# APPROACHING GREEN'S THEOREM VIA RIEMANN SUMS 

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#### Abstract

We give a proof of Green's theorem which captures the underlying intuition and which relies only on the mean value theorems for derivatives and integrals and on the change of variables theorem for double integrals.


## 1. Introduction

The counterpoint of the discrete and continuous has been, perhaps even since Euclid, the essence of many mathematical fugues. Despite this, there are fundamental mathematical subjects where their voices are difficult to distinguish. For example, although early Calculus courses make much of the passage from the discrete world of average rate of change and Riemann sums to the continuous (or, more accurately, smooth) world of derivatives and integrals, by the time the student reaches the central material of vector calculus: scalar fields, vector fields, and their integrals over curves and surfaces, the voice of discrete mathematics has been obscured by the coloratura of continuous mathematics. Our aim in this article is to restore the balance of the voices by showing how Green's Theorem can be understood from the discrete point of view.

Although Green's Theorem admits many generalizations (the most important undoubtedly being the Generalized Stokes’ Theorem from the theory of differentiable manifolds), we restrict ourselves to one of its simplest forms:

Green's Theorem. Let $S \subset \mathbb{R}^{2}$ be a compact surface bounded by (finitely many) simple closed piecewise $C^{1}$ curves oriented so that $S$ is on their left. Suppose that $\mathbf{F}=\binom{M}{N}$ is a $C^{1}$ vector field defined on an open set $U$ containing $S$. Then

$$
\int_{\partial S} \mathbf{F} \cdot d \mathbf{s}=\iint_{S} \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} d A
$$

By a $\mathbf{C}^{1}$ curve we mean a curve which can be parameterized by a function $\gamma:[a, b] \rightarrow$ $\mathbb{R}^{2}$ such that $\gamma^{\prime}$ is continuous and everywhere non-zero. A curve is piecewise $\mathbf{C}^{1}$ if it can be parameterized by a continuous function $\gamma:[a, b] \rightarrow \mathbb{R}^{2}$ such that the interval $[a, b]$ can be subdivided as

$$
a=x_{0}<x_{1}<\ldots<x_{n}=b
$$

with the restriction of $\gamma$ to each interval $\left[x_{i}, x_{i+1}\right]$ being a $\mathrm{C}^{1}$ curve. When we say that $\mathbf{F}$ is a $\mathbf{C}^{1}$ vector field, we mean that $M, N: U \rightarrow \mathbb{R}$ are differentiable and have continuous derivatives.

Although Green's theorem has many important consequences (including its versions for regions in higher dimensions) in a first course on vector calculus the most significant consequence is that it allows for the operator Curl to be given an easily understood geometric interpretation as the rate of rotation of an infinitessimal paddlewheel in a vector field describing fluid flow. Indeed, it is fair to call the integrand

$$
\operatorname{scurl} \mathbf{F}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}
$$

of the integral on the right in the statement of Green's theorem, the scalar curl of the vector field $\mathbf{F}$.

The proofs of Green's theorem presented in most vector calculus books (e.g. in [C, MT]) first use rather tedious and unenlightening calculations to prove Green's theorem in the case when $S$ is a so-called Type III region (this is a region which contains all vertical and horizontal line segments having endpoints in the region). The standard proofs then use the Sum Property of regions satisfying the conclusion of Green's theorem to extend Green's theorem to surfaces which can be decomposed into Type III regions. A hand-waving appeal to "limit arguments" gives the version of Green's theorem stated above.

Apart from the hand-waving (which is often good pedagogy in introductory texts) we object to the proofs on two other grounds. First, the unenlightening calculations give no indication as to why Green's theorem should hold. Second the approach forces the logic lying behind the intuition of Curl to be delayed unacceptably long. The classic text [S] and others attempt to rectify this by "proving" Green's theorem via arguments which are difficult to make rigorous in an appealing way for beginning vector calculus students. (For example, although scalar curl can be defined as a limit of certain integrals, it is not at all obvious that the limit exists.) Apostol, in the first (but not second) edition of his classic book [A, Theorem 10.43] provides one of the most complete treatments of Green's Theorem. His version relies on a rather difficult decomposition of the surface into finitely many so-called Type I and Type II regions. Although our approach (unlike Apostol's) does not give a version of Green's Theorem as general as that stated above, we do show how some relatively simple combinatorics and linear algebra combine with the intuition of Riemann Sums and the Change of Variables Theorem to give a very general version of Green's Theorem.

