Knots In Blue

Scott A. Taylor

What do you see when you look at the painting *In Blue*?

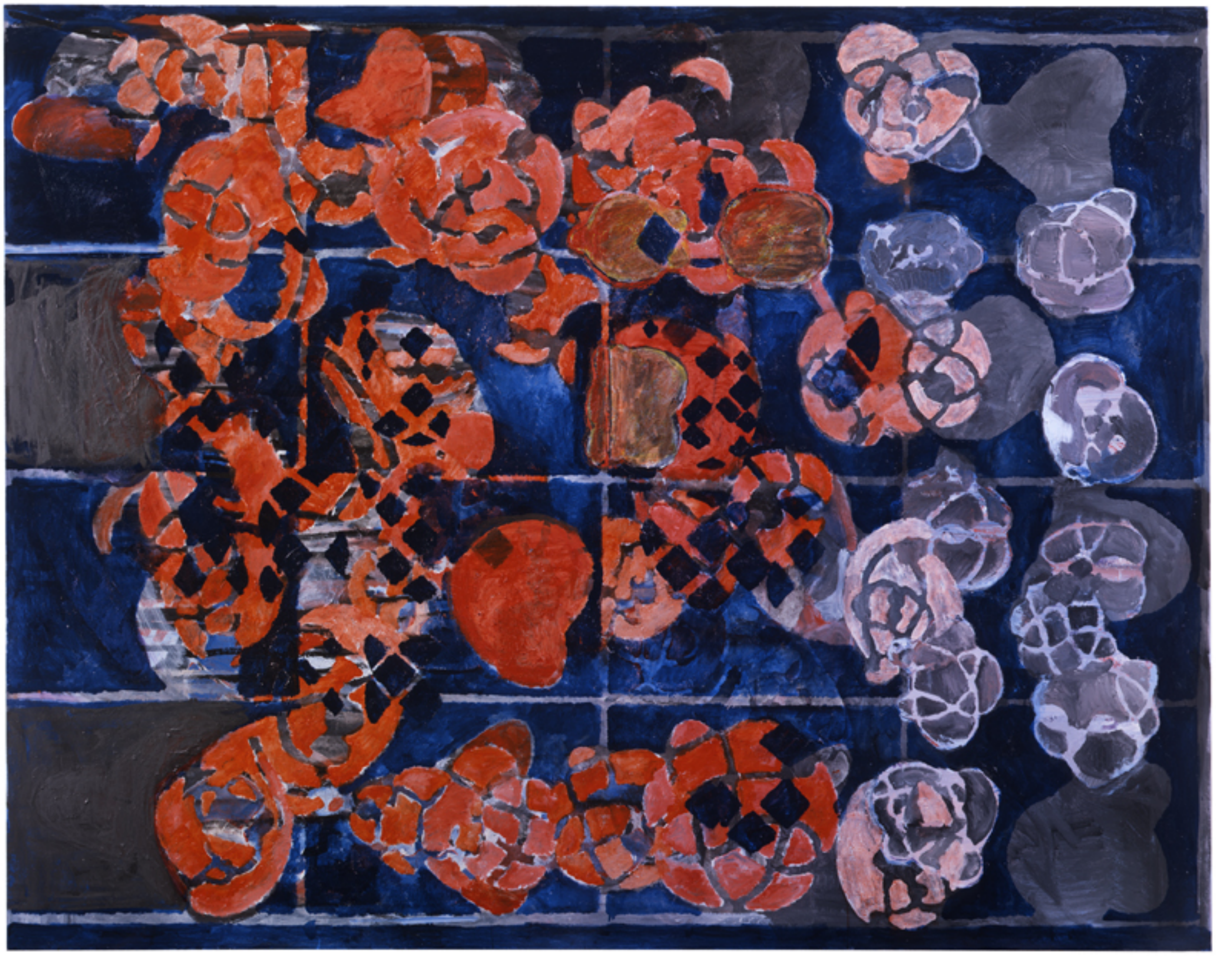

Figure 1 ***In Blue*, 2008. Oil on linen, 88 x 112 inches (223.5 x 284.5 cm). Colby College Museum of Art, The Lunder Collection.**

My 3-year old son sees scary faces; a colleague in psychology responds, "I don't get it!" One of the wonderful properties of good art is that it elicits different and varied reactions from its viewers. Nevertheless, by learning about a branch of mathematics originating from an outlandish 19th century theory of matter, we can find firm footing for understanding and responding to *In Blue*.

