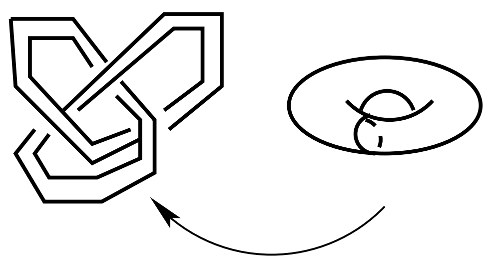
COMBINATORIAL SUTURED MANIFOLD THEORY: PAST AND PRESENT

SCOTT A TAYLOR

ABSTRACT. Gabai's sutured manifold theory has produced many stunning results in knot theory. I will give a brief introduction to Scharlemann's combinatorial version of sutured manifold theory and will then survey some applications, beginning with a seminal theorem of Gabai and ending with more recent work.

1. INTRODUCTION

Theorem 1.1 (Lickorish-Wallace)**.** *Every closed, orientable 3-manifold can be obtained by Dehn surgery on a link in S^3.*



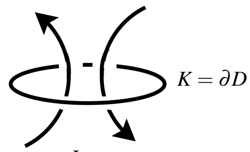
Question: What can we say about 3-manifolds that are related by surgery on a **knot**? What properties do they have in common? What properties might they not have in common?

Theorem 1.2 (Gabai)**.** *Suppose that surgery on a knot $K \subset S^3$ produces $S^1 \times S^2$. Then K is the unknot and the surgery slope is* 0*.*

Studying this question can also provide answers to basic questions about knots:

Theorem 1.3 (Scharlemann)**.** *If L is a composite knot in S^3, then L cannot be unknotted by a single crossing change.*

These are notes for a talk given in the University of Iowa Mathematics Colloquium on April 23, 2009.



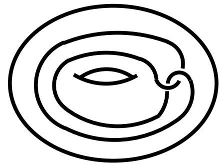
Trying to understand the relationship between knots and 3-manifolds provides a nice excuse for introducing some terminology.

If N is an orientable 3-manifold and $S \subset N$ is an embedded orientable surface, then S is **incompressible** if $\pi_1(S) \hookrightarrow \pi_1(N)$. S is **boundary-parallel** if S bounds a product region $S \times I$ with a subsurface of ∂N. $S \subset N$ is **essential** if it is incompressible, not boundary-parallel, and not an S^2 bounding a 3–ball.

Example If $L \subset S^3$ is a composite knot then $S^3 - \mathring{\eta}(L)$ has an essential annulus. If $L \subset S^3$ is the unknot then $S^3 - \mathring{\eta}(L)$ has an essential disc. Thus, Scharlemann's theorem says that (in the given context) ± 1 surgery on K cannot turn a manifold with an essential (meridional) annulus into a manifold with an essential disc.

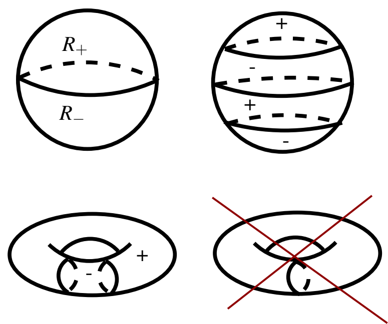
Gabai's theorem was one of the triumphs of his sutured manifold theory. Gabai created sutured manifold theory in order to build interesting foliations of 3-manifolds [G1]. Applying his version of sutured manifold theory to solve other theorems often required the use of deep theorems from foliation theory. Scharlemann found a way to remove sutured manifold theory's dependence on foliation theory creating a purely combinatorial sutured manifold theory [S]. In this talk I will pick one of Gabai's theorems which has turned out to be tremendously important. I will show you Scharlemann's proof of it using combinatorial sutured manifold theory and then will discuss some more recent theorems inspired by this seminal result. The statement given here is significantly weaker than what Gabai actually proves. The theorem is

