sutured manifold hierarchy for $(N(b), \gamma, \beta)$ where each decomposing surface is disjoint from β. We conclude that for all but at most one choice of b, the sutured manifold $(N(b), \gamma, \varnothing)$ and the surface S are not only β–taut, but also $\varnothing$–taut. There are two advantages to this viewpoint. One is that it is possible to see that if the hierarchy is taut then β has infinite order in the fundamental group of $N(b)$ [L2, Theorem A.6]. The other advantage is that, if the hierarchy of the filled manifold is taut, it is not difficult to see that β can be isotoped to lie on the branched surface corresponding to the hierarchy. The analogous statment in Theorem 10.7 is much harder to prove.

That was a sketch of the proof of Gabai's theorem. We now turn to Lackenby's theorem. The surface Q in the statement of Lackenby's theorem is an example of what is called a "parameterizing surface" in (N, γ). (Parameterizing surfaces are defined in Section 5.) Associated to each parameterizing surface is a number called the index (or "sutured manifold norm" [CC]). In the case of Lackenby's theorem, the index of Q is defined to be:

$$I(Q) = |\partial Q \cap \gamma| - 2\chi(Q).$$

Suppose now that $b \subset P$ is the exceptional slope, so that $(N(b), \gamma - b)$ is not taut. Let $Q \subset N$ be a parameterizing surface so that ∂Q intersects γ minimally and $|\partial Q \cap b| > 0$.

In the sutured manifold (N_n, γ_n), the surface Q has decomposed into a parameterizing surface Q_n with $I(Q) \geq I(Q_n)$. The component N'_n of N_n containing P is homeomorphic to $P \times [0,1]$. Some components of Q_n lie in N'_n. Since Q is essential, $Q_n \cap N'_n$ is the union of discs with boundary on $\partial N'_n - P$ and annuli with at least one boundary component on $\partial N'_n - P$. These boundary components must cross the sutures on $\partial N'_n - P$. Analyzing these intersections gives a lower bound on $I(Q_n)$ which is, therefore, a lower bound on $I(Q)$. This lower bound implies the inequality in Lackenby's theorem. As in Gabai's theorem, Lackenby's theorem can be rewritten as a theorem about a sutured manifold (M, γ, β) with β a knot in M. (The knot β is the core of the surgery solid torus with slope b.)

The point of this paper is to develop a theory whereby we can replace the knot β in the the proof of Gabai's and Lackenby's theorems with an arc β. In Theorem 9.5, this arc is the cocore of a 2–handle added to $b \subset \partial N$.

The proof of Theorem 10.7 is inspired by the proof of Lackenby's theorem. For the time being, let $(M, \gamma') = (N[b], \gamma - b)$ and consider the arc $\beta \subset M$ which is the cocore of the attached 2–handle. If we could construct a useful hierarchy of (M, γ', β) disjoint from β, we could adapt Lackenby's combinatorics to obtain a result similar to Theorem 10.7. However, it seems unlikely that such a hierarchy can exist, since although a sequence

$$(M, \gamma', \beta) \xrightarrow{S_1} (M_1, \gamma_1, \beta_1) \xrightarrow{S_2} \ldots \xrightarrow{S_n} (M_n, \gamma_n, \beta_n)$$

can be constructed so that each decomposing surface represents a given homology class, and although it is possible to find such surfaces representing the homology class that are disjoint from β, it may not be possible to find such surfaces giving a β-taut decomposition which are (in the terminology of [S1]) "conditioned". Instead we develop the theory of "band-taut sutured manifolds" to give the necessary control over intersections between β and the decomposing surfaces. Sections 4 and 7 are almost entirely devoted to proving that if (M, γ', β) is a band-taut sutured manifold then there is a so-called "band-taut" sutured manifold hierarchy of M. Section 8 studies the combinatorics of parameterizing surfaces at the end of a band-taut hierarchy and proves a version of Theorem 10.7 for band-taut sutured manifolds. Section 6 reviews Gabai's construction of the branched surface associated to a sequence of sutured manifold decompositions and sets up the technology to prove that the arc β can sometimes be isotoped into the branched surface associated to a taut hierarchy.

In classical combinatorial sutured manifold theory, sutured manifold decompositions are usually constructed so that they "respect" a given parameterizing surface. The framework of "band taut sutured manifolds" requires that we have sutured manifold decompositions that respect each of two, not necessarily disjoint, parameterizing surfaces. Section 5 is devoted to explaining this mild generalization of the classical theory.

Sections 9 and 10 convert the main theorem for the theory of band taut sutured manifolds into theorems for arc-taut and nil-taut sutured manifolds. Section 11 gives the application to tunnel number one knots and links.

Acknowledgements. This paper has its roots in my Ph.D. dissertation [T1], although none of the present work appears there. I am grateful to Qilong Guo who found a gap in [T1], which lead to the development of the concept of "band taut sutured manifold". I am grateful to Marty Scharlemann for his encouragement and helpful comments. Thanks also to the referees for their careful reading and suggestions.

3. Sutured Manifolds

A sutured manifold (M,γ,β) consists of a compact orientable 3-manifold M, a collection of annuli $A(\gamma)\subset\partial M$ whose cores are oriented simple closed curves γ, a collection of torus components $T(\gamma)\subset\partial M$, and a 1-complex β properly embedded in M. Furthermore, the closure of $\partial M-(A(\gamma)\cup T(\gamma))$ is the disjoint union of two surfaces $R_-=R_-(\gamma)$ and $R_+=R_+(\gamma)$. Each component of $A(\gamma)$ is adjacent to both R_- and R_+. The surfaces R_- and R_+ are oriented so that if A is a component of $A(\gamma)$, then the curves $R_-\cap A$, $R_+\cap A$ and $\gamma\cap A$ are all non-empty and are mutually parallel as oriented curves. We denote the union of components of $A(\gamma)-\gamma$ adjacent to $R_\pm$ by $A_\pm$. We let $R(\gamma)=R_-\cup R_+$. We use $R_\pm$ to denote R_- or R_+.

The orientation on ∂R_+ gives an outward normal orientation to R_+ and the orientation on ∂R_- gives an inward normal orientation to R_-. We assign each edge of β an orientation with the stipulation that if an edge has an endpoint in $R_-\cup A_-$ then it is the initial endpoint of the edge and if an edge has an endpoint in $R_+\cup A_+$ then it is the terminal endpoint of the edge. We will only be considering 1-complexes β where this stipulation on the orientation of edges can be attained. (That is, no edge of β will have both endpoints in $R_\pm\cup A_\pm$.)

If (M,γ,β) is a sutured manifold and if $S\subset M$ is a connected surface in general position with respect to β, the **β-norm** of S is

$$x_\beta(S)=\max\{0,-\chi(S)+|S\cap\beta|\}.$$

If S is a disconnected surface in general position with respect to β, the β-norm is defined to be the sum of the β-norms of its components. The norm $x_\varnothing$ is called the **Thurston norm**.

The surface S is **β-minimizing** if, out of all embedded surfaces with the same boundary as S and representing $[S, \partial S]$ in $H_2(M, \partial S)$, the surface S has minimal β-norm. S is **β-taut** if it is β-incompressible (i.e. $S - \beta$ is incompressible in $M - \beta$), β-minimizing, and any given edge of β always intersects S with the same sign.

A sutured manifold is **β-taut** if:

(T0) β is disjoint from $A(\gamma) \cup T(\gamma)$.
(T1) M is β-irreducible
(T2) $R(\gamma)$ (equivalently R_- and R_+) and $T(\gamma)$ are β-taut.

If a 3–manifold or surface is $\varnothing$-taut, we say it is **taut in the Thurston norm** or sometimes, simply, **taut**.

The sutured manifold terminology up until now has been standard (see [S1]). Here is the central new idea of this paper. We say that a sutured manifold (M, γ, β) is **banded** if

(B1) there exists at most one edge $c_\beta \subset \beta$, having an endpoint in $A(\gamma)$. If $c_\beta \neq \varnothing$, one endpoint lies in A_- and the other lies in A_+. The edge c_β is called **the core**.

(B2) If $c_\beta \neq \varnothing$, then there exits a disc D_β, which we think of as an octagon, having its boundary divided into eight arcs, $c_1, c_2, \ldots, c_8$ (in cyclic order), called the **edges** of D_β. The arc c_β is contained in D_β and the interior of D_β is otherwise disjoint from β. We require that:
- c_1 and c_5 are properly embedded in $R_- - \partial\beta$,
- c_2 and c_6 each are properly embedded in $A(\gamma)$, intersect γ exactly once each, and each contains an endpoint of c_β,
- c_3 and c_7 are properly embedded in $R_+ - \partial\beta$,
- c_4 and c_8 each either traverse an edge of $\beta - c_\beta$ or are properly embedded in $A(\gamma)$ and intersect γ exactly once.

Define e_β to be the union of edges of $\beta - c_\beta$ that are traversed by $\partial D_\beta \cap (\beta - c_\beta)$. We have that $|e_\beta| \leq 2$. The disc D_β is called **the band** and the components of e_β are called the **sides** of the band. The sides of a band may lie on zero, one, or two edges of β. The arc $c_1 \cup c_2 \cup c_3$ is called the **top** of the band and the arc $c_5 \cup c_6 \cup c_7$ is called the **bottom** of the band. See Figure 2.

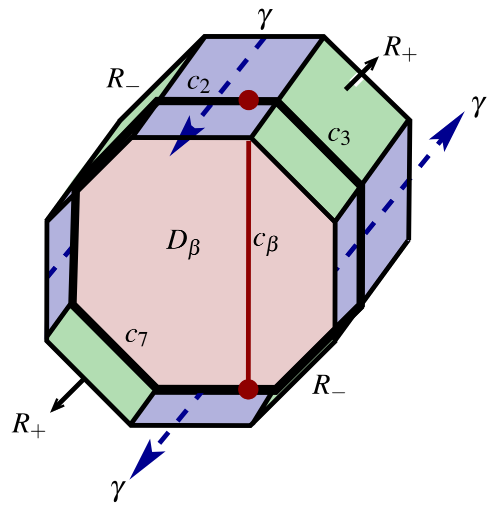

FIGURE 2. The red disc is a band with $e_\beta = \varnothing$. The green surfaces are subsurfaces of $R(\gamma)$ and the blue surfaces are subsurfaces of $A(\gamma)$. The edges of the band are labelled clockwise c_1 through c_8 with c_2 containing the top endpoint of the red arc c_β.

A banded sutured manifold (M, γ, β) is **band-taut** if $(M, \gamma, \beta - c_\beta)$ is $(\beta - c_\beta)$-taut.

Remark. The core of the band c_β is the arc we try to isotope onto the branched surface coming from a sutured manifold hierarchy. When building the hierarchy we will attempt to make each decomposing surface disjoint from c_β. The disc D_β helps to guide the isotopy of (parts of) c_β into the branched surface coming from a sutured manifold hierarchy. That the endpoints of c_β lie in $A(\gamma)$ allow us to use the surface $R(\gamma)$ to modify decomposing surfaces so as to give them algebraic intersection number zero with c_β. Because we want to appeal to as much of the sutured manifold theory developed in [S1] and [S2] as possible, we need ways of appealing to results about taut sutured manifolds. The sides of the band allow us to make use of these results.

4. DECOMPOSITIONS

In classical combinatorial sutured manifold theory, sutured manifolds are decomposed using so-called "conditioned" surfaces and a variety of "product surfaces". We review and expand the classical definitions and then discuss how the surfaces can give decompositions of band-taut sutured manifolds.

4.1. **Sutured manifold decompositions.**

4.1.1. *Decomposing surfaces.* If (M,γ,β) is a sutured manifold, a **decomposing surface** (cf. [S1, Definition 2.3]) is a properly embedded surface $S \subset M$ transverse to β such that:

(D1) ∂S intersects each component of $T(\gamma)$ in a (possibly empty) collection of coherently oriented circles.
(D2) ∂S intersects each component of $A(\gamma)$ in circles parallel to γ (and oriented in the same direction as γ), in essential arcs, or not at all.
(D3) Each circle component of $\partial S \cap A(\gamma)$ is disjoint from γ and no arc component of $\partial S \cap A(\gamma)$ intersects γ more than once.

If S is a decomposing surface, there is a standard way of placing a sutured manifold structure on $M' = M - \mathring{\eta}(S)$. The curves γ' are the oriented double curve sum of γ with ∂S. Let $\beta' = \beta \cap M'$. The sutured manifold (M',γ',β') is obtained by **decomposing** (M,γ,β) using S. We write $(M,\gamma,\beta) \overset{S}{\to} (M',\gamma',\beta')$. If (M,γ,β) is β-taut and if (M',γ',β') is β'-taut, then we say the decomposition is β**-taut**.

If the annuli and tori $A(\gamma) \cup T(\gamma)$ are not disjoint from β, we need to be more precise about the formation of the annuli $A(\gamma')$ in (M',γ',β'). We form annuli $A(\gamma') = \eta(\gamma')$ by demanding that $(A(\gamma) \cup T(\gamma)) \cap M'$ is a subset of $A(\gamma') \cup T(\gamma')$. This requirement ensures that any endpoint of β that lies in $A(\gamma) \cup T(\gamma)$ continues to lie in $A(\gamma') \cup T(\gamma')$. See Figure 3.

4.1.2. *Product Surfaces.* If $e \subset \beta$ is an edge with both endpoints in $R(\gamma)$, a **cancelling** disc for e is a disc properly embedded in $M - \mathring{\eta}(\beta)$ having boundary running once across e and once across $A(\gamma)$. See Figure 4. A **product disc** in a sutured manifold (M,γ,β) is a rectangle P properly embedded in M such that $P \cap \beta = \varnothing$ and $\partial P \cap A(\gamma)$ consists of two opposite edges of the rectangle each intersecting γ once transversally. Notice that the frontier of a regular neighborhood of a cancelling disc is a product disc. A product disc P is **allowable** if no component of $\partial P \cap R(\gamma)$ is β-inessential.

An **amalgamating disc** D in (M,γ,β) is a rectangle with two opposite edges lying on components of β that are edges joining R_- to R_+, one edge in R_+ and one edge in R_-. If ∂D traverses a single edge of β twice, it is a **self amalgamating disc**, otherwise it is a **nonself amalgamating disc**. A self amalgamating disc is **allowable** if both of the arcs $\partial D \cap \partial M$ are β-essential in $R(\gamma)$.

If in (M,γ,β) there is a cancelling disc D for e, we say that the sutured manifold $(M,\gamma,\beta - e)$ is obtained from (M,γ,β) by **cancelling** the arc e. If

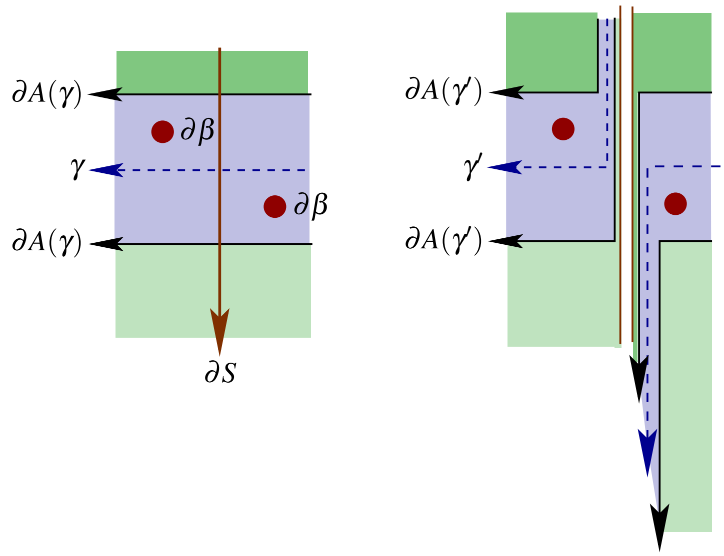

FIGURE 3. Creating sutures in $M - \mathring{\eta}(S)$.

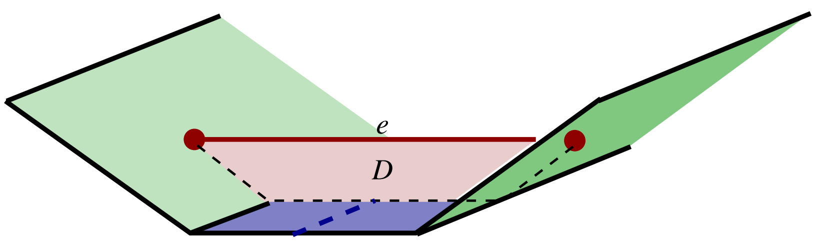

FIGURE 4. A cancelling disc D for an edge e

in (M,γ,β) there is a nonself amalgamating disc with boundary on components β_1 and β_2 of β, we say that the sutured manifolds $(M,\gamma,\beta-\beta_1)$ and $(M,\gamma,\beta-\beta_2)$ are obtained by **amalgamating the arcs** β_1 and β_2.

Lemma 4.3 of [S1] shows that if (M,γ,β) is β-taut, then after cancelling arc e, the sutured manifold is still $(\beta-e)$-taut. The converse is also easily proven. By [S1, Lemma 4.2], if (M,γ,β) is taut, then so is the sutured manifold obtained by decomposing along a product disc in (M,γ,β). By [S1, Lemma 4.3 and Lemma 4.4], if $(M,\gamma,\beta-\beta_1)$ is obtained by amalgamating arcs β_1 and β_2 in the β-taut sutured manifold (M,γ,β), then $(M,\gamma,\beta-\beta_1)$ is $(\beta-\beta_1)$-taut. Later we will review a method for eliminating self amalgamating discs so that tautness is preserved.

A **product annulus** P is an annulus properly embedded in M that is disjoint from β and that has one boundary component in R_- and the other in R_+.

(See [S1, Definition 4.1].) A product annulus is **allowable** if P is not the frontier of a regular neighborhood of an arc in M (this is the same as being "non-trivial" in the sense of [S1, Definition 4.1]). Notice that attaching the two edges of a self-amalgamating disc lying on β produces a product annulus.

4.1.3. *Conditioned and rinsed surfaces.* In addition to decomposing sutured manifolds along product surfaces, we will also need to decompose along more complicated surfaces. We require such surfaces to be "conditioned" [S1, Definition 2.4]. A **conditioned** 1–manifold $C \subset \partial M$ is an embedded oriented 1–manifold satisfying:

- (C0) All circle components of C lying in the same component of $A(\gamma) \cup T(\gamma)$ are oriented in the same direction, and if they lie in $A(\gamma)$, they are oriented in the same direction as the adjacent component of γ.
- (C1) All arcs of $C \cap A(\gamma)$ in any annulus component of $A(\gamma)$ are oriented in the same direction.
- (C2) No collection of simple closed curves of $C \cap R(\gamma)$ is trivial in $H_1(R(\gamma), \partial R(\gamma))$.

Notice that if $z \in H_1(\partial M)$ is non-trivial, then there is a conditioned 1–manifold in M representing z. Furthermore, if C is a conditioned 1–manifold then the oriented double curve sum of C with $\partial R(\gamma)$ is also conditioned.