Terry Winters

Terry Winters, the artist, has long been inspired by concepts and images from the natural sciences, engineering, and mathematics. Knot theory has provided particular inspiration in the last few years. Winters created series in both print and paint with titles directly referencing the subject. Below are the second print from the series *Secret Knots* and the fourth painting from the series *Knotted Graphs*.

Figure 2: *Secret Knots / 2*, 2008. Portfolio of 10 intaglios with photogravure and spit-bite aquatint, 17 1/4 x 20 1/4 inches (sheet size) (43.8 x 51.4 cm). Ed: CP (Colby proof). Colby College Museum of Art, Gift of the artist.

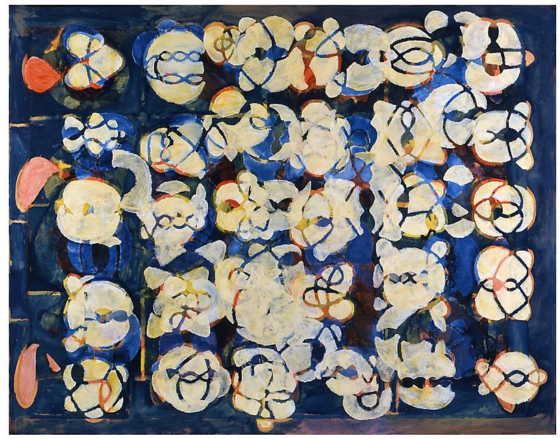

Figure 3: *Knotted Graphs/4*, 2008. Oil on linen, 77 x 98 inches (195.6 x 248.9 cm). Collection Keith and Kathy Sachs.

The print series *Secret Knots*, the painting series *Knotted Graphs*, and the painting *In Blue* each contain images that appear to be curves intersecting themselves and these curves appear in some relation to a grid. In *Secret Knots/2* and *Knotted Graphs/4*, the images themselves form the grid. In *In Blue,* the grid lies in the background and the curves appear to float above it. What are these curves and do they and the grid have any mathematical significance? To answer those questions, we visit Lord Kelvin and Peter Guthrie Tait in Victorian England.

Figure 5: William Thomson, Lord Kelvin (1824-1907).

Figure 4: Peter GuthrieTait (1831-1901).

Knot Theory

In the 1860s, scientists did not have a clear conception of what matter is or what it is made of. Although most everybody believed in the existence of atoms, they did not know what these atoms were. The scientific luminary William Thomson (Lord Kelvin), in an effort to explain all of the physical properties of matter by its movement, proposed the “vortex atom”. In essence, a vortex atom was tube of moving ether. The physical and chemical properties of these vortex atoms were supposedly determined by how they were knotted or linked with each other. Nowadays, we no longer believe in the existence of the ether and we have better models of the atom then Lord Kelvin’s, but we are indebted to him for inspiring the mathematical theory of knots.

Peter Guthrie Tait, a colleague and collaborator of Lord Kelvin’s, was much taken with the theory of vortex atoms and started a systematic listing, or “enumeration”, of knots. His goal was to create a “table of knots”: a table that gave each knot a name and picture (called a “knot diagram”), and listed each knot exactly once. There are infinitely many different knots, so the table would necessarily be incomplete, but Tait made a good start.

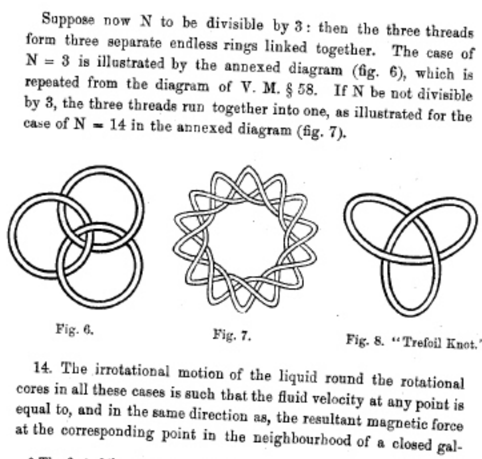
Suppose now N to be divisible by 3: then the three threads form three separate endless rings linked together. The case of N = 3 is illustrated by the annexed diagram (fig. 6), which is repeated from the diagram of V. M. § 58. If N be not divisible by 3, the three threads run together into one, as illustrated for the case of N = 14 in the annexed diagram (fig. 7).

Fig. 6. Fig. 7. Fig. 8. “Trefoil Knot.”