## 2. Scalar Curl

Our proof of Green's Theorem begins by using the traditional mean value theorems for derivatives and integrals to formalize the intuition of scalar curl as an infinitessimal rate of circulation.

Mean Value Theorem for Rectangles. Suppose that $\mathbf{F}=\binom{M}{N}$ is a differentiable vector field defined on a solid rectangle $R \subset \mathbb{R}^{2}$ of positive area with sides parallel to the $x$ and $y$ axes. Then there exist points $\mathbf{x}, \mathbf{y} \in R$ such that

$$
\frac{1}{\operatorname{area}(R)} \int_{\partial R} \mathbf{F} \cdot d \mathbf{s}=\frac{\partial N}{\partial y}(\mathbf{y})-\frac{\partial M}{\partial x}(\mathbf{x})
$$

We observe that if either $M$ or $N$ is the zero function, then the term on the right is the scalar curl of $\mathbf{F}$. Also, if we take a sequence of rectangles $R_{n}$ converging to a point $\mathbf{a}$ then, if $\mathbf{F}$ is of class $\mathrm{C}^{1}$, the term on the right converges to $\operatorname{scurl} \mathbf{F}(\mathbf{a})$. Hence, we have shown that scalar curl is an infinitessimal rate of circulation without using Green's theorem. Of course, we wonder if we can dispense with requiring our regions to be rectangles. The best way to do that does seem to be to first prove Green's theorem (which we do using this Mean Value Theorem).

Proof of the Mean Value Theorem for Rectangles. Suppose that $R=[a, b] \times[c, d]$. Parameterize the top and bottom sides of $\partial R$ as $(t, c)$ and $(t, d)$ for $a \leq t \leq b$, respectively. Parameterize the left and right sides of $\partial R$ as $(a, t)$ and $(b, t)$ for $c \leq t \leq d$ respectively. Note that our parameterizations of the top and left sides of $\partial R$ have the opposite orientations from that induced by $\partial R$. Using the standard definition and properties of integrals over curves, we have:

$$
\begin{equation*}
\frac{1}{\operatorname{area}(R)} \int_{\partial R} \mathbf{F} \cdot d \mathbf{s}=\frac{-1}{b-a} \int_{a}^{b} \frac{M(t, d)-M(t, c)}{d-c} d t+\frac{1}{d-c} \int_{c}^{d} \frac{N(b, t)-N(a, t)}{b-a} d t \tag{1}
\end{equation*}
$$

Since $M$ and $N$ are continuous, the integrands are continuous. By the Mean Value Theorem for Integrals, there exists $\left(x_{0}, y_{0}\right) \in R$ so that the right side of Equation (1) equals

$$
\begin{equation*}
-\frac{M\left(x_{0}, d\right)-M\left(x_{0}, c\right)}{d-c}+\frac{N\left(b, y_{0}\right)-N\left(a, y_{0}\right)}{b-a} \tag{2}
\end{equation*}
$$

Since the functions $M\left(x_{0}, \cdot\right)$ and $N\left(\cdot, y_{0}\right)$ are differentiable on the intervals $[c, d]$ and $[a, b]$ respectively, by the Mean Value Theorem for Derivatives, there exists $\left(x_{1}, y_{1}\right) \in R$ so that Expression (2) equals

$$
-\frac{\partial M}{\partial y}\left(x_{0}, y_{1}\right)+\frac{\partial N}{\partial x}\left(x_{1}, y_{0}\right)
$$

Letting $\mathbf{x}=\left(x_{0}, y_{1}\right)$ and $\mathbf{y}=\left(x_{1}, y_{0}\right)$, we have our mean value theorem.

