Graphs, Surfaces, 3-Manifolds

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with:
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Definition: A \( \theta \)-graph is the connected union of two vertices and three non-separating edges. It is spatial when it is embedded in \( S^3 \) (or any 3-manifold).

Definition: A constituent knot is the result of removing one of the edges from a spatial \( \theta \)-graph.
Definition: Two $\theta$-graphs $X$ and $Y$ can be considered equivalent up to *isotopy* or *neighborhood isotopy*.
I. Motivation and Constructions

Spatial $\Theta$ Graphs: Equivalences

**Definition:** Two $\Theta$-graphs $X$ and $Y$ can be considered equivalent up to *isotopy* or *neighborhood isotopy.*
Definition: Two $\theta$-graphs $X$ and $Y$ can be considered equivalent up to isotopy or neighborhood isotopy.
I. Motivation and Constructions

Spatial $\theta$ Graphs: Fundamental Questions

What are necessary & sufficient conditions for two $\theta$-graphs to be:

- isotopic?
- neighborhood isotopic?
I. Motivation and Constructions

Brunnian $\Theta$ Graphs: Definition

**Definition:** A $\Theta$ graph embedded in $S^3$ is *Brunnian* (minimally knotted, almost unknotted) if it is not planar and if every constituent knot is the unknot.
I. Motivation and Constructions

Brunnian $\theta$ Graphs: Examples

The Kinoshita graph

$\text{hyperbolic exterior}$

$Vol \approx 6.45$

Helaman Ferguson *Knotted Wye*
I. Motivation and Constructions

Brunnian $\theta$ Graphs: Examples

The Kinoshita-Wolcott graphs
I. Motivation and Constructions

Brunnian $\theta$ Graphs: Examples

The Kinoshita-Wolcott graphs
I. Motivation and Constructions

Brunnian $\theta$ Graphs: Examples

*attaching clasps*
I. Motivation and Constructions

Spatial $\theta$ Graphs: Vertex Sums

Facts:
- Up to ordering of edges and choice of vertex, vertex sums are well-defined. (Wolcott)
- There is a prime-decomposition theorem (Motohashi)
- Cobordism classes form a group (Goda)
I. Motivation and Constructions
Brunnian $\theta$ Graphs: Vertex Sums
I. Motivation and Constructions
Brunnian $\theta$ Graphs: Vertex Sums
Observation

$$\theta = \theta_1 \#_v \theta_2$$

is Brunnian if and only if $\theta_1$ and $\theta_2$ are.
Can we build new examples of Brunnian $\theta$-graphs and prove that they are:

- Brunnian?
- Not Kinoshita-Wolcott graphs?
- Vertex Prime?
- Distinct up to isotopy?
- Distinct up to neighborhood isotopy?
- Hyperbolic?
I. Motivation and Constructions

Brunnian $\theta$ Graphs: New Examples

$\rho(A)$

$\sigma(A^{-1})$

$t_1$

$t_2$

$t_3$
I. Motivation and Constructions

Brunnian $\theta$ Graphs: New Examples

$\tau : PB(4) \rightarrow PB(2)$

forget last 2 strands

$A \in \ker \tau$
I. Motivation and Constructions

Brunnian $\Theta$ Graphs: New Examples

$\tau: PB(4) \rightarrow PB(2)$

*forget last 2 strands*

$A \in \ker \tau$

$\rho: PB(4) \rightarrow PB(6)$

*double last 2 strands*
I. Motivation and Constructions

Brunnian $\Theta$ Graphs: New Examples

$\tau : PB(4) \to PB(2)$

forget last 2 strands

$A \in \ker \tau$

$\sigma : PB(4) \to PB(6)$

include into first 4 strands

$\rho : PB(4) \to PB(6)$

double last 2 strands
I. Motivation and Constructions

Brunnian $\theta$ Graphs: New Examples

$\rho(A)$

$\sigma(A^{-1})$

$t_1$

$t_2$

$t_3$

$\theta$-graph
I. Motivation and Constructions

Brunnian $\theta$ Graphs: New Examples

$\rho(A)$

$\sigma(A^{-1})$

$t_1$

$t_2$

$t_3$

$A$

$\theta$-graph

planar
I. Motivation and Constructions
Brunnian θ Graphs: New Examples

$\rho(A)$

$\sigma(A^{-1})$

$A$

$\theta$-graph

$t_1$

planar

$\sigma(A^{-1})$

$t_2$

planar

$t_3$
I. Motivation and Constructions

Brunnian $\theta$ Graphs: New Examples

$\rho(A)$

$\sigma(A^{-1})$

$t_1$

$t_2$

$t_3$

$A$

$\theta$-graph

Planar

Planar

Hyperbolic exterior

$\text{Vol} \approx 21.7651$
I. Motivation and Constructions

Brunnian $\theta$ Graphs: Results

Theorem:
Brunnian $\theta$-graphs are nbhd-isotopic iff they are isotopic.