14. The irrotational motion of the liquid round the rotational cores in all these cases is such that the fluid velocity at any point is equal to, and in the same direction as, the resultant magnetic force at the corresponding point in the neighbourhood of a closed gal-

Figure 6: Knotted vortices from Kelvin's paper *Vortex Statics* (1875)

Here is his method for how to create a knot with 5 crossings (that is, a knotted loop having a diagram where the loop crosses itself 5 times). Begin by drawing five vertices, each with 4 edges protruding. Join the protruding edges without crossing to form a planar graph having 5 vertices, each of degree 4. (We call the graph a “degree 4 planar graph”.) There are multiple ways of doing this, and the different ways often give rise to different knots. Figure 7 shows the initial 5 vertices and Figures 8 and 9 show two different ways of forming degree 4 planar graphs with 5 vertices.

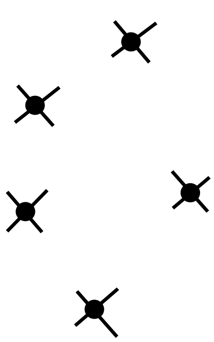

Figure 7: Five vertices, each of degree 4.

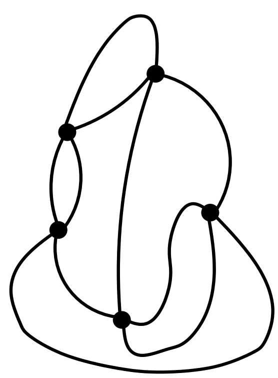

Figure 8: A degree 4 planar graph.

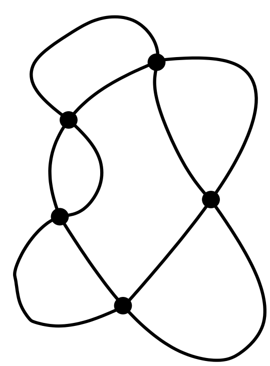

Figure 9: Another degree 4 planar graph.

Now to make a knot (or link) from a degree 4 planar graph, change each vertex of the graph into a crossing. Figure 10 shows the result of changing one vertex of the graph from Figure 9 into a crossing and Figure 11 shows the result of one way of changing all the vertices into crossings.

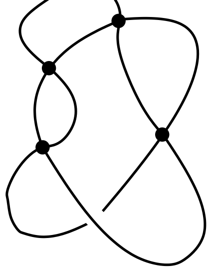

Figure 10: The bottom vertex of the graph in Figure 9 has been changed into a crossing.

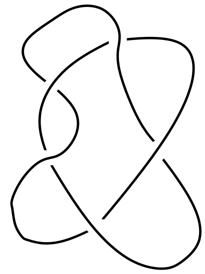

Figure 11: All the vertices of the graph in Figure 9 have been changed into crossings. This knot is an example of an alternating knot.

In summary, the method is this: for each natural number n:

1. Draw all degree 4 planar graphs with n vertices
2. For each of those graphs, draw all possible ways of turning the vertices into crossings.

If you change the vertices into crossings so that each strand travels alternately over and other the strands it crosses, you create what is called an "alternating knot". A knot that can be drawn without any crossings (eg. a circle in the plane) is called an "unknot" or "trivial knot". A knot is "prime" if, roughly speaking, it cannot be created by gluing together two non-trivial knots. William Tait used this method to list all prime, alternating knots with 7 or fewer crossings. Later, using ideas of Kirkman and Little, Tait extended the table to list all such knots with 10 or fewer crossings. Tait organized his alternating knots into a table and gave each knot a label. One version of his table is pictured in Figure 12. Notice that since he is only listing alternating knots he does not need to draw the crossings: he has drawn only the degree 4 planar graph.

The knot we created in Figure 12 received the label 5C, where the 5 denotes the fact that it has 5 crossings and the C denotes the fact that it was the third knot with 5 crossings that he created. The first knot with 5 crossings (the one that should be denoted 5A) is not listed, since it also has a diagram with fewer crossings. (As an aside, it should be noted that the challenge of proving that two diagrams don't represent the same knot is a very challenging one and has inspired much knot theory research.) Nowadays, we give the knot in Figure 12 the label 5_2, since it is the second (prime) knot with 5 crossings listed in the table. Even though this is the knot notation (or should that be *knotation*?) most commonly used, it doesn't convey much information about the knot – after all there's nothing special about the fact that our knot is the *second* knot with five crossings. So, if you invent the perfect labeling scheme, knot theorists will beat a path to your door! Despite the imperfect notation, knot tables are very useful. My particular favorite is the online *Table of Knot Invariants* by Livingston and Cha. There you can explore the mathematical properties of knots with up to 12 crossings.