## 3. Green's Theorem for Sums of Surfaces

We say that the pair $(S, \mathbf{F})$ satisfies Green's theorem if

$$
\int_{\partial S} \mathbf{F} \cdot d \mathbf{s}=\iint_{S} \operatorname{scurl} \mathbf{F} d A
$$

and that $S$ satisfies Green's Theorem if $(S, \mathbf{F})$ satisfies Green's theorem for all $\mathrm{C}^{1}$ vector fields $\mathbf{F}$ on $S$. In this section, we present standard material which allows us to show that a surface satisfies Green's Theorem if it can be decomposed into pieces which do.

Recall that if $\kappa$ is an oriented piecewise $\mathrm{C}^{1}$ curve and if $-\kappa$ denotes $\kappa$ with the opposite orientation then

$$
\begin{equation*}
\int_{\kappa} \mathbf{F} \cdot d \mathbf{s}=-\int_{-\kappa} \mathbf{F} \cdot d \mathbf{s} \tag{3}
\end{equation*}
$$

We can use this simple observation to relate integrals over the boundary of a region to sums of integrals over subregions. Suppose that that a compact surface $S_{0} \subset \mathbb{R}^{2}$ is subdivided by piecewise $\mathrm{C}^{1}$ curves into compact subsurfaces $S_{1}, \ldots, S_{n}$. For each $i$, orient $\partial S_{i}$ so that $S_{i}$ is on the left as $\partial S_{i}$ is traversed, as in Figure 1.


Figure 1. $S_{0}$ is subdivided into surfaces $S_{i}$ for $1 \leq i \leq 6$

Consider a continuous vector field $\mathbf{F}$ on $S_{0}$. For each $i \neq 0, \partial S_{i}$ is divided into piecewise $\mathrm{C}^{1}$ curves. Let $\gamma$ be one of them and observe that, by Equation (3), either both $\int_{\gamma} \mathbf{F} \cdot d \mathbf{s}$ and $-\int_{\gamma} \mathbf{F} \cdot d \mathbf{s}$ show up in the sum $\sum_{i=1}^{n} \int_{\partial S_{i}} \mathbf{F} \cdot d \mathbf{s}$ or $\gamma \subset \partial S_{0}$. Consequently,

$$
\begin{equation*}
\int_{\partial S_{0}} \mathbf{F} \cdot d \mathbf{s}=\sum_{i=1}^{n} \int_{\partial S_{i}} \mathbf{F} \cdot d \mathbf{s} \tag{4}
\end{equation*}
$$

Since double integrals are additive over regions intersecting in a set of measure 0 , we have

Sum Property. Suppose that $S_{0}$ is a compact surface with piecewise $C^{1}$ boundary and that it has been subdivided by piecewise $C^{1}$ curves into subsurfaces $S_{i}$ for $1 \leq i \leq n$. If $\mathbf{F}$ is a $C^{1}$ vector field on $S_{0}$ such that for each $1 \leq i \leq n$, the pair $\left(S_{i}, \mathbf{F}\right)$ satisfies Green's theorem, then the pair $\left(S_{0}, \mathbf{F}\right.$ also satisfies Green's Theorem.

## 4. Green's Theorem for Rectangles

With our mean value theorem for rectangles, we can establish Green's Theorem for rectangles. Let $R=[a, b] \times[c, d]$ be a rectangle in $\mathbb{R}^{2}$ with sides parallel to the $x$ and $y$ axes and let $\mathbf{F}=\binom{M}{N}$ be a $C^{1}$ vector field defined on $R$.
It is easiest to proceed by considering separately the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{F}$. Let $\mathbf{F}^{\mathbf{i}}=\binom{M}{0}$ and let $\mathbf{F}^{\mathbf{j}}=\binom{0}{N}$, so that $\mathbf{F}=\mathbf{F}^{\mathbf{i}}+\mathbf{F}^{\mathbf{j}}$. By the additivity properties of scalar curl, integrals over curves, and double integrals if $\left(R, \mathbf{F}^{\mathbf{i}}\right)$ and $\left(R, \mathbf{F}^{\mathbf{j}}\right)$ satisfy Green's Theorem, then $(R, \mathbf{F})$ does as well. Henceforth, let $\mathbf{F}^{*}$ be either $\mathbf{F}^{\mathbf{i}}$ or $\mathbf{F}^{\mathbf{j}}$.
Subdivide the intervals $[a, b]$ and $[c, d]$ like so:

$$
\begin{aligned}
& a=x_{0}<x_{1}<\ldots<x_{n}=b \\
& c=y_{0}<y_{1}<\ldots<y_{n}=d
\end{aligned}
$$

so that all of the rectangles $R_{i j}=\left[x_{i}, x_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$ have the same length $\Delta x$ and the same height $\Delta y$. (This is not strictly necessary but makes the exposition easier.) By Equation (4)

$$
\int_{\partial R} \mathbf{F}^{*} \cdot d \mathbf{s}=\sum_{i, j} \int_{\partial R_{i j}} \mathbf{F}^{*} \cdot d \mathbf{s}
$$

The summands on the right each measure the circulation of $\mathbf{F}^{*}$ around the boundary of a rectangle and so by the mean value theorem for rectangles, in each $R_{i, j}$ there exists $\mathbf{x}_{i j}$ such that $\int_{\partial R_{i j}} \mathbf{F} \cdot d \mathbf{s}=\operatorname{scurl} \mathbf{F}\left(\mathbf{x}_{i j}\right) \Delta x \Delta y$. Consequently,

$$
\int_{\partial R} \mathbf{F} \cdot d \mathbf{s}=\sum_{i, j} \operatorname{scurl} \mathbf{F}\left(\mathbf{x}_{i j}\right) \Delta x \Delta y
$$

The sum on the right hand side is a Riemann sum (see Figure 2 for an example). Since $\mathbf{F}$ is $\mathbf{C}^{1}$ and scurl $\mathbf{F}$ is continuous,

$$
\int_{\partial R} \mathbf{F} \cdot d \mathbf{s}=\lim _{(\Delta x, \Delta y) \rightarrow(0,0)} \sum_{i, j} \operatorname{scurl} \mathbf{F}\left(\mathbf{x}_{i j}\right) \Delta x \Delta y=\iint_{R} \operatorname{scurl} \mathbf{F} d A .
$$

We've established:
Green's Theorem for Rectangles. If $R$ is a rectangle with sides parallel to the $x$ and $y$ axes then $R$ satisfies Green's Theorem.

By the Sum Property, we also deduce that Green's Theorem holds for any compact surface $S \subset \mathbb{R}^{2}$ whose boundary is the union of finitely many horizontal and vertical line segments. Such surfaces are not necessarily Type III surfaces, although they are, of course, finite unions of Type III surfaces.

Taking stock of where we are: we've done more work than is traditional but accomplished less since we do not even have Green's Theorem for all Type III surfaces. In the next section we use the Change of Variables Theorem and some linear algebra to rectify this. On the other hand, our proof of Green's Theorem incorporates


Figure 2. A rectangle with the points $\mathbf{x}_{i j}$ in each subrectangle $R_{i j}$
both the intuition behind scalar curl and the sum property and is not burdened with overly tiresome calculations.

## 5. Bending our Surfaces

The derivative of a differentiable vector field $\mathbf{F}=\binom{M}{N}$ is the $2 \times 2$ matrix

$$
D \mathbf{F}=\left(\begin{array}{ll}
\partial M / \partial x & \partial M / \partial y \\
\partial N / \partial x & \partial N / \partial y
\end{array}\right)
$$

Noting that the terms of scurl $\mathbf{F}$ can be found in this matrix, we define the scalar curl of any $2 \times 2$ matrix to be

$$
\operatorname{scurl}\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=c-b
$$