A decomposing surface $S \subset M$ is **conditioned** if ∂S is conditioned and if, additionally,

- (C3) Each edge of β intersects $S \cup R(\gamma)$ always with the same sign.

A surface S in a banded 3-manifold (M, γ, β) is **rinsed** if S is conditioned in $(M, \gamma, \beta - c_\beta)$, if S has zero algebraic intersection with c_β, and if every separating closed component of S bounds with a closed component of $R(\gamma)$ a product region intersecting β in vertical arcs.

4.2. **Band-taut decompositions.** An arbitrary decomposition of a banded sutured manifold may not create a banded sutured manifold. In this section, we show how certain surfaces can be used to usefully decompose band-taut sutured manifolds.

The easiest instance of such a decomposition is if (M, γ, β) is a banded sutured manifold and if E is a cancelling disc with interior disjoint from D_β for a component β_1 of e_β. Let E' be the product disc in M that is the frontier of a regular neighborhood of E. The disc E' intersects D_β in either one or two arcs. Those arcs join the top of D_β the bottom of D_β. If there are two arcs (which happens if c_4 and c_8 run along the same edge of β), one arc joins c_1 to c_7 and the other joins c_3 to c_5. If there is a single arc, it either joins c_1 to c_7 or joins c_3 to c_5. Let (M', γ', β') be the result of decomposing (M, γ, β) using E'. The disc D_β is decomposed into 2 or 3 discs, one of which D'_β contains $c_\beta = c_{\beta'}$. The disc D'_β is clearly a band and $|e'_\beta| < |e_\beta|$. In effect, we have cancelled an edge of e_β and the new band runs along a suture instead of along the edge. We call E' a **band-decomposing product disc**. Since cancelling an edge and decomposing along a product disc disjoint from D_β preserve tautness, decomposing a band-taut sutured manifold along either a band-decomposing disc or a product disc disjoint from D_β preserves band-tautness.

We also need ways of decomposing along other product surfaces or conditioned surfaces in ways that preserve band-tautness. To that end, suppose that a decomposing surface S in a banded sutured manifold (M, γ, β) is transverse to D_β. We say that S is a **band-decomposing surface** if it is either a band-decomposing product disc, a product disc disjoint from D_β or if it satisfies:

(BD) Either $e_\beta = \varnothing$ and c_β is isotopic in D_β relative to its endpoints into ∂M or all of the following are true:
  (1) there exists a properly embedded arc c in D_β joining the top of D_β to the bottom of D_β that is disjoint from S
  (2) each point of the intersection between ∂S and the top of D_β has the same sign as the sign of intersection between γ and c_2.
  (3) each point of the intersection between ∂S and the bottom of D_β has the same sign as the sign of intersection between γ and c_6.
  (4) each point of the intersection between ∂S and c_4 has the same sign.
  (5) each point of the intersection between ∂S and c_8 has the same sign.

Suppose that $(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$ is a $(\beta - c_\beta)$-taut sutured manifold decomposition where (M, γ, β) is a banded sutured manifold and S is a band-decomposing surface. We say that the decomposition is **band-taut** if (M, γ, β) and (M', γ', β') are each band-taut and one of the following holds:

(BT1) $e_\beta = \varnothing$, c_β is isotopic (relative to its endpoints) in D_β into ∂M and $D_{\beta'} = c_{\beta'} = \varnothing$, or

(BT2) $c_{\beta'}$ is a properly embedded arc in $D_\beta - S$, such that the initial endpoint of $c_{\beta'}$ lies in $A_-(\gamma') \cap \partial D_\beta$ and the terminal endpoint of $c_{\beta'}$

lies in $A_+(\gamma') \cap \partial D_\beta$ and $D_{\beta'}$ is the component of $D_\beta \cap M'$ containing $c_{\beta'}$.

Lemma 4.1. *Suppose that (M, γ, β) is band-taut with $D_\beta \neq \varnothing$. If a band-decomposing surface S satisfying (BD) has been isotoped relative to ∂S so as to minimize $|S \cap D_\beta|$ then every component of $S \cap D_\beta$ is an arc joining either the top or bottom of D_β to c_4 or c_8.*

Proof. That no component of $S \cap D_\beta$ is a circle follows from an innermost disc argument. By condition (1) of (BD), no arc component joins c_4 to c_8. By conditions (4) and (5) of (BD), no arc component joins c_4 to itself or joins c_8 to itself. Since $|S \cap D_\beta|$ is minimized and since S is a decomposing surface, no arc of $S \cap D_\beta$ has both endpoints on the same edge of D_β. By conditions (2) and (3), no arc joins the top of D_β to itself and no arc joins the bottom of D_β to itself. We need only show, therefore, that each arc joins the top or bottom of D_β to c_4 or c_8.

Due to the orientations of R_- and R_+, the orientations of γ at $\gamma \cap c_2$ and $\gamma \cap c_6$ point in the same direction. Suppose that ζ is an arc of $S \cap D_\beta$ joining the top of D_β to the bottom of D_β. The orientation of S induces a normal orientation of ζ in D_β. The normal orientation of S induces orientations of the endpoints of ζ that are normal to D_β and point in opposite directions. This violates either condition (2) or (3) of (BD). Hence, no arc of $S \cap D_\beta$ joins the top of D_β to the bottom of D_β. $\square$

Lemma 4.2. *Suppose that (M, γ, β) is a band-taut sutured manifold and that S is a band decomposing surface satisfying (BD) and defining a taut decomposition $(M, \gamma, \beta - c_\beta) \xrightarrow{S} (M', \gamma', \beta' - c_\beta)$. Then after an isotopy of S (rel ∂S) to minimize $|S \cap D_\beta|$, there are $D_{\beta'} \subset D_\beta$ and $c_{\beta'} \subset D_{\beta'}$ so that the decomposition is band-taut. Furthermore, if (BT1) does not hold, each component of $(D_\beta \cap M') - D_{\beta'}$ is a product disc or cancelling disc.*

Proof. If $e_\beta = \varnothing$ and if c_β is isotopic into ∂M in D_β, define $D_{\beta'} = c_{\beta'} = \varnothing$. Assume, therefore, that (BT1) does not hold.

Let d_T and d_B be the top and bottom of D_β respectively. Since all points of intersection of S with d_T have the same sign as the intersection of γ with c_2, each component of $d_T - S$ contains exactly one point of $\gamma' \cap D_\beta$. Similarly, each component of $d_B - S$ contains exactly one point of $\gamma' \cap D_\beta$. By Lemma 4.1, every component of $S \cap D_\beta$ joins the top or bottom of D_β to c_4 or c_8. This implies both that each component of $e_\beta - S$ has an endpoint in both $R_-(\gamma')$ and $R_+(\gamma')$ and that there is exactly one component $D_{\beta'}$ of $D_\beta \cap M'$ containing both a point of d_T and a point of d_B. It is not hard to see that $D_{\beta'}$

is a band containing an arc c'_β satisfying the definition of core. See Figures 5 and 6.

Similarly, each component of $D_\beta - S$ other than $D_{\beta'}$ intersects $c_4 \cup c_8$ in at most one arc. If such a component does intersect $(c_4 \cup c_8)$ it is a product disc or cancelling disc (depending on whether or not c_4 or c_8 lies in e_β). If such a component does not intersect $(c_4 \cup c_8)$ then it is adjacent to exactly two arcs of $S \cap D_\beta$ and so is a product disc in M'. □

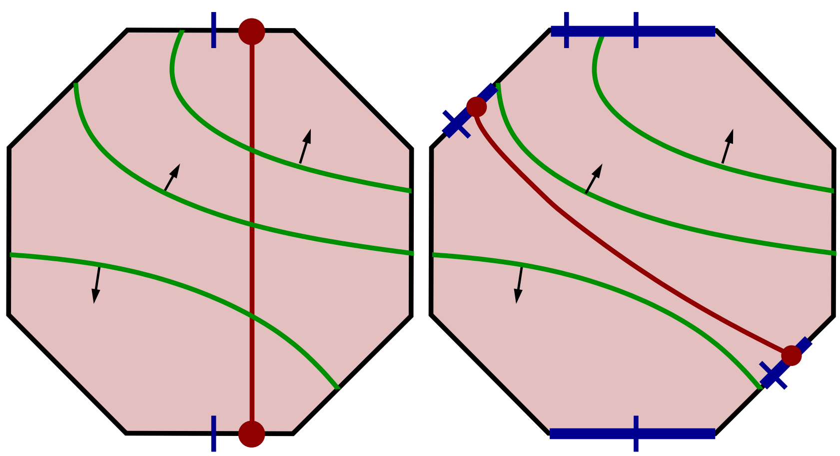

FIGURE 5. The left image shows the core c_β and the right image shows the core $c_{\beta'}$. The core c_β can be isotoped to an arc $c_{\beta'}$ disjoint from S and with endpoints in $A(\gamma')$. The sutures γ' are marked on $\partial D'_\beta$ in the rightmost figure. The endpoint of c_β lying in $A_\pm$ is isotoped to an endpoint of $c_{\beta'}$ lying in $A_\pm(\gamma')$. In the rightmost figure, the intersection of the annuli $A_\pm(\gamma')$ with ∂D_β are highlighted in blue.

We note the following:

Lemma 4.3. *Suppose that $(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$ is a band-taut decomposition. Then there is an isotopy of c_β relative to its endpoints to an arc c such that the closure of $c \cap \mathring{M}'$ is $c_{\beta'}$. Furthermore, $c_{\beta'}$ joins the components of $c_\beta \cap \partial D_{\beta'}$.*

Proof. By the definition of band-taut decomposition, S is a band-decomposing surface. If the decomposition satisfies (BT1), then by definition, there is an

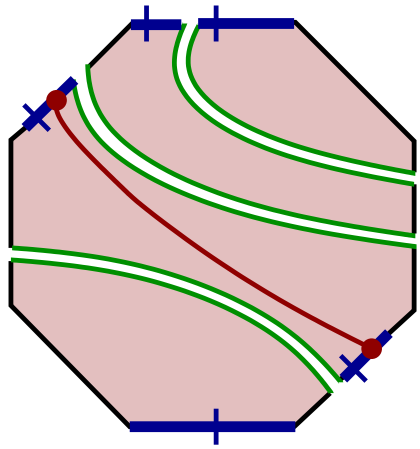

FIGURE 6. The band $D_{\beta'}$ is the component of $D_\beta \cap M'$ containing $c_{\beta'}$.

isotopy of c_β into ∂M and we have our conclusion. Suppose, therefore, that the decomposition satisfies (BT2).

If the decomposition is by a band decomposing disc or a product disc disjoint from the band then no isotopy of c_β is necessary as $c_{\beta'} = c_\beta$. If the decomposition is by a surface satisfying (BD), this follows immediately from the observation that c_β is isotopic to $c_{\beta'}$ by a proper isotopy in D_β that does not move the endpoints of c_β along edges c_4 or c_8 of D_β. See Figure 7. □

4.3. **Cancelling discs, amalgamating discs, product discs and product annuli.** The previous section provided a some criteria for creating decompositions of banded sutured manifolds using surfaces that satisfy (BD). Product surfaces, however, may not satisfy (BD). This section shows how to create a band-taut decomposition if there is a cancelling disc, amalgamating disc, product disc, or product annulus in a band-taut sutured manifold.

4.3.1. *Finding disjoint product surfaces.* We begin by showing that if there is a cancelling disc, amalgamating disc, product disc, or product annulus in a band-taut sutured manifold, then there is one disjoint from the band.

Lemma 4.4. *Suppose that (M, γ, β) is band-taut. If $\beta - c_\beta$ has a*

(1) cancelling disc or allowable product disc;
(2) nonself amalgamating disc; or

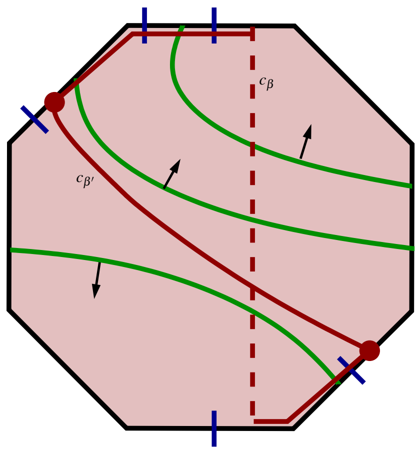

FIGURE 7. The isotopy of c_β to $c_{\beta'}$ can be slightly modified to an isotopy of c_β relative to its endpoints such that after the isotopy $c_\beta - c_{\beta'}$ lies in $\partial D_\beta \cap \partial M$. The solid arc is the union of two subarcs in ∂D_β and the arc $c_{\beta'}$.

(3) allowable self amalgamating disc

then one of the following occurs:

- *there is, respectively, a*
  (1) cancelling disc or allowable product disc;
  (2) cancelling disc, allowable product disc, or nonself amalgamating disc; or
  (3) cancelling disc, allowable product disc, nonself amalgamating disc, or allowable self-amalgamating disc
  that is disjoint from D_β, or
- *$e_\beta = \varnothing$, the boundary of D_β is a $(\beta - c_\beta)$-inessential circle in ∂M, and c_β is isotopic in D_β into ∂M (rel ∂c_β).*

Proof. The proofs with each of the three hypotheses are nearly identical, so we prove it only under hypothesis (3). Let E be a cancelling disc, allowable product disc, nonself amalgamating disc, or allowable self amalgamating disc chosen so that out of all such discs, D_β and E intersect minimally. By an isotopy of E, we can assume that all intersections between ∂D_β and ∂E occur in $R(\gamma)$. An innermost circle argument shows that there are no circles of intersection between D_β and E. Similarly, we may assume that if a component of $D_\beta \cap E$ intersects c_β then it is an arc in D_β joining distinct edges of D_β.

Claim 1: No component of $D_\beta \cap E$ joins an edge of D_β to itself.

Suppose that there is such a component, and let ξ be an outermost such arc in D_β with Δ the disc it cobounds with a subarc of $D_\beta \cap R(\gamma)$. Boundary compress E using Δ to obtain two discs E_1 and E_2. Since E was a cancelling disc, product disc, or amalgamating disc, one of E_1 or E_2 is a cancelling disc, product disc, or amalgamating disc and the other one is a disc with boundary completely contained in $R(\gamma)$. Suppose that E_2 is this latter disc. Since E_2 is disjoint from $(\beta - c_\beta)$ and since $R(\gamma)$ is $(\beta - c_\beta)$-incompressible, the boundary of E_2 is β-inessential in $R(\gamma)$. Thus, E can be isotoped in the complement of β to E_1. This isotopy reduces $|D_\beta \cap E|$ and so we have contradicted our choice of E.

Claim 2: No arc component of $D_\beta \cap E$ joins edge c_1 to edge c_3 or edge c_5 to edge c_7.

Suppose that there is such an arc. Without loss of generality, we may assume that the arc joins side c_1 to c_3. Let ξ be an arc that, out of all such arcs, is closest to c_2. Let Δ be the disc in D_β that it cobounds with c_2. Boundary-compress E using Δ to obtain the union E' of two discs. In E, the arc ξ joins $\partial E \cap R_-$ to $\partial E \cap R_+$. If E is a cancelling disc, one component of E' is a cancelling disc and the other is a product disc. If E is a product disc, both components of E' are product discs. If E is an amalgamating disc, both components of E' are cancelling discs. Each component of E' intersects D_β fewer times than does E, so we need only show that if E is an allowable product disc, then at least one component of E' is an allowable product disc.

Assume that E is an allowable product disc. This implies that it is a $(\beta - c_\beta)$-boundary compressing disc for M. Since E' is obtained by boundary compressing E, at least one component E_1 of E' is a $(\beta - c_\beta)$-boundary compressing disc for ∂M. If it were not allowable, it could be isotoped in the complement of $(\beta - c_\beta)$ to have boundary lying entirely in $R(\gamma)$, this would contradict the fact that $R(\gamma)$ is $(\beta - c_\beta)$-incompressible. Thus, E_1 is an allowable product disc.

Claim 3: No arc component of $D_\beta \cap E$ joins edge c_1 to edge c_7 or edge c_3 to edge c_5.

Suppose that there is such an arc. Without loss of generality, we may assume that it joins edges c_1 and c_7. Out of all such arcs, choose one ξ that is as close as possible to edge c_8. Boundary compress E using the subdisc of D_β cobounded by c_8 and ξ to obtain E'.

If E is a cancelling disc, let E_2 be the component of E' containing c_8 and let E_1 be the other component. If c_8 is an edge of β, then E_2 is an amalgamating disc and E_1 is a cancelling disc. In this case, note that E_1 (after a small isotopy to be transverse to D_β) intersects D_β fewer times than does E. This contradicts our choice of E. If c_8 is not an edge of β, then E_2 is a cancelling disc and E_1 is a product disc. If E_1 is not allowable, it can be isotoped in the complement of $(\beta - c_\beta)$ to have boundary contained entirely in $R(\gamma)$. Since $R(\gamma)$ is $(\beta - c_\beta)$-incompressible, the boundary of this disc must be $(\beta - c_\beta)$-inessential in $R(\gamma)$. It is then easy to see that there is an isotopy reducing the number of intersections between E and D_β, a contradiction.

If E is an allowable product disc, then each component of E' is either a cancelling disc or a product disc. As before, if a component of E' is a product disc, it must be allowable.

If E is an amalgamating disc, then each component of E' is either an amalgamating disc or a cancelling disc. Let E_1 and E_2 be the components of E'. Suppose that each of E_1 or E_2 is a self amalgamating disc. We must show that at least one of them is allowable. Since Δ runs along c_8, each component of E' runs along c_8 at least once. Since each component of E' is a self amalgamating disc, this implies that E is a self-amalgamating disc for c_8. By hypothesis, E is allowable. Thus, at least one loop of $E' \cap R_-$ is $(\beta - c_\beta)$-essential and at least one loop of $E' \cap R_+$ is $(\beta - c_\beta)$-essential. If the union of these loops is either $\partial E_1 \cap R(\gamma)$ or $\partial E_2 \cap R(\gamma)$, then either E_1 or E_2 is allowable. Thus, we may assume that one arc of $E_i \cap R(\gamma)$ is $(\beta - c_\beta)$-essential and the other one is inessential for both $i = 1$ and $i = 2$. Gluing the two arcs of $\partial E_i \cap c_8$ together we obtain a product annulus with one end $(\beta - c_\beta)$-essential and the other end $(\beta - c_\beta)$-essential. Capping the inessential end off, creates a disc disjoint from $(\beta - c_\beta)$ with boundary a $(\beta - c_\beta)$-essential loop in $R(\gamma)$. That is, the disc is a $(\beta - c_\beta)$ compressing disc for $R(\gamma)$, a contradiction. Hence, if both E_1 and E_2 are self-amalgamating discs at least one of them is allowable. Since each of E_1 and E_2 intersects D_β fewer times than does E, we have contradicted our choice of E.

Claim 4: No arc component of $D_\beta \cap E$ joins side c_3 to side c_7 or side c_1 to side c_5.