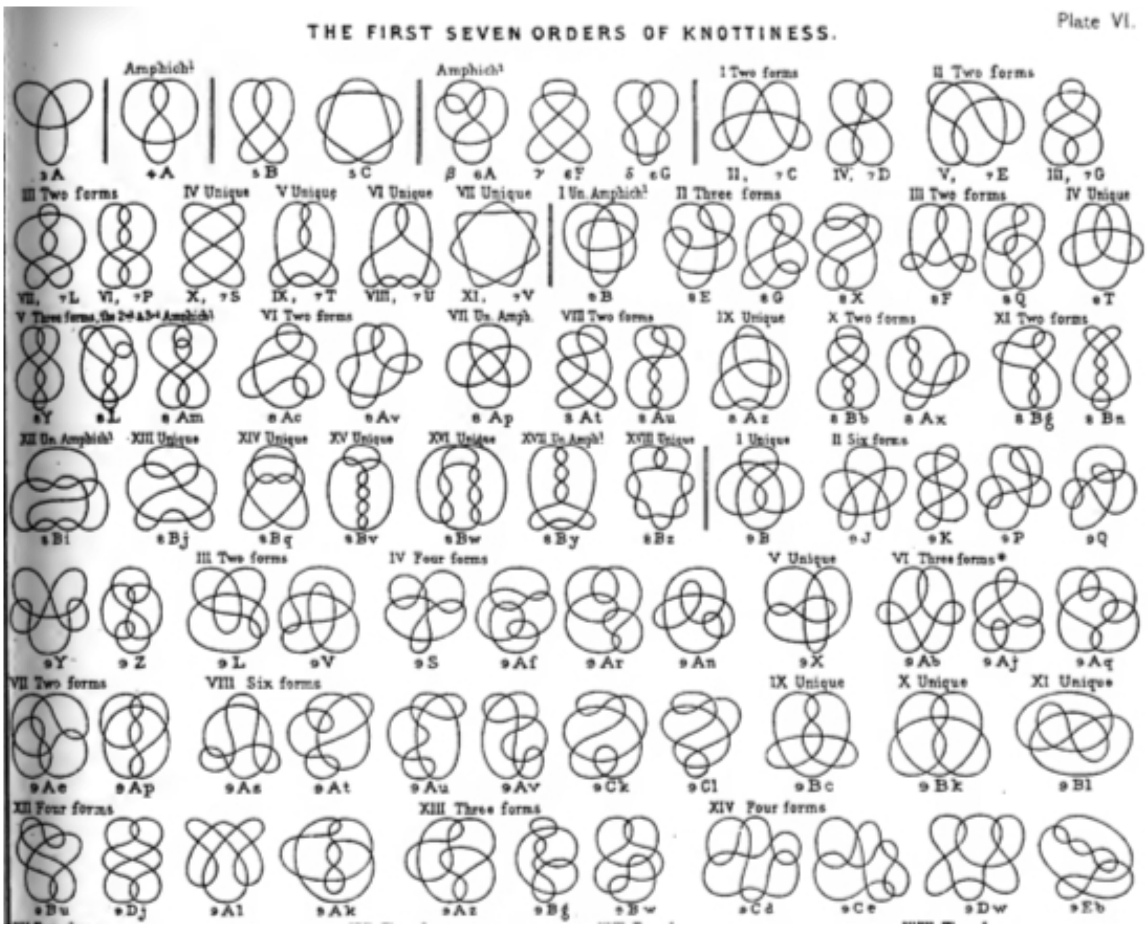

Figure 12: A portion of Tait's knot table. The phrase "seven orders of knottiness" does not refer to the number of crossings, but rather to how many crossings need to be changed to produce a diagram of the unknot.

Secret Knots and *In Blue*

The images in Terry Winters' recent work bears a striking resemblance to the drawings in Tait's knot table. For example, Figure 13 is a detail from *Secret Knots/2* with a degree 4 planar graph overlaid. Figure 14 shows the alternating knot that can be created from the graph. Using forms from *In Blue* or *Knotted Graphs* you could also create knots, although the forms in those paintings often have portions obscured, making the task more difficult.