In particular, $\operatorname{scurl} \mathbf{F}=\operatorname{scurl} D \mathbf{F}$. An easy computation shows that, for $2 \times 2$ matrices $A$ and $B$,

$$
\begin{equation*}
\operatorname{scurl}\left(B^{T} A B\right)=(\operatorname{scurl} A) \operatorname{det} B \tag{5}
\end{equation*}
$$

where $B^{T}$ is the transpose of $B$ and and $\operatorname{det} B$ is the determinant of $B$.
Equation (5) allows us to study the relationship between surfaces related by certain kinds of distortions. Consider compact, connected surfaces $\widehat{S}$ and $S$ in $\mathbb{R}^{2}$, each bounded by piecewise $\mathrm{C}^{1}$ curves. Suppose that $H: \widehat{S} \rightarrow S$ is a continuous bijection of class $\mathrm{C}^{2}$ (i.e. all second partial derivatives exist and are continuous) and with the property that $\operatorname{det} D H$ is non-zero on $\widehat{S}$. We call $H$ a diffeomorphism from $\widehat{S}$ to $S$. Since $H$ is $\mathrm{C}^{2}$, the entries of $D H$ are continuous. Thus, the $\operatorname{sign} \varepsilon$ of $\operatorname{det} D H$ is either always positive or always negative. If it is the former, we say that $H$ is orientation-preserving; and if the latter, that $H$ is orientation-reversing.

Example 5.1. Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be $C^{2}$ functions with the property that $f(x)<g(x)$ for all $x$. Let $a<b$ and $c<d$ be real numbers. Define

$$
H(x, y)=\left(x, \frac{g(x)-f(x)}{d-c}(y-c)+f(x)\right)
$$

Then $H: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ is a $C^{2}$ function with the property that $\operatorname{det} D H(x, y)>0$ for all $(x, y)$. Let $\widehat{S}=[a, b] \times[c, d]$ and let $S=H(\widehat{S})$. Then the restriction of $H$ to $\widehat{S}$ is an orientation preserving diffeomorphism from the rectangle $\widehat{S}$ to $S$.

If we choose

$$
f(x)= \begin{cases}x^{3} \sin (1 / x) & x \neq 0 \\ 0 & x=0\end{cases}
$$

and $g(x)=f(x)+1$, then $H$ is a diffeomorphism from the square $\widehat{S}=[0,1] \times[0,1]$ to a region whose boundary has infinitely many oscillations (Figure 3). This region cannot be subdivided into finitely many Type III regions.


Figure 3. The square $\widehat{S}$ is on the left and the distorted surface $S$ is on the right. The oscillations in $\partial S$ have been exaggerated for effect.

Given a $C^{1}$ vector field $\mathbf{F}$ on $S$, we can "pull" it back to a $C^{1}$ vector field $\widehat{\mathbf{F}}$ on $\widehat{S}$ defined by

$$
\widehat{\mathbf{F}}=D H^{T}(\mathbf{x}) \mathbf{F}(H(\mathbf{x}))
$$

for all $\mathbf{x} \in \widehat{S}$. The vector field $\widehat{\mathbf{F}}$ is essentially the vector field $\mathbf{F}$ moved to the surface $\widehat{S}$ and adjusted to account for the distortion caused by $D H$. Figure 4 shows an example where the surfaces are related by a rotation.
We seek to show that $S$ satisfies Green's Theorem if and only if $\widehat{S}$ does. We begin by examining the relationship between the line integrals.
Let $\widehat{\gamma}:[a, b] \rightarrow S$ be a $C^{1}$ curve, and let $\gamma=H \circ \widehat{\gamma}$. By definition, we have

$$
\int_{\widehat{\gamma}} \widehat{F} \cdot d \mathbf{s}=\int_{a}^{b}\left(D H^{T}(\gamma(t)) \mathbf{F}(\gamma(t))\right) \cdot \widehat{\gamma}^{\prime}(t) d t
$$



Figure 4. The square $\widehat{S}$ (on the left) and the diamond $S$ (on the right) and their vector fields $(1,0)$ and $(1 / \sqrt{2}, 1 / \sqrt{2})$ respectively are related by the diffeomorphism which rotates the plane by $\pi / 4$ radians clockwise.