Suppose that there is such an arc. Without loss of generality, we may assume that the arc joins side c_3 to side c_7. Out of all such arcs choose one ξ that is outermost on E. Boundary compress D_β using the outermost disc in E bounded by ξ. This converts D_β into two discs, D_4 and D_8 containing c_4 and c_8 respectively. The disc D_4 also contains the edge c_6 and the disc D_8 also contains the edge c_1. Since both c_2 and c_6 are contained in $A(\gamma)$, the discs D_4 and D_8 are either cancelling discs or product discs. We must show that if they are both product discs, then at least one of them is allowable.

Assume to the contrary, that both D_4 and D_8 are product discs that are not allowable. Since product discs that are not allowable can be isotoped in the complement of $(\beta - c_\beta)$ to have boundary lying in $R(\gamma)$, and since $R(\gamma)$ is $(\beta - c_\beta)$-incompressible, both D_4 and D_8 are discs with $(\beta - c_\beta)$-inessential boundary in ∂M. The disc D_β can be reconstructed by banding the discs D_4 and D_8 together using an arc lying entirely in R_+. Since M is $(\beta - c_\beta)$-irreducible and since c_β lies in D_β, the arc c_β is isotopic in in D_β into ∂M relative to its endpoints. □

We now study how cancelling discs, amalgamating discs, product discs, and product annuli give band-taut sutured manifold decompositions.

4.3.2. *Eliminating cancelling discs, product discs, and nonself amalgamating discs.*

Lemma 4.5. *Suppose that (M,γ,β) is a connected band taut sutured manifold other than a 3-ball containing a single suture in its boundary and a single arc of $\beta - c_\beta$. If M contains a cancelling disc or allowable product disc, then there is a band-taut decompositions*

$$(M,\gamma,\beta) \xrightarrow{S} (M',\gamma',\beta')$$

The decomposition is a decomposition by a product disc, possibly satisfying (BT1).

Proof. If $e_\beta = \varnothing$ and if c_β is parallel into ∂M along D_β, let $D_{\beta'} = c_{\beta'} = \varnothing$. The decomposition by the given cancelling disc or allowable product disc is then band-taut. Thus, by Lemma 4.4, we may assume that there is a cancelling disc or allowable product disc P that is disjoint from D_β. If P is a cancelling disc, let S be the frontier of a regular neighborhood of P in M. Notice that S is an allowable product disc since M is not a 3-ball with a single suture in its boundary and a single arc in $\beta - c_\beta$. If P was a cancelling disc for a component of e_β, then S is a band-decomposing product disc. If P is an allowable product disc, let $S = P$. Decomposing a taut sutured manifold using a product disc is a taut decomposition, so it is evident that the decomposition $(M,\gamma,\beta) \xrightarrow{S} (M',\gamma',\beta')$ is band-taut. □

Notice that if (M,γ,β) has a cancelling disc for an edge $e \subset \beta$, then the decomposition given by Lemma 4.5 cuts off from M a 3–ball having a single suture in its boundary and containing the edge e and the cancelling disc. We then cancel the arc e. (The reason for decomposing along S is that at the end of the hierarchy we will want to ignore all arc cancellations. Decomposing along the product disc S before cancelling ensures that the cancellable arc is in its own component of the sutured manifold at the end of the hierarchy.)

As a final remark in this subsection, we note that if (M, γ, β) is a band-taut sutured manifold and if there is a nonself amalgamating disc P disjoint from D_β we can eliminate a component of $P \cap (\beta - e_\beta)$ from β and preserve band tautness. If P runs across two components of e_β, then we view the elimination of one of the components of e_β as a melding together of the two sides of D_β. That is, the band D_β is isotoped so that both sides run across the same component of e_β and the other component of e_β is eliminated. Thus, amalgamating arcs preserves band-tautness.

4.3.3. *Eliminating allowable self amalgamating discs.* In the construction of a band-taut hierarchy, it will be necessary to eliminate allowable self amalgamating discs as in [S2, Lemma 2.4]. We briefly recall the essentials.

We make use of a trick which allows us to convert between arcs and sutures [S2, Definition 2.2]. If $e \subset \beta$ is an edge with one endpoint in R_- and one in R_+, then we **convert** e to a suture γ_e by letting $M' = M - \mathring{\eta}(e)$ and letting $\gamma_e \subset \partial M'$ be a meridian of e. Lemma 2.2 of [S2] shows that (M, γ, β) is β-taut if and only if $(M', \gamma \cup \gamma_e, \beta - e)$ is $(\beta - e)$-taut.

Suppose that P is an allowable self amalgamating disc in (M, γ, β) disjoint from D_β. Gluing the components of $\partial P \cap \beta$ together along the edge of β they traverse and isotoping it off β creates a product annulus P_A. Notice that since P was disjoint from D_β, the annulus P_A can be created so that it is disjoint from D_β. Furthermore, if a boundary component of P_A is inessential in $R(\gamma)$, the disc in $R(\gamma)$ it bounds is also disjoint from D_β.

If both components of ∂P_A are essential in $R(\gamma)$, we decompose along P_A. The parallelism of the edge $P \cap \beta$ into P_A becomes a cancelling disc in the decomposed manifold and we cancel the arc $P \cap \beta$ as in Subsection 4.3.2.

If a component of ∂P_A is inessential in $R(\gamma)$, we choose one such component δ and let Δ be the disc in $R(\gamma)$ that it bounds. Let D be the pushoff of $P_A \cup \Delta$ so that it is properly embedded. We decompose along D. As described in [S2, Lemma 2.4], after amalgamating arcs and converting an arc to a suture, the decomposed sutured manifold is equivalent to the sutured manifold obtained by decomposing along P_A. By [S1, Lemma 4.2] and [S2, Lemmas 2.3 and 2.4], if (M, γ, β) is band taut, so is the decomposed sutured manifold.

4.4. **Decomposing by rinsed surfaces.** In the previous sections, we have seen how surfaces satisfying (BD) can be used to construct decompositions of banded sutured manifolds and how the presence of a product surface can be used to construct a band-taut decomposition of a band taut sutured manifold. In this section we show that, in the presence of non-trivial second homology, we can find a rinsed surface giving a band-taut decomposition of a band-taut sutured manifold. We begin with some preliminary lemmas that simplify the search for such a surface.

Lemma 4.6. *Suppose that S is a conditioned or rinsed surface in (M,γ,β). Then the surface S_k obtained by double curve summing S with k copies of $R(\gamma)$ for any $k \geq 0$, is conditioned or rinsed, respectively. Furthermore, if S is rinsed and satisfies conditions (2) and (3) of (BD) in the definition of band decomposing surface, then S_k does also.*

Proof. By induction, it suffices to prove the lemma when $k = 1$. We have already observed that ∂S_k is conditioned. Since S satisfies (C3), it is obvious that S_k does also.

If S is rinsed, then the algebraic intersection number of S with c_β is zero. Since $R(\gamma)$ is disjoint from c_β, the surface S_k also has this property. Suppose that F is a closed component of S_k (with $k = 1$). Since S satisfies condition (C2), no component of S_k is a separating closed surface intersecting S. Any closed component of S_k must, therefore, be parallel to a component of $R(\gamma)$ and bounds a region of parallelism intersecting β only in vertical arcs. Consequently, if S is rinsed, so is S_k. Finally, if S satisfies conditions (2) and (3) of (BD), it follows immediately from (C0) and (C1) that S_k also satisfies (2) and (3) of (BD). □

Lemma 4.7. *Suppose that (M,γ,β) is band-taut and that S is a rinsed surface satisfying conditions (2) and (3) of (BD) in the definition of band-decomposing surface. Then after an isotopy relative to ∂S to minimize the pair $(|S\cap D_\beta|, |S\cap c_\beta|)$ with respect to lexicographic order, the surface S is a band decomposing surface satisfying (BD).*

Proof. If c_4 or c_8 lies in $A(\gamma)$, since ∂S is conditioned, condition (4) or (5) of (BD) is satisfied for that component. If c_4 or c_8 lies in e_β, then by condition (C3) in the definition of rinsed, condition (4) or (5) of (BD) is satisfied for that component. Thus, we need only show that S satisfies condition (1) of (BD).

Each arc of $S\cap D_\beta$ intersects c_β at most once, by our initial isotopy of S. Suppose that a component ζ of $S\cap D_\beta$ joins c_4 to c_8. Since S has algebraic intersection number zero with c_β, there exists another arc ζ' intersecting

c_β but with opposite sign. By conditions (2) and (3) of (BD), at least one endpoint of ζ' must lie on c_4 or c_8. Without loss of generality, assume it to be c_4. Since S always intersects c_4 with the same sign, ζ and ζ' intersect c_4 with the same sign. Since the signs of intersection of each of ζ and ζ' with c_β are the same or opposite of their intersections with c_4, and since they intersect c_4 with the same sign, ζ and ζ' intersect c_β with the same sign. This contradicts the choice of ζ'. Hence no arc joins c_4 to c_8. Thus, every arc joins either the top or bottom of D_β to either c_4 or c_8. A similar argument shows that if ζ and ζ' are arcs each with an endpoint on c_8 (or each with an endpoint on c_4) and each intersecting c_β then they intersect c_β with the same sign. It follows that if $S \cap c_\beta$ is non-empty, then precisely one of the following happens:

(1) There are equal numbers of arcs joining c_5 to c_8 as there are arcs joining c_1 to c_4 and there are no other arcs.
(2) There are equal numbers of arcs joining c_3 to c_8 as there are arcs joining c_7 to c_4 and there are no other arcs.

It follows that conclusion (1) of (BD) holds and so S is a band-decomposing surface. □

The previous two lemmas produce a rinsed band decomposing surface from a given rinsed surface. The next lemma produces a rinsed band decomposing surface from a given homology class.

Lemma 4.8. *Suppose that (M, γ, β) is a band-taut sutured manifold and that $y \in H_2(M, \partial M)$ is non-zero. Then there exists a rinsed band-decomposing surface S in M representing $\pm y$.*

Proof. Let C be a conditioned 1–manifold in ∂M representing ∂y. Isotope C so that

(a) Each circle component of $C \cap A(\gamma)$ is contained in a collar of $\partial R(\gamma)$ that is disjoint from ∂c_β
(b) Each arc component of $C \cap A(\gamma)$ is disjoint from $\partial D_\beta \cap A(\gamma)$.

Let Σ be a surface representing y with $\partial \Sigma = C$. Discard any separating closed component of Σ. The surface Σ is a decomposing surface. We now proceed to modify it to obtain the surface we want. We begin by arranging for the surface to have non-positive algebraic intersection number with c_β.

Let i be the algebraic intersection number of Σ with c_β. If $i > 0$, let $\overline{\Sigma}$ be the result of reversing the orientation of Σ and let $\overline{C} = \partial \overline{\Sigma}$. Notice that Σ

represents $-y$ and that the algebraic intersection between $\overline{\Sigma}$ and c_β is non-positive. If $\overline{C}$ is not conditioned, perform cut and paste operations of $\overline{\Sigma}$ with copies of subsurfaces of $R(\gamma)$ to produce a surface Σ' having conditioned boundary C' and satisfying (a) and (b). Since $R(\gamma)$ is disjoint from c_β, the algebraic (and geometric) intersection number of Σ' with c_β is the same as the algebraic (and geometric) intersection number of $\overline{\Sigma}$ with c_β. This number is, therefore, negative.

We may, therefore, assume without loss of generality that we have a surface Σ such that $C = \partial\Sigma$ satisfies (a) and (b), and:

(c) Σ is a conditioned surface representing $\pm y$
(d) The algebraic intersection number i of Σ with c_β is non-positive.

By Lemma 4.6 and the fact that $R(\gamma)$ is disjoint from c_β, replacing Σ with the double curve sum Σ_k of Σ with k copies of $R(\gamma)$ does not change (a), (b), (c), or (d).

We now explain why we may also assume that Σ satisfies conditions (2) and (3) of (BD). Suppose that $\partial\Sigma$ intersects the top of D_β in points of opposite intersection number. At least one of those points must lie in c_1 or c_3. After possibly increasing k and isotoping a circle component of $\Sigma \cap A(\gamma)$ into $R(\gamma)$ (not allowing it to pass through c_β) we may band together points of $\partial\Sigma \cap c_1$ or $\partial\Sigma \cap c_3$ having opposite intersection number to guarantee that all points of $\Sigma \cap (c_1 \cup c_2 \cup c_3)$ have the same intersection number as $\gamma \cap c_2$. Perform any additional necessary cut and paste operations with subsurfaces of $R(\gamma)$ to ensure that Σ is conditioned. Since $R(\gamma)$ is disjoint from c_β, this does not change i. Since each intersection point of $\partial R(\gamma)$ with the top of D_β has the same sign as $\gamma \cap c_2$, we still have the property that $\partial\Sigma$ intersects the top of D_β with the same sign as $\gamma \cap c_2$. Similarly, we can guarantee that $\partial\Sigma$ also always intersects the bottom of D_β with the same sign as $\gamma \cap c_6$. This implies that we may assume that Σ and Σ_k (for $k \geq 1$) satisfies conditions (2) and (3) of (BD) in the definition of band-decomposing surface. By tubing together points of opposite intersection number, we may also assume that the geometric intersection number of Σ and Σ_k with each edge of β is equal to the algebraic intersection number. From Σ discard every closed separating component that does not bound a product region with $R(\gamma)$ intersecting β in vertical arcs. Thus, by Lemma 4.7, we may assume that Σ and Σ_k (for $k \geq 0$), in addition to satisfying (a) - (d), satisfy every requirement for being a rinsed band decomposing surface except that the algebraic intersection number of Σ and Σ_k with c_β may possibly be negative. We now show how to trade (a) for the property that Σ has algebraic intersection number 0 with c_β. We will then have proved our lemma.

Let $\rho_\pm$ be the surface $R_\pm \cup (A_\pm - \mathring{\eta}(\gamma))$. Let S be the surface obtained by taking the double curve sum of Σ_k with i copies of ρ_-. Since A_- has intersection with c_β consisting of a single point with sign $+1$, S has zero algebraic intersection number with c_β. Tube together points with opposite intersection number in the intersection of S with each component of $\beta - c_\beta$. Discard any closed separating component. Isotope S slightly so that all circle components of $S \cap A(\gamma)$ are disjoint from γ. After discarding any closed separating components of S, we have constructed a rinsed band-decomposing surface representing $\pm y$. □

Our next two results, which are based on [S1, Theorems 2.5 and 2.6], are the key to constructing band-taut decompositions. As usual, we let S_k denote the oriented double curve sum of S with k copies of $R(\gamma)$. Recall that S and S_k (for any $k \geq 0$) represent the same class in $H_2(M, \partial M)$. For reference, we begin by stating [S1, Theorem 2.5]. (As this is an important theorem for us, we note that the surface R in Scharlemann's theorem need not be $R(\gamma)$. Indeed, the statement of the theorem does not mention sutured manifolds.)

Theorem (Theorem 2.5 of [S1])**.** *Given:*

(a) A β-taut surface $(R, \partial R)$ in a β-irreducible 3–manifold $(M, \partial M)$
(b) a properly embedded family C of oriented arcs and circles in $\partial M - \eta(\partial R)$ which is in the kernel of the map

$$H_1(\partial M, \eta(\partial R)) \to H_1(M, \eta(\partial R))$$

induced by inclusion.
(c) y in $H_2(M, \partial M)$ such that $\partial y = [C]$,

then there is a surface $(S, \partial S)$ in $(M, \partial M)$ such that

(i) $\partial S - \eta(\partial R) = C$
(ii) for some integer m, $[S, \partial S] = y + m[R, \partial R]$ in $H_2(M, \partial M)$,
(iii) for any collection R' of parallel copies of components of R (similarly oriented), the double curve sum of S with R' is β-taut,
(iv) any edge of β which intersects both R and S intersects them with the same sign.

Notice that conclusion (iv) actually follows immediately from conclusion (iii).

Theorem 4.9. *Suppose that (M, γ, β) is a band taut sutured manifold and that $y \in H_2(M, \partial M)$ is non-zero. Then there exists a rinsed band-decomposing surface $(S, \partial S) \subset (M, \partial M)$ satisfying (BD) such that:*

(i) S represents $\pm y$ in $H_2(M, \partial M)$.

(ii) $[S, \partial S] = \pm y + k[R(\gamma), \partial R(\gamma)]$ *for some* $k \geq 0$.

(iii) for any collection R' *of parallel copies of components of* $R(\gamma)$*, the double curve sum of* S *with* R' *is* $(\beta - c_\beta)$*-taut.*

Proof. Let Σ be the rinsed band-decomposing surface obtained from Lemma 4.8. Let S be the surface given by [S1, Theorem 2.5]. (To apply it we let $\beta - c_\beta$ be the 1–complex in the hypothesis of that theorem and $R = R(\gamma)$ and $C = \partial\Sigma - \mathring{\eta}(\partial R)$.)

Discard any closed separating component of S and isotope S relative to its boundary to minimize $|S \cap D_\beta|$. Our conclusions (i) - (iiii) coincide with conclusions (i) - (iii) of Scharlemann's theorem. Since Σ and S are homologous in $H_2(M, \partial\Sigma \cup \eta(\partial R(\gamma))$, S has algebraic intersection number 0 with c_β. The criteria for S to be a rinsed, band decomposing surface follow easily from the construction, Lemma 4.7, and that $\partial S = \partial\Sigma$ outside a small neighborhood of $\partial R(\gamma)$ and inside ∂S is oriented the same direction as $\partial R(\gamma)$. □

Corollary 4.10. *Suppose that* (M, γ, β) *is a band-taut sutured manifold with* $y \in H_2(M, \partial M)$ *non-zero. Then there exists a rinsed band-decomposing surface* $S \subset M$ *representing* $\pm y$ *such that for all non-negative* $k \in \mathbb{Z}$*, the surface* S_k *gives a band-taut decomposition* $(M, \gamma, \beta) \overset{S_k}{\to} (M', \gamma', \beta')$*. Furthermore, if (BT1) does not hold, then each component of* $(D_\beta \cap M') - D_{\beta'}$ *is a product disc or cancelling disc.*

Proof. Let S be the surface provided by Theorem 4.9. Let (M', γ', β') be the result of decomposing (M, γ, β) using S_k. By Lemmas 4.6 and 4.7, S_k is a rinsed band-decomposing surface satisfying (BD).

Since the double curve sum of S with $(k+1)$ copies of $R(\gamma)$ is β-taut, $R(\gamma')$ is $(\beta' - c_\beta)$-taut, the decomposition

$$(M, \gamma, \beta) \overset{S_k}{\to} (M', \gamma', \beta')$$

is $(\beta - c_\beta)$-taut. By Lemma 4.2, the decomposition is band-taut and if (BT1) does not hold then each component of $(D_\beta \cap M') - D_{\beta'}$ is a product disc or cancelling disc. □

5. Parameterizing Surfaces

Let (M, γ, β) be a sutured manifold with β having endpoints disjoint from $A(\gamma) \cup T(\gamma)$. (That is, (M, γ, β) satisfies (T0).) A **parameterizing surface** is an orientable surface Q properly embedded in $M - \mathring{\eta}(\beta)$ satisfying:

(P1) $\partial Q \cap A(\gamma)$ consists of spanning arcs each intersecting γ once
(P2) no component of Q is a sphere or disc disjoint from $\beta \cup \gamma$.