Theorem 1.4 (Gabai 1987)**.** *Suppose that $V = S^1 \times D^2$ is a solid torus and that $K \subset V$ is a knot of winding number 0. Assume that K is not contained in a 3-ball and that there are no essential tori in its exterior. Then only the trivial surgery on K will produce a solid torus.*

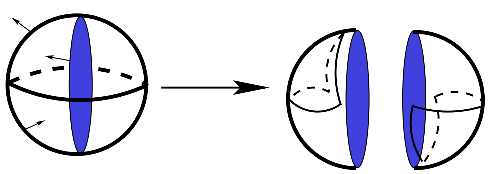


2. Sutured Manifold Theory

A sutured manifold is a compact, orientable 3-manifold N with oriented simple closed curves γ on its boundary. The curves γ are called "sutures". We require that $\partial N - \gamma$ consist of two surfaces $R_+(\gamma)$ and $R_-(\gamma)$ so that $\partial R_+(\gamma) = \partial R_-(\gamma) = \gamma$. $R_+(\gamma)$ is given the orientation with outward pointing normal vector and $R_-(\gamma)$ is given the orientation with inward pointing normal vector. Here are some pictures of sutured manifolds.

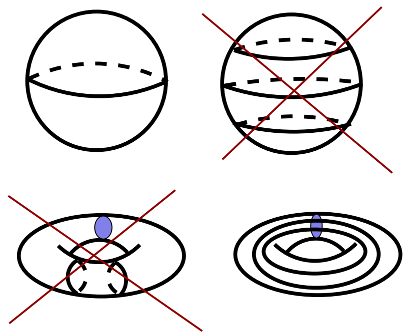


If $S \subset N$ is an oriented surface (transverse to γ), we can cut N open along S to get a new sutured manifold (N', γ'):

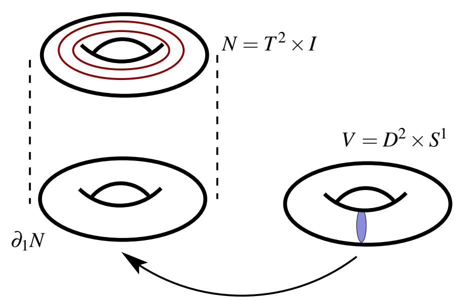


Sutured manifold theory arguments (typically) make extensive use of hierarchies, a concept with a long history in 3-manifold topology. The basic idea is that we chop our manifold up into pieces. Examine the pieces and reassemble, keeping track of what happens along the way. The usual tool in sutured manifold theory is a **taut sutured manifold hierarchy**.

Suppose that $S \subset N$ is a properly embedded, oriented compact surface. If S is connected, its **Thurston norm** is $\max\{0, -\chi(S)\}$. S is **norm-minimizing**, if out of all embedded surfaces representing $[S, \partial S] \in H_2(N, \partial S)$, S has minimal Thurston norm. S is **taut** if it incompressible and norm-minimizing. The sutured manifold (N, γ) is **taut** if $R_\pm(\gamma)$ are taut and if every embedded S^2 in N is the boundary of a 3-ball in N. Here are some examples of taut and not-taut manifolds:



Here is another important example. Consider $N = T^2 \times I$ with sutures γ on one boundary component $\partial_0 N$ and no sutures on the other boundary component $\partial_1 N$.



Notice that (N,γ) is taut. Gluing a solid torus V to $\partial_1 N$ creates a new sutured manifold $(\widehat{N},\gamma)$. In fact, $\widehat{N}$ is a solid torus with sutures on its boundary. Thus, there is exactly one way of gluing V to $\partial_1 N$ to create a non-taut sutured manifold.

We can now state the two fundamental theorems of sutured manifold theory, both originally due to Gabai.

Theorem $\downarrow$. *Every taut sutured manifold (N_0,γ_0) has a taut sutured manifold hierarchy*

$$(N_0,\gamma_0) \xrightarrow{S_1} (N_1,\gamma_1) \xrightarrow{S_2} (N_2,\gamma_2) \xrightarrow{S_3} \dots \xrightarrow{S_n} (N_n,\gamma_n).$$

The hierarchy stops when every surface in N_n separates. That is, when $H_2(N_n,\partial N_n) = 0$. In particular, every boundary component of N_n is a 2–sphere.