By the chain rule [B, Theorem 8.15],

$$
\widehat{\gamma}^{\prime}(t)=\frac{d}{d t} H^{-1}(\gamma(t))=D H^{-1}(\gamma(t)) \gamma^{\prime}(t) .
$$

Since for any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2}, \mathbf{v} \cdot \mathbf{w}=\mathbf{v}^{T} \mathbf{w}$ and since for any two $2 \times 2$ matrices $A$ and $B,(A B)^{T}=B^{T} A^{T}$, we have

$$
\left(D H^{T}(\gamma(t)) \mathbf{F}(\gamma(t))\right) \cdot \widehat{\gamma}^{\prime}(t)=\mathbf{F}(\gamma(t)) \cdot \gamma^{\prime}(t) .
$$

Consequently,

$$
\begin{equation*}
\int_{\widehat{\gamma}} \widehat{\mathbf{F}} \cdot d \mathbf{s}=\int_{\gamma} \mathbf{F} \cdot d \mathbf{s} . \tag{6}
\end{equation*}
$$

If $\widehat{\gamma}:[a, b] \rightarrow \mathbb{R}^{2}$ is a $C^{1}$ parameterization of a portion of $\partial \widehat{S}$, then $\gamma=H(\widehat{\gamma})$ is a $\mathrm{C}^{1}$ parameterization of a portion of $\partial S$, possibly with the wrong orientation. It will have the wrong orientation if and only if $H$ is orientation-reversing. Applying Equation (6) to the $\mathrm{C}^{1}$ portions of $\partial \widehat{S}$, we see that

$$
\begin{equation*}
\int_{\partial \widehat{S}} \widehat{\mathbf{F}} \cdot d \mathbf{s}=\varepsilon \int_{\partial S} \mathbf{F} \cdot d \mathbf{s} . \tag{7}
\end{equation*}
$$

We now consider the relationship between the scalar curl of $\widehat{\mathbf{F}}$ and the scalar curl of $\mathbf{F}$. Suppose that $H(x, y)=\left(H_{1}(x, y), H_{2}(x, y)\right)$ and let

$$
Q_{1}=\left(\begin{array}{ll}
\frac{\partial^{2} H_{1}}{\partial x^{2}} & \frac{\partial^{2} H_{1}}{\partial \partial x} \\
\frac{\partial^{2} H_{1}}{\partial x \partial y} & \frac{\partial^{2} H_{1}}{\partial y^{2}}
\end{array}\right) \quad \text { and } \quad Q_{2}=\left(\begin{array}{ll}
\frac{\partial^{2} H_{2}}{\partial x^{2}} & \frac{\partial^{2} H_{2}}{\partial y \partial x} \\
\frac{2^{2} H_{2}}{\partial x \partial y} & \frac{\partial^{2} H_{2}}{\partial y^{2}}
\end{array}\right) .
$$

Notice that by the equality of mixed second partial derivatives [B, Theorem 8.24], $\operatorname{scurl} Q_{1}=\operatorname{scurl} Q_{2}=0$.

A computation shows that $D \widehat{\mathbf{F}}$ is equal to

$$
M(H) Q_{1}+N(H) Q_{2}+\left(D H^{T}\right) D(\mathbf{F}(H))
$$

The scalar curl for matrices is linear and so

$$
\operatorname{scurl} \widehat{\mathbf{F}}=\operatorname{scurl}\left(D H^{T} D(\mathbf{F}(H))\right)=\operatorname{scurl}\left(D H^{T} D \mathbf{F}(H) D H\right)
$$

where we've used the chain rule to obtain the second equality. Thus, by Equation (5), $\operatorname{scurl} \widehat{\mathbf{F}}=(\operatorname{scurl} \mathbf{F}) \operatorname{det} D H$ and so

$$
\begin{equation*}
\operatorname{scurl} \widehat{\mathbf{F}}=\varepsilon(\operatorname{scurl} \mathbf{F})|\operatorname{det} D H| \tag{8}
\end{equation*}
$$