For a parameterizing surface Q, let $\mu(Q)$ denote the number of times that ∂Q traverses an edge of β. Define the **index** of Q to be:

$$I(Q) = \mu(Q) + |\partial Q \cap \gamma| - 2\chi(Q).$$

Remark. In the definition of index given in [S1, Definition 7.4], there is also a term denoted $\mathscr{K}$ that is the sum of values of a function defined on the interior vertices of β. As Scharlemann remarks, that the function can be chosen arbitrarily, and in this paper we will always choose it to be identically zero. Also, Scharlemann allows parameterizing surfaces to contain spherical components. No harm is done to [S1] by forbidding them and some simplicity is gained since spherical components have negative index. Lackenby in [L1] has a similar convention.

We define a parameterizing surface in a banded sutured manifold (M, γ, β) to be a parameterizing surface Q in $(M, \gamma, \beta - c_\beta)$.

If Q is a parameterizing surface and if $S \subset M$ is a decomposing surface, we say that S and Q are **normalized** if they have been isotoped in a neighborhood of $A(\gamma) \cup T(\gamma)$ to intersect minimally and if no component of $S \cap Q$ is an inessential circle on Q. It is clear that if S is β-taut then S and Q can be normalized without increasing the index of Q. Furthermore, it is not difficult to see that if $Q_1, \ldots, Q_n$ are parameterizing surfaces, not necessarily disjoint, then a β-taut decomposing surface S and $Q_1, \ldots, Q_n$ can be simultaneously normalized by an isotopy of S and each Q_i.

Suppose that $(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$ is a β-taut decomposition and that $Q \subset M$ is a parameterizing surface. If S is a conditioned surface normalized with respect to Q, we say that the decomposition **respects** Q if $Q \cap M'$ is a parameterizing surface.

The next lemma is a simple extension of [S1, Section 7]. Recall that S_k denotes the oriented double curve sum of S with k copies of $R(\gamma)$.

Lemma 5.1. *Suppose that $(M, \gamma, \beta) \xrightarrow{S_k} (M', \gamma', \beta')$ is β-taut decomposition with S a conditioned surface and that $Q_1, \ldots, Q_n$ are parameterizing surfaces in (M, γ, β) such that S_k and each Q_i are normalized. Then for k large enough, the decomposition of (M, γ, β) using S_k respects each Q_i and the index of each Q_i does not increase under the decomposition.*

Proof. Scharlemann [S1, Lemma 7.5] shows that for each i, there exists $m_i \in \mathbb{N}$ such that if $k_i \geq m_i$, and if S_{k_i} is normalized with respect to a parameterizing surface Q_i, then $Q_i \cap M'$ is a parameterizing surface with index no larger than Q_i. Since for each k, S_k can be normalized simultaneously with $Q_1, \ldots, Q_n$, we simply need to choose $k \geq \max(m_1, \ldots, m_n)$. □

Suppose that S is a product disc, product annulus, or disc with boundary in $R(\gamma)$. We say that a parameterizing surface $Q^c \subset M$ is obtained by **modifying** Q relative to S if Q^c is obtained by completely boundary compressing Q using outermost discs of $S - Q$ bounded by outermost arcs having both endpoints in $R_\pm$, normalizing Q and S, and then removing all disc components with boundary completely contained in $R(\gamma)$. Scharlemann proves [S1, Lemma 7.6] that modifying a parameterizing surface does not increase index. Lackenby [L1] points out that if Q^c is compressible so is Q.

Lemma 5.2. *Suppose that $(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$ is a β-taut decomposition with S a product disc, product annulus, or disc with boundary in $R(\gamma)$. Let $Q_1, \ldots, Q_n$ be parameterizing surfaces. Then after replacing each Q_i with Q_i^c, each of the surfaces $Q_i^c \cap M'$ is a parameterizing surface in M' with index no larger than the index of Q_i.*

Proof. This is nearly identical to the proof of Lemma 5.1, but uses [S1, Lemma 7.6]. □

We say that the decomposition described in Lemma 5.2 **respects** Q.

We now assemble some of the facts we have collected.

Theorem 5.3. *Suppose that (M, γ, β) is a band-taut sutured manifold and that $y \in H_2(M, \partial M)$ is non-zero. Suppose that $Q_1, \ldots, Q_n$ are parameterizing surfaces in M. Then there exists a band-taut decomposition*

$$(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$$

respecting each Q_i with S a rinsed band-decomposing surface representing $\pm y$.

Proof. By Corollary 4.10, there exists a band-taut decomposition

$$(M, \gamma, \beta) \xrightarrow{S} (M', \gamma', \beta')$$

with S a rinsed band-decomposing surface representing $\pm y$. By Lemma 5.1, if we replace S with S_k for large enough k, we may assume that the decomposition respects each Q_i. □

Similarly we have:

Theorem 5.4. *Suppose that (M,γ,β) is a band taut sutured manifold and that there exists an allowable product disc or allowable product annulus in M'. Let $Q_1,\dots,Q_n$ be parameterizing surfaces in M. Then there exists an allowable product disc or allowable product annulus P, such that, after modifying each Q_i, the decomposition given by P is band-taut and respects each Q_i.*

Proof. This follows immediately from Lemma 4.5 and Lemma 5.2. □

6. SUTURED MANIFOLD DECOMPOSITIONS AND BRANCHED SURFACES

In [G3, Construction 4.16], Gabai explains how to build a branched surface $B(\mathscr{H})$ from a sequence of sutured manifold decompositions

$$\mathscr{H} : (M,\gamma,\beta) \stackrel{S_1}{\to} (M_1,\gamma_1,\beta_1) \stackrel{S_2}{\to} \dots \stackrel{S_n}{\to} (M_n,\gamma_n,\beta_n)$$

Essentially, the branched surface is the union $\bigcup_{i=1}^{n} S_i$ with the intersections smoothed. We will call $B(\mathscr{H})$ the branched surface **associated** to the sequence $\mathscr{H}$.

Lemma 6.1. *If $\mathscr{H}$ is a sequence of band-taut sutured manifold decompositions, there is an isotopy of c_β (relative to ∂c_β) to an arc a such that the closure of $a \cap \mathring{M}_n$ is c_{β_n} and so that $a - c_{\beta_n}$ is embedded in $\partial M \cup B(\mathscr{H})$. Furthermore, there is a proper isotopy of c_β in M to $(a \cap B(\mathscr{H})) \cup c_{\beta_n}$.*

Proof. By the definition of "band-taut" decomposition, the decomposition

$$(M_i,\gamma_i,\beta_i) \stackrel{S_{i+1}}{\to} (M_{i+1},\gamma_{i+1},\beta_{i+1})$$

defines an isotopy ϕ_i in D_{β_i} of c_{β_i} (relative to its endpoints) to an arc a_i such that the intersection of a_i with the interior of M_{i+1} is the core $c_{\beta_{i+1}}$. If the decomposition is of the form (BT1), then the isotopy moves c_{β_i} into $\partial M \cup B(\mathscr{H})$. By Lemma 4.3, the intersection of a_i with ∂D_{β_i} consists of arcs, each joining an endpoint of c_{β_i} to an endpoint of $c_{\beta_{i+1}}$. Each of these arcs, if not a single point, intersects $D_{\beta_{i+1}}$ in an arc with one endpoint on $\partial c_{\beta_{i+1}}$ and the other on a point of $\partial S_{i+1} \cap \partial D_{\beta_i}$.

Each ϕ_i is also a homotopy of c_β. Their concatanation is a homotopy ϕ of c_β. We desire to show ϕ can be homotoped to provide an isotopy ϕ' in D_β of the arc c_β (relative to ∂c_β) to an arc a so that a intersects the interior of M_n in c_{β_n} and $a - c_{\beta_n}$ is embedded in $B(\mathscr{H})$.

To that end, suppose that i is the smallest index such that the isotopy of c_{β_i} to $c_{\beta_{i+1}}$ makes c_β non-embedded. This implies that a_i intersects a_{i-1}. The arc a_{i-1} lies in $\partial D_{\beta_{i-1}}$ and the arc a_i lies in ∂D_{β_i}. The boundary of D_{β_i} is the union of portions of $\partial D_{\beta_{i-1}}$ with components of $S_i \cap D_{\beta_{i-1}}$. The arcs a_{i-1} and a_i, therefore, intersect in closed intervals lying in $\partial D_{\beta_{i-1}} \cap \partial D_{\beta_i}$. There are at most two intervals of overlap and each interval of overlap has one endpoint lying on ∂S_i. These intervals of overlap can each be homotoped to be a point of $\partial S_i \cap \partial D_\beta$. This homotopy deforms the concatenation of the isotopy from $c_{\beta_{i-1}}$ to c_{β_i} with the isotopy from c_{β_i} to $c_{\beta_{i+1}}$ to be an isotopy of c_β such that c_β intersects the interior of M_{i+1} in $c_{\beta_{i+1}}$ and, after the isotopy, $c_\beta - c_{\beta_{i+1}}$ is embedded in $B(\mathscr{H})$. By induction on i, we create the desired isotopy of c_β to a.

The intersection $a \cap \partial M$ consists of at most two arcs, each with an endpoint at ∂c_β and with the other endpoint at ∂c_{β_1}. There is, therefore, also a proper isotopy of c_β to $(a \cap B(\mathscr{H})) \cup c_{\beta_n}$. $\square$

7. Band Taut Hierarchies

Let (M, γ, β) be a β–taut sutured manifold and suppose that $U \subset T(\gamma)$. A β-taut sutured manifold **hierarchy** (cf. [S2, Definition 2.1]) relative to U is a finite sequence

$$\mathscr{H} : (M, \gamma) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (M_n, \gamma_n)$$

of β-taut decompositions for which

(i) each S_i is either a conditioned surface, a product disc, a product annulus whose ends are essential in $R(\gamma_{i-1})$, or a disc whose boundary is β-essential in $R(\gamma_{i-1})$ and each S_i is disjoint from U.
(ii) $H_2(M_n, \partial M_n - U) = 0$.

If $U = \varnothing$, then we simply call it a β-taut sutured manifold hierarchy.

We say that the hierarchy **respects** a parameterizing surface $Q \subset M$ if each decomposition in $\mathscr{H}$ respects Q. (Implicitly, we assume that Q may be modified by isotopies and ∂-compressions during the decompositions as in Section 5.)

Suppose that (M, γ, β) is a band-taut sutured manifold. A **band-taut hierarchy** for M is a $(\beta - c_\beta)$-taut sutured manifold hierarchy $\mathscr{H}$ for $(M, \gamma, \beta - c_\beta)$ with each decomposition $(M_{i-1}, \gamma_{i-1}, \beta_{i-1}) \xrightarrow{S_i} (M_i, \gamma_i, \beta_i)$ a band-taut decomposition.

Theorem 7.1. *Suppose that (M,γ,β) is a band-taut sutured manifold and that $Q_1,\ldots,Q_n$ are parameterizing surfaces in M. Then the following are all true:*

(1) *there exists a band-taut sutured manifold hierarchy*

$$\mathscr{H}:(M,\gamma)\xrightarrow{S_1}(M_1,\gamma_1)\xrightarrow{S_2}\ldots\xrightarrow{S_n}(M_n,\gamma_n)$$

for M respecting each Q_i.

(2) *Each surface S_i is a band-decomposing surface and if S_i is conditioned then it is also rinsed.*

(3) *If $y\in H_2(M,\partial M)$ is non-zero, S_1 may be taken to represent $\pm y$*

(4) *There is a proper isotopy of c_β in M to an arc disjoint from S_1.*

(5) *Let $B(\mathscr{H})$ be the branched surface associated to $\mathscr{H}$. There is an isotopy of c_β in D_β relative to ∂c_β to an arc a such that the the closure of the arc $a\cap\mathring{M}_n$ is c_{β_n}, the arc $a-c_{\beta_n}$ is embedded in $\partial M\cup B(\mathscr{H})$. Furthermore, there is a proper isotopy of c_β in D_β to an embedded arc in $B(\mathscr{H})$.*

Proof. Let S_1 be the surface provided by Theorem 5.3 representing $\pm y$ and giving a band-taut sutured manifold decomposition $(M,\gamma,\beta)\xrightarrow{S_1}(M_1,\gamma_1,\beta_1)$ respecting Q. Let Q_1 be the parameterizing surface in (M_1,γ_1,β_1) resulting from Q.

If $H_2(M_1,\partial M_1-U)=0$, we are done. Otherwise, define S_2 according to the instructions below. In the description below, it should always be assumed that if S is chosen at step (i), then step (k) for all $k>i$ will not be applied.

(1) If $e_{\beta_1}=\varnothing$ and if D_{β_1} is a boundary parallel disc in $M-(\beta-c_\beta)$, then let $S_2=c_{\beta_2}=D_{\beta_2}=\varnothing$. The decomposition by S_2 is of the form (BT1).

(2) If $(M_1,\gamma_1,\beta_1-c_{\beta_1})$ contains a cancelling disc or product disc, it contains one disjoint from D_{β_1} (Lemma 4.4). Let S_2 be either a product disc disjoint from D_{β_1} or the frontier of a regular neighborhood of a cancelling disc disjoint from D_{β_1}. If S_2 is the frontier of a cancelling disc, after decomposing along S_2, cancel the edge of β_1 adjacent to the cancelling disc.

(3) If $(M_1,\gamma_1,\beta_1-c_{\beta_1})$ contains a nonself amalgamating disc, amalgamate an arc component of β_1 and let $S_2=\varnothing$. This does not affect the fact that (M_1,γ_1,β_1) is a band-taut sutured manifold by [S1, Lemmas 4.3 and 4.4].

(4) If $(M_1,\gamma_1,\beta_1 - c_{\beta_1})$ contains an allowable product disc choose one S_2 that is disjoint from D_{β_1} (Lemma 4.4). Modify Q_1 so that S_2 respects Q_1. Decomposing along S_2 gives a band-taut decomposition by Theorem 5.4.
(5) If $(M_1,\gamma_1,\beta_1 - c_{\beta_1})$ has an allowable self amalgamating disc, choose one that is disjoint from D_{β_1}. This is possible by Lemma 4.4. If the associated product annulus has both ends essential in $R(\gamma)$, let S_2 be that annulus. Otherwise, let S_2 be the disc obtained by isotoping the disc obtained by capping off the annulus with a disc in $R(\gamma)$ so that it is properly embedded in M. Modify Q_1 so that S_2 respects Q_1.
(6) If $(M_1,\gamma_1,\beta_1 - c_{\beta_1})$ has no product discs, cancelling discs, or allowable nonself amalgamating discs, let S_2 be the surface obtained by applying Theorem 5.3 to an nontrivial element of $H_2(M_2,\partial M_2 - U)$.

Decompose (M_1,γ_1,β_1) using S_2 to obtain (M',γ_2',β_2'). If S_2 was a disc with boundary in $R(\gamma_1)$, amalgamate arcs and convert an arc to a suture as in Section 4.3.3. By the results of that section, this elimination of nonself amalgamating discs preserves the fact that the resulting sutured manifold (M_2,γ_2,β_2) is band-taut. Let Q_2 be the resulting parameterizing surface in M_2.

If $H_2(M_2,\partial M_2 - U) = 0$ we are done. Otherwise, a sutured manifold (M_3,γ_3,β_3) can be obtained from (M_2,γ_2,β_2) by a method analogous to how we obtained (M_2,γ_2,β_2) from (M_1,γ_1,β_1). Repeating this process creates a sequence of band-taut sutured manifold decompositions

$$\mathscr{H} : (M,\gamma,\beta) \stackrel{S_1}{\to} (M_1,\gamma_1,\beta_1) \stackrel{S_2}{\to} (M_2,\gamma_2,\beta_2) \stackrel{S_3}{\to} \dots$$

respecting Q.

By the proofs of [S1, Theorem 4.19] and [S2, Theorem 2.5], the sequence

$$(M_1,\gamma_1,\beta_1 - c_{\beta_1}) \stackrel{S_2}{\to} (M_2,\gamma_2,\beta_2 - c_{\beta_2}) \stackrel{S_3}{\to} \dots$$

must terminate in $(M_n,\gamma_n,\beta_n - c_{\beta_n})$ with $H_2(M_n,\partial M_n - U) = 0$. Consequently, $\mathscr{H}$ is finite. This sequence with all arc cancellations and amalgamations ignored is the desired hierarchy. If $c_{\beta_i} \neq \varnothing$ but $c_{\beta_{i+1}} = \varnothing$ then, by the definition of band taut decomposition, c_{β_i} can be isotoped in D_{β_i} (rel ∂c_{β_i}) in ∂M_i. Conclusions (3) and (4) follow from Lemma 6.1 $\square$

Many arguments in sutured manifold theory require showing that a hierarchy remains taut after removing some components of β. We will need the following theorem, which is a slight generalization of what is stated in [S2] (and is implicit in that paper and in [S1]).

Theorem 7.2 ([S2, Lemma 2.6]). *Suppose that*

$$(M,\gamma,\beta) = (M_0,\gamma_0,\beta_0) \xrightarrow{S_1} (M_1,\gamma_1,\beta_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (M_n,\gamma_n,\beta_n)$$

is a sequence of β-taut sutured manifold decompositions in which

(1) no component of M is a solid torus disjoint from $\gamma \cup \beta$
(2) each S_i is either a conditioned surface, a product disc, a product annulus with each boundary component essential in $R(\gamma_{i-1})$, or a disc D such that
  (a) $\partial D \subset R(\gamma_{i-1})$
  (b) If ∂D is β-inessential in $R(\gamma_{i-1})$ then D is disjoint from β.
(3) If a closed component of S_i separates, then it bounds a product region with a closed component of $R(\gamma)$ intersecting β in vertical arcs.

Then if (M_n,γ_n,β_n) is β_n-taut so is every decomposition in the sequence.

Proof. The only difference between this and what is found in [S2] is that we allow closed components of S_i to be parallel to closed components of $R(\gamma)$. Decomposing along such a component creates a sutured manifold equivalent to the original and so if the sutured manifold after the decomposition is (β_{i+1})-taut, the original must be (β_i)-taut. □

Remark. The reason for stating this generalization of [S2, Lemma 2.6] is that in creating a sutured manifold hierarchy that respects a parameterizing surface we may need to decompose along the double curve sum S_k of a conditioned surface S with some number of copies of $R(\gamma)$. If S is disjoint from a closed component of $R(\gamma)$ then some components of S_k will be closed and separating.

The next corollary is immediate:

Corollary 7.3. *Suppose that (M,γ,β) is a band-taut sutured manifold such that no component of M is solid torus disjoint from $(\beta - c_\beta) \cup \gamma$ and that*

$$\mathscr{H} : (M,\gamma) \xrightarrow{S_1} (M_1,\gamma_1) \xrightarrow{S_2} \dots \xrightarrow{S_n} (M_n,\gamma_n)$$

is the band-taut sutured manifold hierarchy given by Theorem 7.1. If (M_n,γ_n) is $\beta_n - (e_{\beta_n} \cup c_{\beta_n})$-taut, then (M,γ) is $\beta - (e_\beta \cup c_\beta)$-taut.