This is usually used in conjunction with

Theorem $\uparrow$. *Suppose that*

$$(N_0,\gamma_0) \xrightarrow{S_1} (N_1,\gamma_1) \xrightarrow{S_2} (N_2,\gamma_2) \xrightarrow{S_3} \dots \xrightarrow{S_n} (N_n,\gamma_n).$$

is a sequence of sutured manifold decompositions such that **conditions** *then if (N_n, γ_n) is taut, so is (N_0, γ_0).*

3. Gabai's Theorem

Theorem 3.1 (Gabai)**.** *Suppose that $K \subset V$ is a knot of winding number zero which is not contained in a 3–ball. Suppose that every torus in the exterior of K is inessential. Then only the trivial surgery on K will produce a non-taut manifold.*

Let $N = V - \mathring{\eta}(K)$ and $\gamma = \varnothing$. Notice that since ∂N consists of tori and since K is not contained in a 3–ball, (N, γ) is taut. By Theorem ↓ there exists a taut sutured manifold hierarchy

$$(N,\gamma) \xrightarrow{S_1} (N_1,\gamma_1) \xrightarrow{S_2} (N_2,\gamma_2) \xrightarrow{S_3} \dots \xrightarrow{S_n} (N_n,\gamma_n).$$

for (N, γ). However, construct this hierarchy so that none of the surfaces S_i intersect $\partial\eta(K)$. This means that the hierarchy will stop when $H_2(N_n, \partial N_n - \partial\eta(K)) = 0$. That is, when every non-separating surface in intersects $\partial\eta(K)$. This homology condition and the fact that (N_n, γ_n) is taut implies that every component but one of N_n is a 3-ball with a single suture. The other component has two torus boundary components $\partial\eta(K)$ and one other one T. Since K had winding number zero in V, $T \neq \partial V$. Thus, T has sutures γ' on it. The hypothesis that N contains no essential tori implies that the component N' of N_n containing T is $T \times I$. Attaching a solid torus to $\partial\eta(K)$ in N' creates a solid torus V'. Since $\gamma_n \cap T \neq \varnothing$ all but one way of attaching the solid torus creates a taut sutured manifold. Thus, by Theorem ↑, all but one way of performing Dehn surgery on K in N produces a non-taut manifold. □

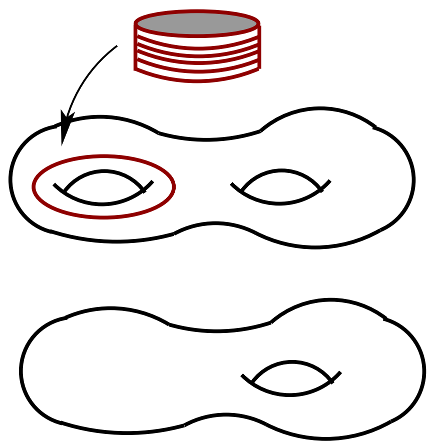
4. More Recent Work

This theorem of Gabai's inspired a remarkable theorem of Lackenby. The proof of Lackenby's theorem is very complicated and so I won't discuss the proof. But I would be remiss if I did not bring your attention to this result. I will state a somewhat simplified version of this result.

Theorem 4.1 (Lackenby 1997)**.** *Let M be a compact, connected, orientable 3-manifold such that every $S^2 = \partial B^3$. Let $K \subset M$ be a knot which does not lie in a 3-ball. Suppose that $H_2(M - \mathring{\eta}(K), \partial M) \neq 0$. Let r be an essential curve on $\partial\eta(K)$ and let Δ denote the minimal number of times r intersects a meridian of $\eta(K)$. Then there exists a constant $C(M)$ such that if Dehn surgery on K with slope r produces a manifold having an S^2 which does not bound a homology ball then $\Delta \leq C(M)$.*

I'd like to talk about an even more recent result, which is actually much easier to prove than Lackenby's result. This result uses sutured manifold theory to draw conclusions about the effects of 2–handle addition, a generalization of Dehn surgery.