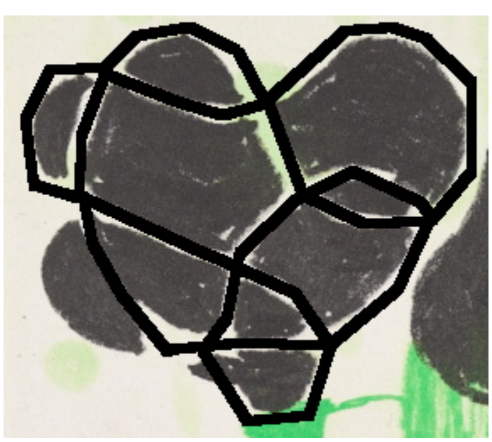

Figure 10: A detail from *Secret Knots/2* with a degree 4 planar graph overlaid.

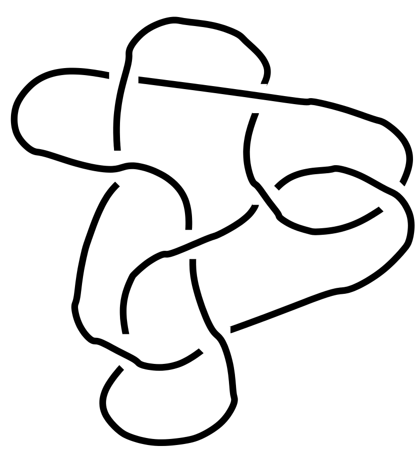

Figure 11: The knot 8_8 can be created from the graph in Figure 14.

So what do I see when I look at *In Blue*? I see knots escaping from their table. And as they do so, they begin to lose their identity. In many figures, the crossings have lost their over/under information and in some portions of the strands are missing or obscured. But not all identity has been lost; one of the crossings of the knot in the upper-right corner can still be determined from the brushstrokes, though you'll have to come see the painting in person to verify that!

In addition to referencing knot tables, *In Blue* hints at other connections to knot theory. The basket-like images in the center look like "Lissajous curves". On the right, the dark shaded regions bounded by the knot strands are reminiscent of surfaces bounded by knots. The proofs of many theorems about knots rely on such surfaces. But all these connections are only hinted at, none are made explicit.

Terry Winters has said:

> "I like the suspense of things that seem real but you're not quite sure what they are. Abstraction can be used as a process to build those real-world pictures. […] Science is a quantifiable and verifiable measurement. It's a factual subject and a good place to start. The imaginary dimensions of painting can be built on those facts. […] What I'm trying to do is engineer the pictures to the point where those figural components are there, but not quite there. A tension develops between them becoming legible and illegible, or drifting off from one thing to the next. That, for me, is part of what keeps them moving."
> (Interview in the *Brooklyn Rail*, Dec. 2008/Jan. 2009)

The knots in *In Blue* embody this ambiguity: floating, drifting, unconfined, their identity in flux, their strands fusing; they "are there, but not quite there."

Further Reading

The painting *In Blue* as well as the print series *Secret Knots* can be found at the Colby College Museum of Art. The paint series *Knotted Graphs* was exhibited at the Matthew Marks Gallery in New York City. A catalogue of the exhibition, which includes *In Blue*, can be purchased.

Colin Adam's text *The Knot Book* (2004, American Mathematical Society) is a fun introduction to various aspects of knot theory.

For more on Kelvin's theory of vortex atoms see pages 212-215 of *Lord Kelvin* by Sharlin (1979, The Pennsylvania State University Press) and pages 417-438 of *Energy and Empire* by Smith and Wise (1989, Cambridge University Press).

Andrew Ranicki has a wonderful page on the history of knot theory that includes many of the original papers of William Thomson (Lord Kelvin) and Peter Guthrie Tait. The image of Tait's knot table was taken from the paper *On Knots II*, which you can find on Ranicki's site.
http://www.maths.ed.ac.uk/~aar/knots/

The MacTutor History of Mathematics Archive has biographies of many mathematicians, including William Thomson and Peter Guthrie Tait. The images of Thomson and Tait above were taken from this site.
http://www-history.mcs.st-and.ac.uk/

Livingston and Cha's online table of knots can be found at: http://www.indiana.edu/~knotinfo/

The interview and discussion with Terry Winters in *The Brooklyn Rail* can be found at:
http://www.brooklynrail.org/2008/12/art/in-conversation-terry-winters

Jim Hoste has contributed to two very enjoyable survey articles on knot enumeration. Both can be found on his personal website at: http://pzacad.pitzer.edu/~jhoste/HosteWebPages/pub.html

> *The first 1,701,936 knots*, with Morwen Thistlethwaite and Jeff Weeks, Math. Intelligencer 20, no. 4 (1998) 33--48.
>
> *The enumeration and classification of knots and links*, Handbook of Knot Theory, W. Menasco and M. Thistlethwaite, eds, Elsevier (2005) 209--232.

Acknowledgements

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