Before analyzing the parameterizing surface at the end of the hierarchy, we present one final lemma in this section. Recall that if $b \subset \partial M$ is a simple closed curve and if $Q \subset M$ is a surface, then a b-boundary compressing disc for Q is a disc whose boundary consists of an arc on Q and a sub-arc of b. Suppose that $\beta \subset M$ is an edge with both endpoints on ∂M and that b is a meridian of β in the boundary of $M - \mathring{\eta}(\beta)$. If Q is a surface in $M - \mathring{\eta}(\beta)$, we define a **$\beta$-boundary compressing disc** for Q in M to be a b-boundary compressing disc for Q in $M - \mathring{\eta}(\beta)$.

The next lemma gives a criterion for determining when a compressing disc or β-boundary compressing disc for a parameterizing surface at the end of a hierarchy can be pulled back to such a disc for a parameterizing surface in the initial sutured manifold.

Lemma 7.4. *Suppose that*

$$(M,\gamma,\beta) \xrightarrow{S} (M',\gamma',\beta')$$

is sutured manifold decomposition respecting a parameterizing surface Q with β a single arc. Assume that $\mu(Q) \geq 1$. Let Q' be the resulting parameterizing surface in M'. Let β'_0 be a component of β'. Then if the surface Q' has a compressing disc or β'_0-boundary compressing disc with interior disjoint from β' then the surface Q has a compressing disc or β-boundary compressing disc.

Proof. Let D be a compressing or β'_0-boundary compressing disc for Q'. Either $Q' = Q - \mathring{\eta}(S)$ or $Q' = Q^c - \mathring{\eta}(S)$ is obtained by first modifying Q to Q^c. If $Q' = Q - \mathring{\eta}(S)$, then $Q' \subset Q$ and D is also a compressing or β-boundary compressing disc for Q. We may, therefore, assume that if $Q' = Q^c - \mathring{\eta}(S)$, then Q^c has a compressing or β-boundary compressing disc E. By a small isotopy we may assume that $E \cap \partial M = \varnothing$.

By the construction of Q^c, Q can be obtained from Q^c by tubing Q^c to itself and to discs with boundary in $R(\gamma)$ using tubes that are the frontiers of regular neighborhoods of arcs in $R(\gamma)$. The disc E is easily made disjoint from those tubes by a small isotopy, and so E remains a compressing or β-boundary compressing disc for Q. □

8. Combinatorics

We begin this section with a sequence of lemmas concerning sutured manifolds that are at the end of a band-taut hierarchy. We consider only the situation in which $e_\beta = \beta - c_\beta$. Recall from the definition of banded sutured manifold that $|e_\beta| \leq 2$.

Lemma 8.1. *Suppose that (M,γ,β) is a connected band-taut sutured manifold with $H_2(M,\partial M) = 0$ and $\beta - c_\beta = e_\beta$. Assume that $\partial M \neq \varnothing$. Then ∂M is the union of one or two spheres, each component of ∂M contains exactly one disc component of R_- and exactly one disc component of R_+, and one of the following holds:*

(1) $e_\beta = \varnothing$, $|\gamma| = 1$, *and* M *is a 3–ball.*
(2) $|e_\beta| = 1$, ∂M *is a single sphere, and* e_β *has endpoints in the disc components of* $R(\gamma)$.
(3) $|e_\beta| = 2$, ∂M *is a single sphere, one edge* e *of* e_β *has endpoints in the disc components of* $R(\gamma)$ *and the other edge of* e_β *has endpoints either in the same disc components, in which case* $|\gamma| = 1$, *or in the adjacent annulus components of* $R(\gamma)$.
(4) $|e_\beta| = 2$, $M = S^2 \times [0,1]$, *the edges of* e_β *are fibers in the product structure of* M. *Each component of* ∂M *contains a single suture.*

Proof. Since $H_2(M, \partial M) = 0$, by the "half lives, half dies" theorem of algebraic topology, the boundary of M must be the union of spheres. Since M is e_β-irreducible, if there is a component of ∂M that is disjoint from e_β, then ∂M must be that sphere and $|e_\beta| = 0$. Since (M, γ, e_β) is e_β-taut, this implies conclusion (1). Assume, therefore, that $|e_\beta| \in \{1, 2\}$ and that each component of ∂M is adjacent to a component of e_β.

If M contains a sphere intersecting e_β exactly once, then since $R(\gamma)$ is e_β-incompressible, ∂M must be the union of two spheres, $|e_\beta| = 1$ and $|\gamma| = 0$. This contradicts the definition of banded sutured manifold. Thus, each component of ∂M contains at least two endpoints of e_β. Furthermore, each disc component of $R(\gamma)$ must contain an endpoint of e_β since $R(\gamma)$ is e_β-incompressible. We conclude that ∂M has no more than two components.

If ∂M has two components, then each of them must contain two endpoints of e_β and so $|e_\beta| = 2$. In each component of ∂M the endpoints of e_β are contained in disc components of $R(\gamma)$. Each component of $D_\beta \cap \partial M$ crosses γ exactly once and so each component of ∂M contains exactly one suture. The frontier of a regular neighborhood of $D_\beta \cup \partial M$ is a sphere in $M - e_\beta$ which must bound a 3–ball in $M - e_\beta$. Thus, $M = S^2 \times [0,1]$ and the components of e_β are fibers. This is conclusion (4).

We may assume, therefore, that ∂M is a single sphere. Suppose that R_- (say) has two disc components R_1 and R_2. The discs R_1 and R_2 must each contain an endpoint v_1 and v_2, respectively, of e_β. Since (M, γ) is e_β-taut, each component of e_β has one endpoint in R_- and one in R_+. Thus, v_1 and v_2 belong to different components of e_β. (Consequently, $|e_\beta| = 2$.) Let w_1 and w_2 be the other endpoints of e_β (lying in R_+) so that v_i and w_i are endpoints of the same edge of e_β.

If R_+ has a disc component, then one of w_1 or w_2 must lie in it. Without loss of generality, suppose it is w_1. A component of $\partial D_\beta \cap \partial M$ joins w_1 to v_2 and crosses γ once. The union of the disc component of R_+ containing

w_1 with R_2 is a sphere and so ∂M contains more than one component, a contradiction. This implies that if $R_\pm$ contains two discs, then $R_\mp$ cannot contain any. Since ∂M is a sphere, $R(\gamma)$ contains exactly two discs. Hence, all other components of $R(\gamma)$ are annuli. We see, therefore, that if $|\gamma|$ is even then $R_\pm$ contains two discs and all other components of $R(\gamma)$ are annuli and if $|\gamma|$ is odd then each of R_- and R_+ contains a disc and all other components of $R(\gamma)$ are annuli.

If $|e_\beta| = 1$, then since one endpoint of e_β is in R_- and the other is in R_+ and since each disc component of $R(\gamma)$ contains an endpoint, conclusion (2) holds. Assume, therefore, that $|e_\beta| = 2$. Suppose, for the moment, that some disc component D of $R(\gamma)$ contains two endpoints of e_β. These endpoints must belong to different components of e_β. Each component of $\partial D_\beta \cap \partial M$ joins endpoints of e_β and crosses γ once. Thus, the other endpoints of e_β are in the component of $R(\gamma)$ adjacent to D. This component must, therefore be a disc and so (3) holds. We may assume, therefore, that each disc component of $R(\gamma)$ contains exactly one endpoint of e_β.

Suppose that $|\gamma|$ is odd. Let $D_\pm$ be the disc component of $R_\pm$. Each contains an endpoint $v_\pm$ of e_β. If v_- and v_+ do not belong to the same arc of e_β, then the other endpoint of the arc containing v_+ lies in the component of $R(\gamma)$ adjacent to D_-, since each component of $D_\beta \cap M$ intersects γ exactly once. But this component must lie in R_+ and so a component of e_β has both endpoints in R_+, a contradiction. Thus, v_- and v_+ are endpoints of the same component of e_β, and the fact that each component of $D_\beta \cap \partial M$ intersects γ exactly once immediately implies conclusion (3).

Suppose that $|\gamma|$ is even. Then both disc components of $R(\gamma)$ lie, without loss of generality, in R_-. Each contains exactly one endpoint of e_β. All other components of R_- are annuli disjoint from β. Thus, $x_{e_\beta}(R_-) = 0$. The surface R_+ is the union of annuli, one or two of which contain the two endpoints of $e_\beta \cap R_+$. Thus, $x_{e_\beta}(R_+) = 2$. The union $R_- \cup A(\gamma)$ is a surface with boundary equal to ∂R_+ and homologous to R_+ in $H_2(M, \partial R_+)$. Consequently, R_+ is not x_{e_β}-minimizing, and, therefore, not e_β-taut. This contradicts our hypotheses. Hence, $|\gamma|$ cannot be even and so each of $R_\pm$ contains a single disc. $\square$

Lemma 8.2. *Suppose that (M, γ, β) is a connected band-taut manifold such that $H_2(M, \partial M) = 0$, ∂M is connected and non-empty, and $\beta - c_\beta = e_\beta$. Then the number of sutures $|\gamma|$ is odd and there is an edge component e of e_β such that (M, γ, e) is e-taut. Furthermore, if (M, γ) is not $\varnothing$-taut, then either $|\gamma| \geq 3$ or M is a non-trivial rational homology ball.*

Proof. Lemma 8.1 implies that either (M,γ) is a 3–ball with a single suture in its boundary or one of the following occurs:

- $|e_\beta| = 1$, and R_- and R_+ each contain a single disc. The intersection of these discs with e_β is the endpoints of an edge e of e_β.
- $|e_\beta| = 2$, R_- and R_+ each contain a single disc. Unless $|\gamma| = 1$, there is a component e of e_β such that the intersection of the disc components of $R(\gamma)$ with e_β is the endpoints of e. If $|\gamma| = 1$, then that intersection contains all the endpoints of e_β.

Let e be an edge of e_β having endpoints in the disc components of $R(\gamma)$. Since e does not have both endpoints in $R_\pm$, and since $R(\gamma)$ contains exactly two disc components, γ consists of an odd number of parallel sutures on the sphere ∂M.

We claim that (M,γ,e) is e-taut. Since $R(\gamma)$ has two disc components and $|\gamma|$ is odd, R_- and R_+ are each e-minimizing. Suppose, first, that S is an e-reducing sphere. Choose S to minimize $|S \cap D_\beta|$. An innermost circle argument shows that $S \cap D_\beta$ is empty, and so S is disjoint from e_β. Since M is e_β-taut, S bounds a ball disjoint from e, a contradiction. Suppose, therefore, that S is a compressing disc for $R(\gamma) - e$ that is disjoint from e. Since ∂M is a 2-sphere, there is a 2-sphere in M intersecting e a single time. Hence, M contains a non-separating S^2, contradicting the assumption that $H_2(M,\partial M) = 0$. Thus, (M,γ,e) is e-taut.

If M is a 3-ball with a single suture in its boundary, then (M,γ) is $\varnothing$-taut. Thus, either $|\gamma| \geq 3$ or M is not a 3-ball. The relative long exact sequence for $H_2(M,\partial M)$ shows that $H_2(M) = 0$ and that $H_1(M)$ is isomorphic to $H_1(M,\partial M)$. Duality for manifolds with boundary shows that $H^1(M) = 0$ since $H_2(M,\partial M) = 0$. The universal coefficient theorem shows that $H^1(M)$ is isomorphic to the direct sum of the free part of $H_1(M)$ and the torsion part of $H_0(M)$. Thus, $H_1(M)$ is finite. This implies that if $M \neq B^3$, then M is a non-trivial rational homology ball, as desired. □

The presence of a parameterizing surface can give us more information.

Lemma 8.3. *Suppose that (M,γ,e) is a connected e-taut sutured manifold with e an edge. Assume that $H_2(M,\partial M) = 0$. Suppose that $Q \subset M$ is a parameterizing surface having no compressing or e-boundary compressing disc. If $\mu(Q) \geq 1$, then one of the following holds:*

(1) M is a 3–ball, $|\gamma| = 1$ and e is boundary-parallel by a component of Q.
(2) M is a punctured lens space and e is a core of M.

(3) $I(Q) \geq 2\mu(Q)$.

Proof. Since Q is a parameterizing surface, no component has negative index. Removing all components of Q that are disjoint from e does not increase index. Since Q is incompressible, no component of ∂Q is an inessential circle in $\partial M - e$. Since ∂M is the union of 2–spheres and since Q has no e-boundary compressing disc, each arc component of $\partial Q \cap \partial M$ joins distinct endpoints of e. Since $\mu(Q) \geq 1$, there is at least one such arc component. Thus, no component of ∂Q is an essential circle in $\partial M - e$. Therefore, each component of $\partial Q \cap \partial M$ is an arc joining the endpoints of e. Isotope Q so as to minimize $\partial Q \cap \gamma$. This does not increase $I(Q)$. Since e is an edge and (M, γ, e) is e-taut, $|\gamma|$ must be odd.

Case 1: $|\gamma| = 1$.

If some component Q_0 of Q is a disc intersecting γ once, then it is a cancelling disc for e. This implies that (M, γ) is $\varnothing$-taut. M is, therefore, a 3–ball and e is boundary parallel by a component of Q.

Suppose, therefore, that some component of Q is a disc intersecting γ more than once (and, therefore, running along e more than once). Compressing the frontier of $\eta(\partial M \cup e)$ using that disc produces a 2–sphere which must bound a 3–ball. Hence, M is a punctured lens space with core e.

If no component of Q is a disc, then no component of Q has positive euler characteristic, and so $I(Q) \geq \mu(Q) + |\partial Q \cap \gamma| = 2\mu(Q)$.

Case 2: $|\gamma| \geq 3$.

There are at most $\mu(Q)$ components of Q and so $-2\chi(Q) \geq -2\mu(Q)$. We have, therefore, $I(Q) \geq \mu(Q) + |\partial Q \cap \gamma| - 2\mu(Q)$. Since $|\gamma| \geq 3$ and since all sutures are parallel, each arc of $\partial Q \cap \partial M$ intersects γ at least 3 times. Thus,

$$I(Q) \geq \mu(Q) + 3\mu(Q) - 2\mu(Q) = 2\mu(Q)$$

as desired. □

The next theorem is the key result of the paper. It applies the combinatorics of the previous lemmas to the last term of a band-taut hierarchy.

Theorem 8.4. *Suppose that (M, γ, β) is a band taut sutured manifold with $e_\beta = \beta - c_\beta$. Assume that e_β has components e_1 and e_2. Let Q_1 and Q_2 be parameterizing surfaces in (M, γ, e_β) with $Q_1 \cap e_2 = Q_2 \cap e_1 = \varnothing$. We allow the possibility that $e_i = Q_i = \varnothing$ for $i \in \{1, 2\}$.*

Then one of the following occurs:

(1) Some Q_i has a compressing or e_i-boundary compressing disc in (M,γ,e_i).
(2) $|e_\beta| = 2$ and M contains an S^2 intersecting each edge of e_β exactly once.
(3) For some i, $(M,e_i) = (M_0',\beta_0')\#(M_1',\beta_1')$ where M_1' is a lens space and β_1' is a core of M_1'.
(4) (M,γ) is $\varnothing$-taut. The arc c_β can be properly isotoped onto a branched surface $B(\mathscr{H})$ associated to a taut sutured manifold hierarchy $\mathscr{H}$ for M. Also, a proper isotopy of c_β in M takes c_β to an arc disjoint from the first decomposing surface of $\mathscr{H}$. That first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(M,\partial M)$.
(5) Either

$$I(Q_1) \geq 2\mu(Q_1) \text{ or } I(Q_2) \geq 2\mu(Q_2).$$

Proof. By Theorem 7.1, there exists a band-taut hierarchy

$$\mathscr{H} : (M,\gamma,\beta) \xrightarrow{S_1} \ldots \xrightarrow{S_n} (M_n,\gamma_n,\beta_n)$$

respecting Q_1 and Q_2 with S_1 representing $\pm y$. By that theorem, there is a proper isotopy of c_β in M to an arc disjoint from S_1. Let c_{β_n} be the core of the band in M_n. By Theorem 7.1, there is an isotopy of c_β so that $c_\beta - c_{\beta_n}$ is embedded in the union of ∂M with the branched surface $B(\mathscr{H})$. At each stage of the hierarchy, each component of $D_{\beta_i} - S_i$ not containing $c_{\beta_{i+1}}$ is a cancelling disc, product disc, or amalgamating disc (Lemma 4.2), and the hierarchy is constructed so as to eliminate all such discs. Thus, we may assume that each component of $\beta_n - (e_{\beta_n} \cup c_{\beta_n})$ is an arc in a 3-ball component of M_n having a single suture in its boundary; that 3-ball is disjoint from all other components of β_n. Deleting such arc components preserves the $(\beta_n - c_{\beta_n})$-tautness of (M_n,γ_n). Henceforth, we ignore such components.

Either conclusion (1) of our theorem occurs, or by Lemma 7.4, Q_i does not have a compressing or e_i-boundary compressing disc in $(M_n,\gamma_n,\beta_n - (c_{\beta_n} \cup e_j)$ (with $j \neq i$). We assume that Q_i does not have such a disc.

The manifold M_n has $H_2(M_n,\partial M_n) = 0$. Let M' denote the component of M_n containing c_{β_n}. Let $\gamma' = \gamma \cap M'$. We have $H_2(M',\partial M') = 0$. By Lemma 8.1, ∂M_n is the union of one or two spheres and one of the following holds:

(a) $e_{\beta_n} = \varnothing$, $|\gamma'| = 1$, and M' is a 3–ball.
(b) $|e_{\beta_n}| = 1$, $\partial M'$ is a single sphere, each of R_- and R_+ contains a single disc, and e_{β_n} has endpoints in the disc components of $R(\gamma)$.

(c) $|e_{\beta_n}| = 2$, $\partial M'$ is a single sphere, one edge e of e_{β_n} has endpoints in the disc components of $R(\gamma')$ and the other edge of e_{β_n} has endpoints either in the same disc components, in which case $|\gamma| = 1$ or in the adjacent annulus components of $R(\gamma')$.
(d) $|e_{\beta_n}| = 2$, $M' = S^2 \times [0,1]$, the edges of e_{β_n} are fibers in a product structure of M'. Each component of $\partial M'$ contains a single suture.

If (d) occurs then we have conclusion (2) of our theorem. Assume, therefore, that neither (1) nor (2) of our theorem occur.

If (a) occurs, then (M_n, γ_n) is $\varnothing$-taut and by Corollary 7.3, the sequence $\mathscr{H}$ is $\varnothing$-taut. The disc D_{β_n} is isotopic into $\partial M'$ and so the hierarchy $\mathscr{H}$ can be extended by a decomposition satisfying (BT1) with empty decomposing surface. This gives conclusion (4).