We are now in a position to apply the change of variables theorem from multivariable calculus. Without loss of generality, we may assume that $S$ and $\widehat{S}$ are connected (if not, do what follows for each component). By the change of variables theorem [M, Theorem 17.1] (applied to the interiors of $S$ and $\widehat{S}$ ), for any $\mathrm{C}^{1}$ function $f: S \rightarrow \mathbb{R}$ :

$$
\iint_{S} f d A=\iint_{\widehat{S}} f \circ H|\operatorname{det} D H| d A
$$

Letting $f=\operatorname{scurl} \mathbf{F}$ and applying Equation (8), we obtain:

$$
\begin{equation*}
\varepsilon \iint_{S} \operatorname{scurl} \mathbf{F} d A=\iint_{\widehat{S}} \operatorname{scurl} \widehat{\mathbf{F}} d A \tag{9}
\end{equation*}
$$

Combining Equations (7) and (9) results in:
Equivalence under Diffeomorphism. The pair (S,F) satisfies Green's theorem if and only if the pair $(\widehat{S}, \widehat{\mathbf{F}})$ does. In particular, if $\widehat{S}$ satifies Green's Theorem then $S$ also does.

Together with Green's Theorem for Rectangles and the Sum Property, this shows that if $S$ is any surface which can be decomposed by piecewise $\mathrm{C}^{1}$ curves into subsurfaces each of which is diffeomorphic (by a $C^{2}$ diffeomorphism) to a rectangle then $S$ satisfies Green's Theorem. By the classification of surfaces up to piecewise $\mathrm{C}^{2}$ diffeomorphism, it follows that Green's theorem holds for all compact surfaces $S \subset \mathbb{R}^{2}$ with piecewise $C^{2}$ boundary (i.e. surfaces with boundary having parameterizations with continuously differentiable and non-vanishing first derivatives) and all $C^{1}$ vector fields $\mathbf{F}$. Rather than delving into these details, however, we content ourselves with noticing that the surface $S$ from Example 5.1 satisfies Green's Theorem. Since $S$ cannot be decomposed into Type III regions we have finally succeeded in surpassing the traditional approach.

## 6. Acknowledgments

A version of the Equivalence Theorem appears in the online supplement to [HHGM], however the absence of linear algebra makes it appear rather mysterious. Apostol's proof of [A, Theorem 11.36] is similar to our proof of the Equivalence Theorem. It would be surprising if the Mean Value Theorem for Rectangles were genuinely
new, but we have not been able to find it in the literature. In an expanded version [BPT], we develop the combinatorial ideas present in our proof of Green's Theorem and use them to give a brief introduction to simplicial cohomology and deRham cohomology. We thank Otto Bretscher and Fernando Gouvêa for helpful conversations and Colby College for supporting the work of the first two authors.

## REFERENCES

[A] Tom M. Apostol, Mathematical analysis: a modern approach to advanced calculus, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1957. MR0087718 (19,398e)
[B] Andrew Browder, Mathematical analysis, Undergraduate Texts in Mathematics, SpringerVerlag, New York, 1996. An introduction. MR1411675 (97g:00001)
[BPT] Jennie Buskin, Philip Prosapio, and Scott A. Taylor, The 3 stooges of vector calculus and their impersonators: A viewer's guide to the classic episodes, available at arXiv: 1301. 1937.
[C] Susan J. Colley, Vector Calculus, Pearson, Boston, MA, 2011.
[HHGM] Deborah Hughes-Hallett, Andrew M. Gleason, and William G. McCallum, Calculus: Single and Multivariable, 2013.
[MT] Jerrold Marsden and Anthony Tromba, Vector Calculus, W.H. Freeman, 2003.
[M] James R. Munkres, Analysis on manifolds, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1991. MR1079066 (92d:58001)
[S] H.M. Schey, Div, Grad, Curl and All That: An Informal Text on Vector Calculus, W.W. Norton \& Company, Redwood City, CA, 2005.