Let $a \subset \partial N$ be an essential simple closed curve. A 3–manifold $N[a]$ can be formed by attaching a 2–handle $D^2 \times I$ to $\eta(a)$. If a and b are two curves on ∂N, let $\Delta = \Delta(a,b)$ be the minimal number of intersections between a and b (up to isotopy).



Theorem 4.2 (Taylor)**.** *Suppose that N is a compact orientable 3-manifold and that $F \subset \partial N$ is a genus 2 component. Assume that N does not contain an essential sphere, disc, annulus, or torus. Let $a,b \subset F$ be non-isotopic essential separating simple closed curves. If $N[a]$ has an essential sphere and if $N[b]$ has an essential sphere, disc, annulus, or torus then $\Delta = 4$, and $N[b]$ has an essential annulus with one boundary component on a component of $\partial N[b]$ with genus at least 2.*

Since all of the 3-manifolds under consideration have boundary, Thurston's geometrization theorem for Haken manifolds implies (roughly speaking) that the property of having a complete hyperbolic structure is equivalent to there not being any essential sphere, discs, annuli, or tori.

A separating curve is an example of what Scharlemann and Wu [SW] call a "basic curve". They prove that if N does not have an essential sphere, disc, annulus, or torus, if one of a and b is basic, if $N[a]$ has an essential S^2 and if $N[b]$ has an essential disc then a and b can be isotoped to be disjoint. (The curves a and b are allowed to be on components of ∂N of genus greater than 2.) They conjecture that if both a and b are basic and if both $N[a]$ or $N[b]$ contain an essential sphere, disc, annulus, or torus then $\Delta \leq 5$. This theorem give some evidence for their conjecture.

Both this theorem and Lackenby's theorem use another piece of sutured manifold technology: the parameterizing surface. Here is a sketch of this last theorem. I discussed other applications of these techniques in yesterday's topology seminar.

The manifold (N,a) is a taut sutured manifold. Take a taut sutured manifold hierarchy which is always disjoint from $\eta(a)$. (Proving that such a hierarchy exists takes some work.) If attaching a 2–handle to a at the end of the hierachy produces a taut sutured manifold, we should be able to conclude that $(N[a],\varnothing)$ is taut, contradicting our hypothesis that $N[a]$ has an essential S^2. This should follow from Theorem $\uparrow$, but it takes some work to prove that we can apply Theorem $\uparrow$. We do end up concluding that attaching a 2–handle to a at the end of the hierarchy produces a non-taut manifold.

Suppose that $\overline{Q} \subset N[b]$ is an essential sphere, disc, annulus, or torus. Let $Q = \overline{Q} \cap N$ and watch how Q is chopped up during the hierarchy. Q is an example of a parameterizing surface. There is a number called the index associated to a parameterizing surface which never increases during a hierarchy. Using this number and some combinatorial arguments the proof can be concluded.

REFERENCES

[G1] Gabai, David. Foliations and the topology of 3-manifolds. *J. Differential Geom.* 18 (1983), no. 3, 445–503.

[G2] Gabai, David. Foliations and the topology of 3-manifolds. II. *J. Differential Geom.* 26 (1987), no. 3, 461–478.

[SW] Scharlemann, Martin and Wu, Ying Qing. Hyperbolic manifolds and degenerating handle additions. *J. Austral. Math. Soc. Ser. A* 55 (1993), no. 1, 72–89.

[L] Lackenby, Marc. Dehn surgery on knots in 3-manifolds. *J. Amer. Math. Soc.* 10 (1997), no. 4, 835–864.

[S] Scharlemann, Martin. Sutured manifolds and generalized Thurston norms. *J. Differential Geom.* 29 (1989), no. 3, 557–614.

[T] Taylor, Scott. 'Adding 2–handles to sutured manifolds' preprint.

E-mail address: `sataylor@colby.edu`