Assume, therefore, that $|e_{\beta_n}| \geq 1$. By Lemma 8.2, $|\gamma'|$ is odd and there exists an edge e of e_{β_n} such that e has both endpoints in disc components of $R(\gamma')$ and (M', γ', e) is e-taut. If M' is a 3–ball and if e is boundary-parallel then, once again, we have conclusion (4). Assume, therefore, that conclusion (4) does not occur.

The edge e is a subarc of e_i for some $i \in \{1,2\}$. Let Q'_i be the parameterizing surface in M' resulting from Q_i. By hypothesis, $\mu(Q'_i) \geq 1$ and Q'_i does not have any compressing or e_i-boundary compressing discs. By Lemma 8.3, one of the following occurs:

(i) M' is a punctured lens space and e is a core of M'.
(ii) $I(Q'_i) \geq 2\mu(Q'_i)$

If (i) happens then we have conclusion (3) of our theorem. If (ii) happens, then using the facts that $I(Q_i) \geq I(Q'_i)$ and $\mu(Q_i) = \mu(Q'_i)$ we have $I(Q_i) \geq 2\mu(Q_i)$. This is conclusion (5) of our theorem. □

9. From Arc-Taut to Band-Taut

We begin this section by constructing a band taut sutured manifold from an arc-taut sutured manifold (that is, a β-taut sutured manifold where β is an arc).

Let (M, γ, β_1) be a β_1-taut sutured manifold with β_1 an edge having endpoints in components of $R(\gamma)$ with boundary. Let c_β be obtained by isotoping the endpoints of β_1 into components of A_- and A_+ adjacent to the components of $R(\gamma)$ containing the endpoints of β_1. Let β_2 be the arc obtained by continuing to isotope c_β so that its endpoints are moved across

γ and into $R(\gamma)$. Let D_β be the disc of parallelism between β_1 and β_2 that contains c_β. Let $\beta = \beta_1 \cup c_\beta \cup \beta_2$. We call (M, γ, β) the **associated banded sutured manifold**.

Lemma 9.1. *If (M, γ, β_1) is a β_1-taut sutured manifold with β_1 an edge, then (M, γ, β) is a band-taut sutured manifold.*

Proof. Without loss of generality, we may assume that M is connected. Recall that $e_\beta = \beta_1 \cup \beta_2$. We desire to show that (M, γ, e_β) is e_β-taut. Clearly, since $M - \beta_1$ is irreducible, $M - e_\beta$ is irreducible. Since e_β is disjoint from $T(\gamma)$, $T(\gamma)$ is taut. It remains to show that $R_\pm$ is e_β-taut.

Let S be a e_β-taut surface with $\partial S = \partial R_\pm$ and $[S, \partial S] = [R_\pm, \partial R_\pm]$ in $H_2(M, \partial R_\pm)$. Out of all such surfaces, choose S to intersect D_β minimally.

Since S is e_β-taut and since D_β is a disc, no component of $S \cap D_\beta$ is a circle or an arc with both endpoints on the same component of e_β. Since $\partial S = \partial R_\pm$, the intersection $S \cap D_\beta$ contains exactly two arcs having an endpoint on ∂M. Since S and $R_\pm$ are homologous, the algebraic intersection number of each surface with each component of e_β is the same. Since S is e_β-taut, the geometric intersection number of S with each component of e_β equals the absolute value of the algebraic intersection number. Consequently, S intersects each component of e_β exactly once. This implies that $S \cap D_\beta$ consists exactly of two arcs each joining ∂M to e_β and S intersects both components of e_β.

Suppose, for a moment, that some disc component R_1 of $R_\pm$ is disjoint from β_1 but not from e_β. Let R_2 be the component of $R_\mp$ adjacent to R_1. Since R_1 is adjacent to β_2, R_2 must be adjacent to β_1. Consequently, R_1 is a β_1-compressing disc for R_2. This contradicts the fact that R_2 is β_1-incompressible. We conclude that no component of $R_\pm$ is a disc disjoint from β_1 but not from β_2. Consequently,

$$x_{e_\beta}(R_\pm) = x_{\beta_1}(R_\pm) + 1.$$

Without loss of generality, we may assume that S contains no sphere component disjoint from e_β. Thus, if S_0 is a component of S, then either $x_{e_\beta}(S_0) = -\chi(S_0) + |S_0 \cap e_\beta|$ or S_0 is a disc disjoint from e_β. Suppose that S_0 is a disc disjoint from e_β and let R be the component of $R_\pm$ with $\partial S \subset \partial R$. Since R is β_1-incompressible, R must be a disc disjoint from β_1. By the previous paragraph, R is also disjoint from β_2. This implies that the component of ∂M containing R is a 2-sphere disjoint from e_β and containing a single suture. Since M is β_1 irreducible, this implies that M is a 3-ball disjoint from β_1 and having a single suture in its boundary, a contradiction. Thus, no component of S is a disc disjoint from e_β and $x_{e_\beta}(S) = -\chi(S) + |S \cap e_\beta|$.

Similarly, if S_0 is a component of S then either $x_{\beta_1}(S_0) = -\chi(S_0) + |S_0 \cap \beta_1|$ or S_0 is a disc disjoint from β_1. The component of $R_\pm$ containing ∂S_0 is β_1-incompressible and so must be a disc disjoint from β_1. As before, this implies that M is a 3-ball disjoint from β_1 with a single suture in its boundary. This contradicts our hypotheses and so $x_{\beta_1}(S) = -\chi(S) + |S \cap \beta_1|$. Consequently,

$$x_{e_\beta}(S) = x_{\beta_1}(S) + 1.$$

Since $R_\pm$ is β_1-minimizing, we have

$$x_{\beta_1}(R_\pm) \leq x_{\beta_1}(S).$$

Hence,

$$x_{e_\beta}(R_\pm) - 1 \leq x_{e_\beta}(S) - 1,$$

and so

$$x_{e_\beta}(R_\pm) \leq x_{e_\beta}(S).$$

Since S is e_β-minimizing, $R_\pm$ must be as well.

If $R_\pm$ were e_β-compressible by a compressing disc D, the boundary of D would have to be β_1-inessential in $R_\pm$. Since β_2 has only one endpoint in $R_\pm$, the union of D with a disc contained in $R_\pm$ produces a sphere intersecting β_2 exactly once. Since β_1 and β_2 are parallel, there is a sphere intersecting β_1 exactly once transversally. The components of $R(\gamma)$ containing the endpoints of β_1 are, therefore, β_1-compressible, a contradiction. Thus $R_\pm$ is e_β-incompressible and so (M, γ, β) is band-taut.

□

If (M, γ, β_1) has a parameterizing surface Q_1, the isotopy of β_1 to β_2 gives an isotopy of Q_1 to a parameterizing surface Q_2 for (M, γ, β_2). The next two results give conditions guaranteeing the existence of such an isotopy that does not increase the index of the parameterizing surface. First, we define some notation for the statement of the lemmas.

Let $v_\pm$ be the endpoints of β_1. Let $\alpha_\pm$ be the path from $v_\pm$ to the endpoints of β_2 defined by the isotopy of β_1 to β_2. Let $\gamma_\pm$ be the components of γ intersecting $\alpha_\pm$. Let $n_\pm$ be the number of arc components of ∂Q_1 in a neighborhood of $v_\pm$. Some arc components may belong to edges of $\partial Q_1 \cap R(\gamma)$ parallel to $\alpha_\pm$. Let $m_\pm$ be the number of those arcs plus the number of circle components of $\partial Q \cap \gamma$ parallel to $\gamma_\pm$.

Lemma 9.2. *Assume that any component of $\partial Q_1 \cap R(\gamma)$ intersecting $\alpha_\pm$ is a circle parallel to $\gamma_\pm$. Then there is an isotopy of Q_1 to a parameterizing surface Q_2 for (M,γ,β_2) so that*

$$I(Q_2) = I(Q_1) + (n_- + n_+) - 2(m_- + m_+).$$

Proof. Each arc component of $\partial Q_1 \cap R_\pm$ contributing to $m_\pm$ can be isotoped to lie entirely in $R_\mp$. Each other arc component of $\partial Q_1 \cap R_\pm$ after the isotopy of Q_1 to Q_2 crosses γ an additional time. Any component of $\partial Q \cap R(\gamma)$ intersecting $\alpha_\pm$ can be isotoped across $A(\gamma)$ without changing the index of Q, since such a component is hypothesized to be parallel to $\gamma_\pm$. □

Figure 8 shows an example of an isotopy which decreases index by 1.

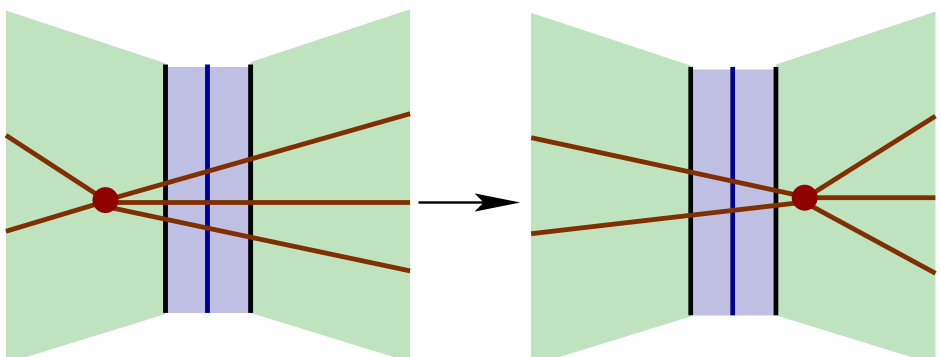

FIGURE 8. An example with $n_\pm = 5$ and $m_\pm = 3$. The isotopy of the endpoint $v_\pm$ across γ reduces the index of the parameterizing surface by 1.

Corollary 9.3. *Suppose that (M,γ,β_1) is a β_1-taut sutured manifold with β_1 an arc having endpoints on annular components of $R(\gamma)$. Suppose also that Q_1 is a parameterizing surface with $\mu(Q_1) \geq 1$ and that those annular components do not contain any inessential arc or circle of $\partial Q_1 \cap R(\gamma)$. Let (M,γ,β) be an associated banded sutured manifold and let Q_2 be a parameterizing surface in (M,γ,β_2) isotopic to Q_1. Then β_2 and Q_2 can be chosen so that $I(Q_1) \geq I(Q_2)$ and $Q_1 \cap \beta_2 = Q_2 \cap \beta_1 = \varnothing$.*

Proof. Let $\rho_\pm$ be the components of $R_\pm$ containing the endpoints of β_1. The surfaces $\rho_\pm - \partial\beta_1$ are thrice punctured spheres. Let $R = \rho_\pm - \partial\beta_1$. By hypothesis, each arc of $\partial Q_1 \cap R$ is an essential arc. In particular, $\partial Q_1 \cap R$ has at most one isotopy class of arcs with both endpoints on a single component of $\partial R(\gamma)$. Choose paths $\alpha_\pm$ from $\partial\beta_1$ to $A_\pm$ disjoint from any arcs with both endpoints on a single component of $\partial R(\gamma)$. If there are no arcs with both endpoints on a single component of $\partial R(\gamma)$, then choose $\alpha_\pm$ to join $\partial\beta_1$ to the component of $\partial\rho_\pm$ containing the greatest number of endpoints of $\partial Q_1 \cap \rho_\pm$. Any arc of $\partial Q_1 \cap R$ having both endpoints at $\partial\beta_1$ forms a loop parallel to both components of $\partial\rho_\pm \cap \partial R(\gamma)$. Hence, we have satisfied the hypotheses of Lemma 9.2. In the notation of that lemma, we have $2m_\pm \geq n_\pm$. Thus, $I(Q_1) \geq I(Q_2)$. A small isotopy makes Q_1 disjoint from β_2 and Q_2 disjoint from β_1. □

We can now use Theorem 8.4 to obtain a theorem for arc taut sutured manifolds where the arc has endpoints in annulus components of $R(\gamma)$.

Theorem 9.4. *Suppose that (M,γ,β) is a β-taut sutured manifold with β a single edge. Let Q be a parameterizing surface in M with $\mu(Q) \geq 1$. Assume that the endpoints of β lie in annulus components $\rho_\pm$ of $R_\pm$ and that no arc or circle of $\partial Q \cap \rho_\pm$ is inessential. Then one of the following is true:*

(1) Q has a compressing or β-boundary compressing disc.
(2) $(M,\beta) = (M'_0,\beta'_0)\#(M'_1,\beta'_1)$ where M'_1 is a lens space and β'_1 is a core of M'_1.
(3) (M,γ) is $\varnothing$-taut. The arc β can be isotoped relative to its endpoints to be embedded on the union of ∂M with a branched surface $B(\mathscr{H})$ associated to a taut sutured manifold hierarchy $\mathscr{H}$ for M. Furthermore, there is a proper isotopy of β in M to an arc disjoint from the first decomposing surface of $\mathscr{H}$. That first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(M,\partial M)$.
(4)
$$I(Q) \geq 2\mu(Q)$$

Proof. Let $e_1 = \beta$. By Corollary 9.3, the endpoints of β_1 can be isotoped across $A(\gamma)$ to create an arc e_2 and an associated banded sutured manifold $(M,\gamma,\widehat{\beta})$. By Lemma 9.1, this sutured manifold is band-taut. By Corollary 9.3, the isotopy can be chosen so that $Q = Q_1$ is isotoped to a surface Q_2 disjoint from e_1 such that $I(Q_2) \leq I(Q_1)$. By a small isotopy, we can make $Q_1 \cap e_2 = Q_2 \cap e_1 = \varnothing$. (The surfaces Q_1 and Q_2 may intersect.) By Theorem 8.4, one of the following happens:

(a) Some Q_i has a compressing or e_i-boundary compressing disc in (M,γ,e_i).
(b) M contains an S^2 intersecting each of e_1 and e_2 exactly once.
(c) For some i, $(M,e_i) = (M'_0,\beta'_0)\#(M'_1,\beta'_1)$ where M'_1 is a lens space and β'_1 is a core of a genus one Heegaard splitting of M'_1.

(d) (M,γ) is $\varnothing$-taut. The arc $c_{\widehat{\beta}}$ can be isotoped relative to its endpoints to be embedded on the branched surface associated to a taut sutured manifold hierarchy for M. Furthermore, there is a proper isotopy of $c_{\widehat{\beta}}$ in M to an arc disjoint from the first decomposing surface of the hierarchy. That first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(M,\partial M)$.

(e) Either
$$I(Q_1) \geq 2\mu(Q_1) \text{ or } I(Q_2) \geq 2\mu(Q_2).$$

Since each (e_i,Q_i) is isotopic to (β,Q), possibility (a) implies conclusion (1) of our theorem. Possibility (b) cannot occur since that would imply that there was a β-compressing disc for $R(\gamma)$. Possibility (c) implies Conclusion (2), since e_i is isotopic to β. Possibility (d) implies Conclusion (3). Possibility (e) implies conclusion (4) since $I(Q) = I(Q_1) \geq I(Q_2)$ and $\mu(Q) = \mu(Q_2) = \mu(Q_1)$. □

We can now prove Theorem 10.7 for the case when the components of $R(\gamma)$ adjacent to b are thrice-punctured spheres. It is really only a slight rephrasing of Theorem 9.4.

Theorem 9.5. *Suppose that (N,γ) is a taut sutured manifold and that $b \subset \gamma$ is a curve adjacent to thrice-punctured sphere components of $R(\gamma)$. Let Q be a parameterizing surface in N with $|Q \cap b| \geq 1$ and with the property that the intersection of Q with the components of $R(\gamma)$ adjacent to b contains no inessential arcs or circles. Let β be the cocore in $N[b]$ of a 2-handle attached along b. Then one of the following is true:*

(1) Q has a compressing or b-boundary compressing disc.
(2) $(N[b],\beta) = (M'_0,\beta'_0)\#(M'_1,\beta'_1)$ where M'_1 is a lens space and β'_1 is a core of a genus one Heegaard splitting of M'_1.
(3) $(N[b],\gamma - b)$ is $\varnothing$-taut. The arc β can be properly isotoped to be embedded on a branched surface $B(\mathscr{H})$ associated to a taut sutured manifold hierarchy $\mathscr{H}$ for $N[b]$. There is also a proper isotopy of β in $N[b]$ to an arc disjoint from the first decomposing surface of $\mathscr{H}$. That first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(N[b],\partial N[b])$.
(4)
$$-2\chi(Q) + |Q \cap \gamma| \geq 2|Q \cap b|.$$

Proof. Let $M = N[b]$. Convert the suture b to an arc β. Since (N,γ) is $\varnothing$-taut, $(M,\gamma - b,\beta)$ is β-taut. The theorem then follows immediately from Theorem 9.4. □

10. Separating Sutures on Genus Two Surfaces

In this section, we prove Theorem 10.7 for the case when b is adjacent to once-punctured tori. The key idea is to create a band-taut sutured manifold by viewing a certain decomposition of the original sutured manifold in three different ways.

We say that a sutured manifold (M,γ,β) is **almost taut** if it satisfies (T1), (T2) from Section 3 and also:

(AT) β is a single edge and either has both endpoints in distinct components of $T(\gamma)$ or has both endpoints in distinct components of $A(\gamma)$.

The strategy is to begin with an arc-taut sutured manifold $M_+ = (M,\gamma,\beta_+)$ where β_+ is an arc having endpoints in distinct torus components of $R(\gamma)$. Convert it to an almost taut sutured manifold $M_0 = (M,\gamma,c_\beta)$ where c_β has endpoints in distinct torus components of $T(\gamma)$, produce a so-called "almost-taut decomposition" of M_0 resulting in an almost taut sutured manifold $M_0' = (M',\gamma',c_\beta')$, convert M_0' to a band-taut sutured manifold (M',γ',β') and then appeal to Theorem 8.4. Along the way we will also have to analyze the behaviour of parameterizing surfaces.

We establish the following notation:

Let $M_+ = (M,\gamma,\beta_+)$ be a sutured manifold, with β_+ an arc having endpoints in torus components $T_- \subset R_-(\gamma)$ and $T_+ \subset R_+(\gamma)$. Let $M_- = (M,\gamma,\beta_-)$ be the sutured manifold resulting from moving T_- into R_+, moving T_+ into R_- and performing a small isotopy of β_+ to an arc β_- disjoint from β_+. Let $M_0 = (M,\gamma,c_\beta)$ be the sutured manifold resulting from moving $T = T_- \cup T_+$ into $T(\gamma)$ and performing a small isotopy of β_+ to an arc c_β that is disjoint from $\beta_+ \cup \beta_-$.

10.1. Preliminary tautness results.

The next lemma is straightforward to prove, and so we omit the proof.

Lemma 10.1. *If M_+ is β_+-taut, then M_0 is almost taut.*

Now suppose that we are given an almost taut sutured manifold $M_0' = (M',\gamma',c_{\beta'})$ with the endpoints of $c_{\beta'}$ in $A(\gamma)$. We create a banded sutured manifold (M',γ',β') as follows. Isotope the endpoints of $c_{\beta'}$ out of $A(\gamma')$ and into $R(\gamma')$ so that one endpoint lies in R_- and the other in R_+. (We require that once the endpoints leave $A(\gamma')$ they do not reenter it during the isotopy.) Since the endpoints of $c_{\beta'}$ lie in distinct components of $A(\gamma')$, up to ambient isotopy of M' (relative to $A(\gamma')$) there are two ways of isotoping $c_{\beta'}$ so that the endpoints lie in $R(\gamma')$. Let $M'_- = (M', \gamma', \beta'_-)$ and $M'_+ = (M', \gamma', \beta'_+)$ denote the two ways of doing this. Perform the isotopies so that $c_{\beta'}$, β'_+, and β'_- are pairwise disjoint. Let β' denote their union, and let $D_{\beta'}$ be an (embedded) disc of parallelism between β'_- and β'_+ that contains $c_{\beta'}$ in its interior. Then (M', γ', β') is a banded sutured manifold. We say that it is a banded sutured manifold **derived** from M'_0. The next lemma gives criteria guaranteeing that the derived sutured manifold is band-taut.

Lemma 10.2. *Suppose that $(M', \gamma', c_{\beta'})$ is a $c_{\beta'}$-almost taut connected sutured manifold and that (M', γ', β') is a derived banded sutured manifold. Suppose that no sphere in M' intersects $c_{\beta'}$ exactly once transversally. If no component of $R(\gamma')$ containing an endpoint of $e_{\beta'} = \beta'_- \cup \beta'_+$ is a disc and if $\chi(R_-) = \chi(R_+)$, then (M', γ', β') is band taut.*

Proof. Since each component of $e_{\beta'}$ is isotopic to $c_{\beta'}$ and since no sphere separates the components of $e_{\beta'}$, $(M', \gamma', e_{\beta'})$ is $e_{\beta'}$-irreducible.

Suppose that $R_\pm$ is $e_{\beta'}$-compressible by a disc D. Since $R_\pm$ is $c_{\beta'}$-incompressible, the boundary of D bounds a disc $D' \subset R_\pm$ containing one or two endpoints of $e_{\beta'}$. If it contains two endpoints, they must be endpoints of different components of $e_{\beta'}$. Then $D \cup D'$ is a sphere in M' intersecting an edge of $e_{\beta'}$ in a single point. Since each edge of $e_{\beta'}$ is isotopic to $c_{\beta'}$, there is a sphere in M' intersecting $c_{\beta'}$ in a single point, contrary to hypothesis.

Let S be a surface representing $[R_\pm, \partial R_\pm]$ in $H_2(M', \partial R_\pm)$ such that:

- S is $e_{\beta'}$-incompressible
- S intersects each edge of $e_{\beta'}$ always with the same sign.

We wish to show that $x_{e_{\beta'}}(R_\pm) \leq x_{e_{\beta'}}(S)$.

Isotope S, relative to ∂S, to minimize the pair $(|D_{\beta'} \cap S|, |c_{\beta'} \cap S|)$ lexicographically. An innermost disc argument shows that S intersects $D_{\beta'}$ in arcs only. An outermost arc argument shows that each of these arcs has an endpoint on ∂M or joins β'_- to β'_+. Since S represents $[R_\pm, \partial R_\pm]$, the algebraic intersection of S with each component of $e_{\beta'}$ is ± 1 and the algebraic intersection of S with $c_{\beta'}$ is zero. The absolute value of the algebraic intersection of S with each edge of $e_{\beta'}$ is equal to the geometric intersection number. Since $|\partial S \cap D_{\beta'}| = 2$, there are two arcs in $S \cap D_{\beta'}$. Since the algebraic intersection number of S with each component of $e_{\beta'}$ is ± 1, each of β'_- to β'_+ is incident to exactly one arc of $S \cap D_{\beta'}$. If an arc of $S \cap D_{\beta'}$ joins β'_- to β'_+, then S would have algebraic intersection number ± 1 with $c_{\beta'}$.

This contradicts the fact that $(S, \partial S)$ is homologous to $(R_\pm, \partial R_\pm)$. Thus, neither arc joins β'_- to β'_+. Similarly, since $S \cap D_{\beta'}$ contains two arcs and since each of β'_- and β'_+ intersects an arc and since they don't intersect the same arc, each arc of $S \cap D_{\beta'}$ joins $\partial M'$ to $e_{\beta'}$. Since S has zero algebraic intersection with $c_{\beta'}$, as in Figure 9, either these arcs are both disjoint from $c_{\beta'}$ or they each intersect $c_{\beta'}$ exactly once.

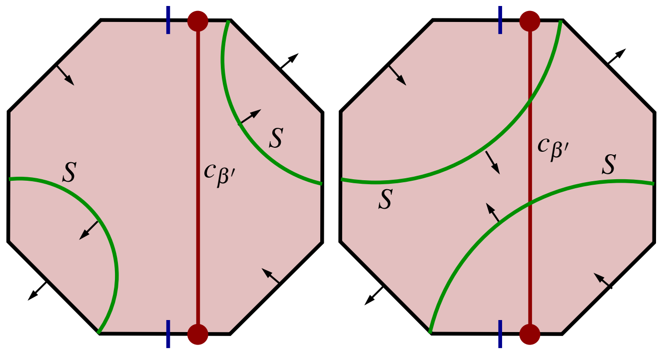

FIGURE 9. The two possible kinds of intersection between S and $D_{\beta'}$ (for the case when S is homologous to R_+).

Case 1: The arcs $S \cap D_{\beta'}$ are disjoint from $c_{\beta'}$.

By the $e_{\beta'}$-irreducibility of M and our hypotheses, we may assume that no component of S is a sphere intersecting $e_{\beta'}$ one or fewer times. Let n_S be the number of components of S that are discs intersecting $e_{\beta'}$ exactly once. Similarly, we may assume that no component of $R(\gamma')$ is a sphere intersecting $e_{\beta'}$ one or fewer times. Recall that no component of $R(\gamma')$ containing an endpoint of $e_{\beta'}$ is a disc. We have

$$x_{e_{\beta'}}(R_\pm) = x_{c_{\beta'}}(R_\pm) + 2$$

and

$$x_{e_{\beta'}}(S) = x_{c_{\beta'}}(R_\pm) + 2 - n_S$$

If a component of S is a disc intersecting $e_{\beta'}$ once, then either it is a $c_{\beta'}$-compressing disc for the component of $R_\pm$ sharing its boundary, or that component is a disc. Since $R_\pm$ is $c_{\beta'}$-taut and since no sphere intersects an edge of $e_{\beta'}$ once, that component of $R_\pm$ must be a disc intersecting $e_{\beta'}$ once, contradicting our hypotheses. Thus, $n_S = 0$. It then follows that since $R_\pm$ is $x_{c_{\beta'}}$-minimizing, $x_{e_{\beta'}}(R_\pm) \leq x_{e_{\beta'}}(S)$. Hence, $R_\pm$ is $e_{\beta'}$-taut.

Case 2: The arcs $S \cap D_{\beta'}$ are not disjoint from $c_{\beta'}$.

Since the endpoints of $c_{\beta'}$ are in different components of $A(\gamma')$, we can isotope S so that ∂S moves across $A(\gamma')$ and so that S is made disjoint from

$c_{\beta'}$. Call the resulting surface S'. We have $\partial S' = \partial R_{\mp}$. The intersection between S' and $D_{\beta'}$ is as in Figure 10. An isotopy of S' relative to $\partial S'$ makes S' disjoint from $c_{\beta'}$.

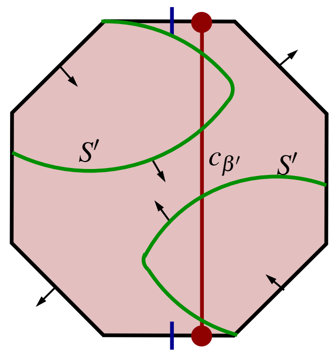

FIGURE 10. The intersection between S' and $D_{\beta'}$ (for the case when S is homologous to R_+).

Let $S'' = S' \cup T(\gamma')$. Since $[S, \partial S] = [R_{\pm}, \partial R_{\pm}]$, we have $[S'', \partial S''] = [R_{\mp}, \partial R_{\mp}]$. We note that S'' is $e_{\beta'}$-incompressible and that it always intersects each edge of $e_{\beta'}$ with the same sign. Consequently, by Case 1 and the fact that R_- and R_+ have the same euler characteristic,

$$x_{e_{\beta'}}(R_{\pm}) = x_{e_{\beta'}}(R_{\mp}) \leq x_{e_{\beta'}}(S'') = x_{e_{\beta'}}(S).$$

Hence, $R_{\pm}$ is $x_{e_{\beta'}}$-minimizing and is, therefore, $e_{\beta'}$-taut.

We have proved that, in either case, $R_{\pm}$ is $e_{\beta'}$-taut. It is easy to show that $T(\gamma')$ is $e_{\beta'}$-taut and, therefore, that $(M, \gamma, e_{\beta'})$ is $e_{\beta'}$-taut. Consequently, $(M', \gamma', e_{\beta'} \cup c_{\beta'})$ is band-taut. $\square$

We say that a sutured manifold decomposition

$$(M, \gamma, c_{\beta}) \xrightarrow{S} (M', \gamma', c_{\beta'})$$

is **almost-taut** if S is disjoint from c_{β} (and so $c_{\beta} = c_{\beta'}$) and both (M, γ, c_{β}) and $(M', \gamma', c_{\beta'})$ are almost taut.

10.2. **Almost taut decompositions.** To create almost taut decompositions, we recall the definition of "Seifert-like" homology class from the introduction: A class $y \in H_2(M, \partial M)$ is **Seifert-like** for the union T of two torus components of ∂M, if the projection of y to the first homology of each component is non-zero. By the "half-lives, half-dies" theorem from algebraic topology, there are non-zero classes in the first homology of each component of T that are the projections of the boundary of classes $y_1, y_2 \in$

$H_2(M, \partial M)$. If neither y_1 nor y_2 is Seifert-like for T, then summing them produces a Seifert-like homology class. Thus, if ∂M has two torus components, there is a class in $H_2(M, \partial M)$ that is Seifert-like for their union. The next two lemmas show how to construct an almost taut decomposition, given a Seifert-like homology class.

Lemma 10.3. *Suppose that (M, γ, c_β) is a c_β-almost taut sutured manifold, with c_β an arc having both endpoints on torus components T of $T(\gamma)$. Let y be a Seifert-like homology class for T. Then there exists a conditioned surface S representing y and disjoint from c_β, such that the double curve sum S_k of S with k copies of $R(\gamma)$ is c_β-taut for any $k \geq 0$. Hence, the decomposition*

$$(M, \gamma, c_\beta) \xrightarrow{S_k} (M', \gamma', c_\beta)$$

is c_β-almost taut for any $k \geq 0$.

Proof. **Claim 1:** There exists a conditioned surface Σ representing y disjoint from c_β.

Standard arguments show that there exists a conditioned surface representing y. Out of all such surfaces, choose one Σ that minimizes $|\Sigma \cap c_\beta|$. By tubing together points of opposite intersection, we may assume that the geometric intersection number of Σ with c_β equals the absolute value of the algebraic intersection number. If this number is non-zero, we may isotope the boundary components of Σ around a simple closed curve on one component of T so as to introduce enough intersections of Σ with c_β of the correct sign so that Σ and c_β have algebraic intersection number zero. This does not change the fact that Σ is conditioned. By tubing together points of opposite intersection, we obtain a surface contradicting our original choice of Σ.

Claim 2: There exists a conditioned surface S representing y that is disjoint from c_β and which has the property that the double curve sum S_k of S with $k \geq 0$ copies of $R(\gamma)$ creates a c_β-taut surface disjoint from c_β.

We apply Theorem 2.5 of [S1] (see page 25). We apply the theorem with $R = R(\gamma)$, $C = \partial\Sigma$, and $y = [\Sigma]$. As noted on page 25, Scharlemann's theorem applies even in the absence of a sutured manifold structure, and so there is no problem with applying it in our situation. Since $R(\gamma)$ is disjoint from c_β, each of the surfaces S_k is disjoint from c_β.

Claim 3: The manifold $(M', \gamma', c_{\beta'})$ obtained by decomposing (M, γ, β), using S_k from Claim 2, is $c_{\beta'}$-almost taut.

Since S_k is disjoint from c_β, we have $c_{\beta'} = c_\beta$. The endpoints of c_β lay in distinct components of $T(\gamma)$, so the endpoints of $c_{\beta'}$ lie in distinct components of $A(\gamma')$. The surface $R(\gamma')$ is the double curve sum of S_k with $R(\gamma)$, i.e. S_{k+1}. Thus, $R(\gamma')$ is $c_{\beta'}$-taut. It follows easily that $(M', \gamma', c_{\beta'})$ is $c_{\beta'}$-almost taut. □

We now show that starting with an arc-taut sutured manifold, converting it to an almost taut sutured manifold, applying an almost-taut decomposition, and then creating a banded sutured manifold can result in a band-taut sutured manifold.

Lemma 10.4. *Suppose that M_+ is β_+-taut and that $y \in H_2(M, \partial M)$ is Seifert-like for T. Let S be a conditioned surface that represents y and that gives an almost taut decomposition:*

$$M_0 \xrightarrow{S} (M', \gamma', c_{\beta'}).$$

Then the banded sutured manifold (M', γ', β') derived from $(M', \gamma', c_{\beta'})$ is band-taut.

Proof. Since M_+ is β_+-taut and since T has one component in $R_-(\gamma)$ and one in $R_+(\gamma)$, $\chi(R_-(\gamma) - T) = \chi(R_+(\gamma) - T)$. Also, since $T \subset M_+$ is β_+-incompressible, no sphere in M intersects $c_{\beta'}$ exactly once transversally. In M', the components of $R(\gamma')$ adjacent to $T \cap M'$ each contain a copy of a component of S, since S had boundary on both components of T. If one of the components of $R(\gamma')$ containing an endpoint of $e_{\beta'}$ is a disc, then some component of S with boundary on T must be a disk. Since S is conditioned and disjoint from c_β, this implies that a component of T is compressible in $M - c_\beta$ and thus in $M - \beta_+$. This contradicts the fact that M_+ is β_+-taut. Therefore, no component of $R(\gamma')$ containing an endpoint of $e_{\beta'}$ is a disc. Thus, by Lemma 10.2, (M', γ', β') is band-taut. □

10.3. **Parameterizing surfaces.** Suppose that M_+ is β_+-taut and that $y \in H_2(M, \partial M)$ is Seifert-like for T. Let S be a conditioned surface representing y and giving an almost taut decomposition:

$$M_0 \xrightarrow{S} (M', \gamma', c_\beta).$$

Let $Q \subset M_+$ be a parameterizing surface.

Lemma 10.5. *Assume that no component of $\partial Q \cap (T - \mathring{\eta}(c_\beta))$ is an inessential arc or inessential circle in $T - c_\beta$. Let T' be a component of $T \cap M'$. The following are true:*

- *$\partial Q \cap T'$ consists of either essential loops in T' or edges joining the components of $\partial T'$ and edges joining an endpoint of c_β to a component of $\partial T'$.*
- *There are equal numbers of edges joining the endpoint of c_β to the two components of $\partial T'$.*

Proof. The lemma follows immediately from the observation that on a component $T_\pm$ of T, each arc of $\partial Q \cap (T_\pm - \mathring{\eta}(c_\beta))$ is an essential loop. Such a loop σ is either disjoint from ∂S or always intersects each component of ∂S with the same sign of intersection. □

We observe that by Lemma 10.1, M_0 is c_β-almost taut. We do not know that M_- is β_--taut. Let S be a conditioned decomposing surface giving an almost taut decomposition $M_0 \xrightarrow{S} (M', \gamma', c_\beta)$. Let (M', γ', β') be the banded sutured manifold derived from $M_0' = (M', \gamma', c_\beta)$. By Lemma 10.4, (M', γ', β') is band-taut. The surface S also gives sutured manifold decompositions of M_+ and M_-, with S disjoint from β_+ and β_- respectively. The resulting sutured manifolds M_-' and M_+' can also be obtained by isotoping the endpoints of c_β' out of $A(\gamma') \subset M'$ and into $R(\gamma) \subset M'$. This gives us the following decompositions:

$$\begin{array}{lcccc} M_+ & \xrightarrow{S} & (M', \gamma', \beta_+') & = & M_+' \\ M_- & \xrightarrow{S} & (M', \gamma', \beta_-') & = & M_-' \end{array}$$

The arcs β_+' and β_-' are obtained by isotoping the arc $c_\beta \subset M'$ so that its endpoints move out of $A(\gamma)$. That is, $\beta_+' \cup \beta_-' = e_{\beta'}$. See Figure 11 for a schematic depiction of the relationship between M_-', M_+', and M_0'.

If $Q_\pm$ is a parameterizing surface in $M_\pm$, then we have the decomposed surfaces $Q_\pm' \subset M_\pm'$. We assume that the ambient isotopy of β_+ to β_- takes Q_+ to Q_- and that $\beta_- \cap Q_+ = \beta_+ \cap Q_- = \varnothing$. We say that the decomposition $M_\pm \xrightarrow{S} M_\pm'$ **respects** Q_+ if Q_-' and Q_+' are parameterizing surfaces.

Lemma 10.6. *Suppose that M_0, $M_\pm$, S, and $Q_\pm$ are as above and that no component of $\partial Q \cap (T - \mathring{\eta}(c_\beta))$ is an inessential arc or inessential circle in $T - c_\beta$. Then, for large enough k, the decompositions*

$$M_- \xrightarrow{S_k} M_-' \quad \text{and} \quad M_+ \xrightarrow{S_k} M_+'$$

respect Q, where S_k is the surface obtained by double-curve summing S with k copies of $R(\gamma) \subset M_0$. If $Q_\pm' \subset M_\pm'$ are the resulting parameterizing surfaces, then $I(Q_\pm') = I(Q)$.

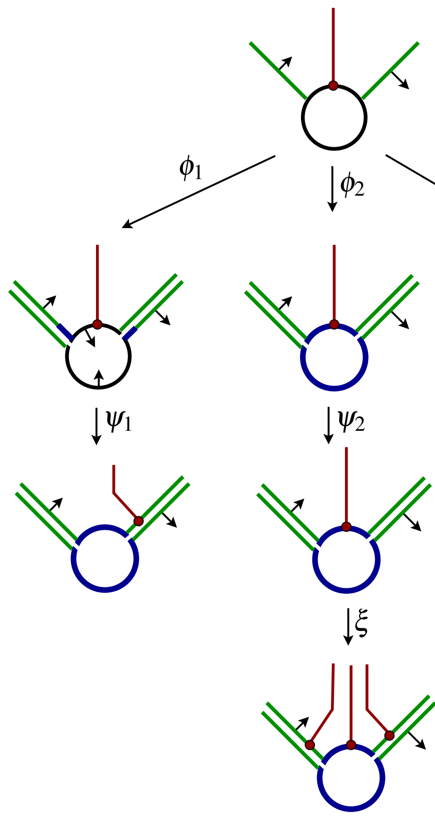

FIGURE 11. This is a schematic depiction of the creation of M'_0, M'_+, M'_-, and the banded sutured manifold (M', γ', β'). The arrows ϕ_1, ϕ_2, and ϕ_3 show the decompositions $M_+ \to M'_+$, $M_0 \to M'_0$ and $M_- \to M'_-$ respectively. The arrows ψ_1 and ψ_3, show how M'_+ and M'_- can be obtained from M'_0 by an isotopy of the sutured manifold structure. The arrow ξ shows how the banded sutured manifold (M', γ', β') is derived from M'_0 and is the result of superimposing the sutured manifolds M'_+, M'_0, and M'_-. In all diagrams, the green lines represent the decomposing surface, blue curves represent annuli $A(\gamma)$, and the circle represents a component of T.

Proof. Fix $k \geq 0$ and let $M'_\pm = (M', \gamma'_\pm, \beta'_\pm)$ be the result of decomposing $M_\pm$ by S_k. Recall that S_k is disjoint from c_β. Since β_-, β_+, and c_β are related by isotopies we may assume that S_k is also disjoint from $\beta_- \cup \beta_+$. Let $M'_0 = (M'_0, \gamma'_0, c'_\beta)$ be the result of decomposing M_0 by S_k. By Lemma 10.5, no component of $\partial Q'_\pm \cap (T \cap M')$ is an inessential loop or arc, and no component has both endpoints on the same boundary component of $T \cap M'$. Thus, if q is a disc component of $Q'_\pm$ having boundary in $R(\gamma'_\pm)$, then either

∂q is disjoint from $T \cap M'$, or q is a $\beta'_\pm$ compressing disk for $T \cap M'$. If the latter happens, then $\partial q \subset \partial Q$. This would imply that q was actually a component of Q that was a β_+-compressing disk for T. This contradicts the fact that M_+ is β_+-taut. Thus, ∂q is disjoint from $T \cap M'$. This implies that $\partial q \subset R(\gamma'_0)$.

The proof of Claim 1 of [S1, Lemma 7.5] shows that for large enough k, no component of $Q'_\pm$ is a disc with boundary in $R(\gamma'_0)$. Hence, $Q'_\pm$ is a parameterizing surface in $M'_\pm$. Claim 2 of [S1, Lemma 7.5] shows that $I(Q') = I(Q)$. □

We can now prove the main result of this paper.

Theorem 10.7. *Suppose that (N, γ) is a taut sutured manifold and that $F \subset \partial N$ is a component of genus at least 2. Let $b \subset \gamma \cap F$ be a simple closed curve such that either each component of $R(\gamma)$ adjacent to b is a thrice punctured sphere or each component of $R(\gamma)$ adjacent to b is a once-punctured torus. Let $M = N[b]$ and let β be the cocore of the 2–handle attached to b. Let $Q \subset N$ be a parameterizing surface. Assume that $|Q \cap b| \geq 1$ and that the intersection of Q with the components of $R(\gamma)$ adjacent to b contains no inessential arcs or circles. Then one of the following is true:*

(1) Q has a compressing or b-boundary compressing disc.
(2) $(N[b], \beta) = (M'_0, \beta'_0) \# (M'_1, \beta'_1)$ where M'_1 is a lens space and β'_1 is a core of a genus one Heegaard splitting of M'_1.
(3) The sutured manifold $(N[b], \gamma - b)$ is $\varnothing$-taut. The arc β can be properly isotoped to be embedded on a branched surface $B(\mathscr{H})$ associated to a taut sutured manifold hierarchy $\mathscr{H}$ for $N[b]$. There is also a proper isotopy of β in $N[b]$ to an arc disjoint from the first decomposing surface of $\mathscr{H}$. If b is adjacent to thrice-punctured sphere components of $R(\gamma)$, that first decomposing surface can be taken to represent $\pm y$ for any given non-zero $y \in H_2(N[b], \partial N[b])$. If b is adjacent to once-punctured tori, the first decomposing surface can be taken to represent y for any given homology class in $H_2(N[b], \partial N[b])$ that is Seifert-like for the corresponding unpunctured torus components of $\partial N[b]$.
(4)
$$-2\chi(Q) + |Q \cap \gamma| \geq 2|Q \cap b|.$$

Proof. By Theorem 9.5, it suffices to prove the statement for the case when b is a separating suture on a genus two surface. Convert b to an arc β_+ (see Section 4.3.3) so that we have the β_+-taut sutured manifold $M_+ = (M, \gamma - b, \beta_+)$. Let T be the components of $R(\gamma - b)$ containing the endpoints of

β_+. Let $y \in H_2(M, \partial M)$ be Seifert-like for T. By the remarks preceding Lemma 10.3, such a y exists. Let $Q_+ = Q$.

Isotope β_+ off itself slightly in two directions to obtain disjoint arcs β_- and c_β. Let $M_0 = (M_0, \gamma, c_\beta)$ be the sutured manifold obtained by moving T into $T(\gamma - b)$ and ignoring $\beta_+ \cup \beta_-$. Let $M_- = (M_-, \gamma, \beta_-)$ be the sutured manifold obtained by swapping the locations of the components of T in $R(\gamma - b)$ and ignoring $\beta_+ \cup c_\beta$. Let Q_- be the parameterizing surface in M_- obtained by isotoping Q_+ using the isotopy taking β_+ to β_-. By a small adjustment of the isotopy, we may assume that $\beta_+ \cap Q_- = \beta_- \cap Q+ = \varnothing$.

Let S be the surface provided by Lemma 10.3, so that the decomposition $M_0 \stackrel{S_k}{\to} M_0'$ is c_β-almost taut for any $k \geq 0$. Choose k large enough so that the decompositions $M_- \stackrel{S_k}{\to} M_-'$ and $M_+ \stackrel{S_k}{\to} M_+'$ respect Q. This is possible by Lemma 10.6. Recall that these decompositions are disjoint from $c_\beta \cup \beta_- \cup \beta_+$, since S_k is obtained by summing S with copies of $R(\gamma) \subset M_0$ (and not $R(\gamma) \subset M_\pm$). Let $Q_1 = Q_+'$ and $Q_2 = Q_-'$ be the resulting parameterizing surfaces in M_+' and M_-' respectively. Note that they are isotopic to each other. By Lemma 10.6, we have $I(Q) = I(Q_-') = I(Q_+')$.

Recall from Lemma 10.4 that the banded sutured manifold (M', γ', β') derived from (M', γ', c_β') is band taut and that the components β_-' and β_+' of e_β are obtained by isotopies of c_β' in M_0'. Let $e_1 = \beta_+'$ and $e_2 = \beta_+'$. By Theorem 8.4 one of the following occurs:

(1) Some Q_i has a compressing or e_i-boundary compressing disc in (M', γ', e_i).
(2) M' contains an S^2 intersecting each of e_1 and e_2 exactly once.
(3) For some i, (M', e_i) has a connect summand that is a lens space and a core.
(4) (M', γ') is $\varnothing$-taut. The arc c_β can be properly isotoped onto a branched surface $B(\mathscr{H}')$ associated to a taut sutured manifold hierarchy for M'.
(5) Either $I(Q_1') \geq 2\mu(Q_1')$ or $I(Q_2') \geq 2\mu(Q_2')$.

If (1) holds, then by Lemma 7.4, Q would have a compressing or b-boundary compressing disc in (M, γ).

If (2) holds, then there is an S^2 in M intersecting β_+ exactly once, contradicting the fact that M_+ is β_+-taut with the endpoints of β_+ in torus components of $R(\gamma) \subset M_+$.

If (3) holds, then since β_+ is isotopic to c_β, there is a (lens space, core) summand of (M, β_+).

If (4) holds, then by Theorem 7.2, since S_k is conditioned (M,γ) is $\varnothing$-taut. By construction the first decomposing surface is disjoint from the arc. Lemma 6.1 shows, in fact, that there is an isotopy of c_β (rel endpoints) to lie on $\partial M' \cup B(\mathscr{H}')$. There is a proper isotopy of c_β in M to lie on $S_k \cup B(\mathscr{H}')$. Thus, there is a branched surface $B(\mathscr{H})$ associated to a taut sutured manifold hierarchy $\mathscr{H}$ for (M,γ) such that there is a proper isotopy of c_β into $B(\mathscr{H})$.

If (5) holds, then since $I(Q) = I(Q_1') = I(Q_2')$ and since $\mu(Q_1) = \mu(Q_2) = \mu(Q)$, we have $-2\chi(Q) + |\partial Q \cap \gamma| \geq 2|\partial Q \cap b|$. □

11. Tunnel Number One Knots

In this section we apply Theorem 10.7 to the study of tunnel number one knots and links. Scharlemann and Thompson [ST, Proposition 4.2], proved that given a tunnel for a tunnel number one knot in S^3, the tunnel can be slid and isotoped to be disjoint from some minimal genus Seifert surface for the knot[1]. We generalize and extend this result in several ways:

- Scharlemann and Thompson's result holds for 2-component tunnel number one links in S^3.
- A similar theorem applies to all tunnel number one knots and 2-component links in any closed, orientable 3–manifold. (Of course, if a 3-manifold contains a tunnel number one knot or link, the 3-manifold has Heegaard genus less than or equal to two.)
- A given tunnel for a tunnel number one knot or link can be properly isotoped to lie on a branched surface arising from a certain taut sutured manifold hierarchy of the knot or link exterior.

We begin with some terminology.

A link C in a closed 3-manifold M is a **generalized unlink** if each component of $\partial(M - \mathring{\eta}(C))$ is compressible in the exterior of C. Suppose that $L_b \subset M$ is a knot or two-component link and that β is an arc properly embedded in the complement of L_b. The arc β is a **tunnel** for L_b if the exterior of $L_b \cup \beta$ is a handlebody. If L_b is a two-component link this implies that β joins the components of L_b. L_b has **tunnel number one** if it has a tunnel and is not a generalized unlink.

[1] It is perhaps worth remarking that [ST, Proposition 4.2] depends on [ST, Lemma 4.1] whose proof relies on sutured manifold theory. Also, we should note, that Scharlemann and Thompson prove, in fact, that in many cases the tunnel can be isotoped onto a minimal genus Seifert surface. We will not address that aspect of their work.

A **generalized Seifert surface** S for a knot or link L_b in a closed manifold M is a compact oriented surface properly embedded in $M - \mathring{\eta}(L_b)$ such that ∂S consists of parallel (as oriented curves) longitudes on each component of $\partial(M - \mathring{\eta}(L_b))$. In particular, ∂S has components on each component of $\partial(M - \mathring{\eta}(L_b))$. If ∂S has a single component on each component of $\partial(M - \mathring{\eta}(L_b))$ then S is a **Seifert surface** for L_b. A generalized Seifert surface is **minimal genus** if it has minimal genus among all generalized Seifert surfaces in the same homology class.

Theorem 11.1. *Suppose that $L_b \subset M$ has tunnel number one and that β is a tunnel for L_b. Assume also that $(M - L_b, \beta)$ does not have a (lens space, core) summand. Then there exist (possibly empty) curves $\widehat{\gamma}$ on $\partial(M - \mathring{\eta}(L_b))$ such that $(M - \mathring{\eta}(L_b), \widehat{\gamma})$ is a taut sutured manifold and the tunnel β can be properly isotoped to lie on the branched surface associated to a taut sutured manifold hierarchy of $(M - \mathring{\eta}(L_b), \widehat{\gamma})$. In particular, if L_b has a (generalized) Seifert surface, then there exists a minimal genus (generalized) Seifert surface for L_b that is disjoint from β.*

Proof. Let $W = \eta(L_b \cup \beta)$ and let $N = M - \mathring{W}$ be the complementary handlebody. Let $H = \partial W$. Let $b \subset H$ be a simple closed curve that is a meridian of β, so that the exterior $N[b]$ of L_b can be obtained by attaching a 2–handle to ∂N along b. The tunnel β is a cocore of that 2–handle.

Claim: $H - b$ is incompressible in N.

Proof of Claim. If b is compressible in N, then (W, N) is a reducible Heegaard splitting for M. Since boundary reducing a handlebody creates a handlebody, L_b must be a generalized unlink. Suppose that D is a compressing disc for $H - b$. If b is separating, then ∂D must be an essential curve in one of the punctured torus components of $H - b$. Compressing that component using D creates a compressing disc for b in N. Thus, b cannot be separating. If b is non-separating then either L_b is a generalized unlink or ∂D is an inessential curve in $\partial N[b]$. In the latter case, ∂D bounds an essential disc in W (obtained by banding together two copies of the disc in W bounded by b), so once again (W, N) is a reducible Heegaard splitting for M and C must be a generalized unlink. Thus, $H - b$ is incompressible in N. $\square$

Let $Q \subset N$ be a pair of properly embedded non-parallel non-separating essential discs, chosen so as to intersect b minimally. As a consequence of the claim, no component of Q is disjoint from b. By the minimality of $|\partial Q \cap b|$, each component of $Q \cap (H - b)$ is an essential arc.

If there were a b-boundary compressing disc D for a component Q_0 of Q, then boundary compressing Q_0 using D results in two discs, each intersecting b fewer times than does Q with at least one of them a compressing disc for H in N. Thus, by the minimality of the intersection between ∂Q and b, Q has no b-boundary compressing disc.

If b is separating, choose $\widehat{\gamma} = \varnothing$. If b is non-separating, we want to choose essential curves $\widehat{\gamma} \subset H - \mathring{\eta}(b)$ with the following properties:

(1) $\widehat{\gamma}$ consists of two essential simple closed curves that are parallel in $\partial N[b]$ and which separate the components of $\partial \eta(b)$.
(2) Each arc component of $Q \cap (H - \mathring{\eta}(b))$ is an arc intersecting $\widehat{\gamma}$ zero or one times.

To see that this can be done, recall that the surface $H' = H - \mathring{\eta}(b)$ is a twice-punctured torus and that $Q \cap H'$ is a collection of essential arcs. We describe how to find $\widehat{\gamma}$ if each component of $Q \cap H'$ joins the components of $\partial \eta(b)$. We leave the other case as an exercise. There are at most four disjoint non-parallel essential isotopy classes $c_1, \ldots, c_4$ of arcs in $\partial Q \cap H'$. An essential simple closed curve γ_1 can be chosen that is disjoint from representatives of two of the arcs (say c_1 and c_2) and that intersects representatives of the other two classes in a single point each. Let γ_2 be a second copy of γ_1, isotoped to be disjoint from γ_1. In $\partial N[b]$, push a sub-arc of γ_2 along arcs of $Q - \gamma_1$ until it crosses an endpoint of β. Then γ_2 intersects c_3 and c_4 exactly once and is disjoint from c_1 and c_2. By isotoping $\widehat{\gamma} = \gamma_1 \cup \gamma_2$ in H' to intersect ∂Q minimally we obtain the desired curves. See Figure 12 for a schematic depiction of the four isotopy classes of arcs and the sutures γ_1 and γ_2.

It is now easy to verify that $(N, \widehat{\gamma} \cup b)$ is a taut sutured manifold and that $|\partial Q \cap \widehat{\gamma}| \leq |\partial Q \cap b|$. Since $-2\chi(Q) = -4$, it is impossible that

$$-2\chi(Q) + |Q \cap (\widehat{\gamma} \cup b)| \geq 2|Q \cap b|$$

Consequently, by Theorem 10.7, β can be isotoped to lie on a branched surface associated to a taut sutured manifold hierarchy of $(N[b], \widehat{\gamma})$.

If L_b has a (generalized) Seifert surface, choose $y \in H_2(N[b], \partial N[b])$ to be a class represented by (generalized) Seifert surfaces for L_b. The first surface S in the sutured manifold hierarchy constructed in the proof of Theorem 10.7 is a conditioned surface representing $\pm y$ that is taut in the Thurston norm of $N[b]$ and is disjoint from β. If Σ is a minimal genus (generalized) Seifert surface for L_b representing $\pm y$, then Σ can be isotoped to have the same boundary as S and (possibly after spinning around $\partial N[b]$ and changing orientation) is homologous to S in $H_2(N[b], \partial S)$. Since S has minimal

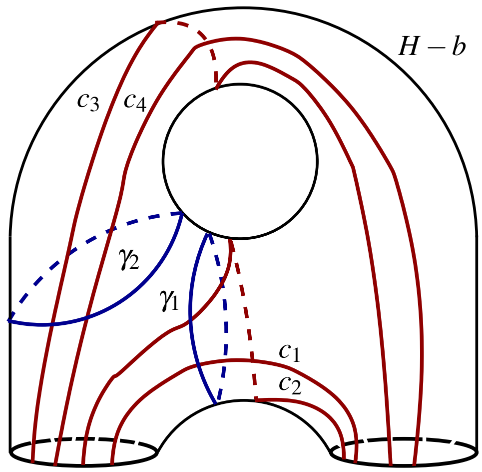

FIGURE 12. The possible isotopy classes of arcs of $\partial Q \cap (H-b)$ (up to homeomorphism of $H-b$) and the sutures γ_1 and γ_2 chosen to intersect those isotopy classes nicely.

Thurston norm among all such surfaces, it is a minimal genus (generalized) Seifert surface for L_b disjoint from β. □

Scharlemann-Thompson's result follows immediately:

Corollary 11.2 (Scharlemann–Thompson)**.** *Suppose that K is a tunnel number one knot or link in S^3 with tunnel β then β can be isotoped to be disjoint from a minimal genus Seifert surface for K.*

REFERENCES

[CC] J. Cantwell and L. Conlon 'The sutured Thurston norm.' `arxiv/0606534`

[G1] D. Gabai 'Foliations and the topology of 3-manifolds.' *J. Differential Geom.* 18 (1983), no. 3, 445–503.

[G2] D. Gabai 'Foliations and the topology of 3-manifolds. II.' *J. Differential Geom.* 26 (1987), no. 3, 461–478.

[G3] D. Gabai 'Foliations and the topology of 3-manifolds. III.' *J. Differential Geom.* 26 (1987), no. 3, 479–536.

[K] E. Kalfagianni ' Surgery n-triviality and companion tori.' *J. Knot Theory Ramifications* 13 (2004), no. 4, 441–456.

[L1] M. Lackenby, 'Surfaces, surgery and unknotting operations.' *Math. Ann.* 308 (1997) 4.

[L2] M. Lackenby, 'Dehn surgery on knots in 3-manifolds.' *J. Amer. Math. Soc.* 10 (1997) 4.

[L3] M. Lackenby, 'Upper bounds in the theory of unknotting operations.' *Topology* 37 (1998), no. 1, 63–73.

[S1] M. Scharlemann, 'Sutured manifolds and generalized Thurston norms.' *J. Differential Geom.* 29 (1989) 3.

[S2] M. Scharlemann, ‘Producing reducible 3–manifolds by surgery on a knot.’*Topology.* 29 (1990) 4.

[ST] M. Scharlemann and A. Thompson, ‘Unknotting tunnels and Seifert surfaces.’ *Proc. London Math. Soc.* 87 (2003), 3 no. 2, 523–544.

[T1] S. Taylor, ‘Boring split links and unknots.’ Dissertation. University of California, Santa Barbara. September, 2008.

[T2] S. Taylor ‘Comparing 2–handle additions to a genus 2 boundary component’. arXiv:0806.1572

E-mail address: `sataylor@colby.edu`