Theorem-in-progress:
Each of these graphs which is not planar is vertex-prime and hyperbolic.

Conjecture:
Infinitely many distinct, non-Kinoshita-Wolcott graphs can be represented this way.

Challenge: Characterize the braids giving Brunnian graphs.
An Aside

Damien Heard’s *Orb*

Available from the CompuTop site:
http://www.math.uiuc.edu/~nmd/computop/
II. Bridge Position

Motivation
Definition: A link is in *bridge position* with respect to a height function if all c.p. are maxima/minima and if all minima are below all maxima. The bridge position is *minimal* if it has the least possible number of maxima. The *bridge number* of a link is the number of maxima when the link is in minimal bridge position.
II. Bridge Position

Motivation

Bridge Surface

\( \mathbb{R} \)
II. Bridge Position

Motivation
II. Bridge Position

Motivation

\[ \omega_h(K) = 10 \]
II. Bridge Position

Motivation

\[ w_h(K) = 10 \]
II. Bridge Position

Motivation

\[ w_h(K) = 10 \]

\[ w_{h'}(K) = 6 \]
II. Bridge Position

Motivation

\[ w_h(K) = 10 \]

\[ w_{h'}(K) = 6 \]

\[ w(K) = \min_h w_h(K) \]
II. Bridge Position

Motivation

\[ w_h(K) = 10 \]

\[ w_{h'}(K) = 6 \]

\[ w(K) = \min_h w_h(K) \]

Definition (Gabai): A link is in \textit{thin position} if it minimizes width. It is in a \textit{locally thin position} if it cannot be thinned.
Theorem (Thompson): If thin position is not minimal bridge position for a link then there is an “essential” meridional planar surface in the link complement.
II. Bridge Position

Motivation

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Theorem (Thompson): If thin position is not minimal bridge position for a link then there is an “essential” meridional planar surface in the link complement.
Definition: A *Heegaard splitting* of a 3-manifold is a handle decomposition with all 0 and 1-handles appearing before all 2 and 3-handles.
II. Bridge Position

Motivation
II. Bridge Position

Motivation

**Theorem** (Scharlemann-Thompson): When a 3-manifold is in thin position all thin surfaces are *essential* and all thick surfaces are *strongly irreducible*.

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![Diagram of thin and thick surfaces]
II. Bridge Position

Thin position for (3-mfld, graph)

(Building on work of Hayashi-Shimokawa & Tomova)

Theorem: (T.- Tomova) Suppose that \( G \subset M \) (s.t. ...) has a c-bridge surface \( H \) (s.t. ...) then \( H \) can be put in “thin position” so that:

- All thin surfaces are c-essential
- All thick surfaces are c-strongly irreducible
II. Bridge Position
Application: Vertex Sums

**Define:** For a $\Theta$ graph in “bridge position” define its *c-disc number* to be half the total number of handles in the corresponding handle decomposition. The *c-disc number* of $\Theta$ is the minimum over all such c-disc numbers

\[
h(\theta) = 0 \quad \quad h(\theta) = 1
\]

\[
h(\theta) = 2 \quad \quad h(\theta) = 3
\]
II. Bridge Position

Application: Vertex Sums

Theorem: \( h(\Theta_1 \#_v \Theta_2) = h(\Theta_1) + h(\Theta_2) \)
**Theorem:** \( h(\Theta_1 \#_v \Theta_2) = h(\Theta_1) + h(\Theta_2) \)

**Conjecture:** If \( \Theta \) is toroidal, then \( h(\Theta) \geq 5 \)
II. Bridge Position

Application: Vertex Sums

\[ h(\theta) = 0 \quad \begin{array}{c}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array} \quad h(\theta) = 1 \quad \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ
\end{array}
\end{array}
\]

\[ h(\theta) = 2 \quad \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \quad \circ \\
\circ \quad \circ \quad \circ
\end{array}
\end{array} \quad h(\theta) = 3 \quad \begin{array}{c}
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \\
\circ \quad \circ \quad \circ \quad \circ
\end{array}
\end{array}
\]

**Theorem:** \[ h(\Theta_1 \ #_v \Theta_2) = h(\Theta_1) + h(\Theta_2) \]

**Conjecture:** If \( \Theta \) is toroidal, then \( h(\Theta) \geq 5 \)

**Conjecture:** If \( \Theta \) is Brunnian and atoroidal, then it is hyperbolic
II. Bridge Position

Application: Vertex Sums

Theorem: $h(\Theta_1 \#_v \Theta_2) = h(\Theta_1) + h(\Theta_2)$

Conjecture: If $\Theta$ is toroidal, then $h(\Theta) \geq 5$

Conjecture: If $\Theta$ is Brunnian and atoroidal, then it is hyperbolic

Corollary: Any Brunnian $\Theta$ graph with $h(\Theta) \leq 3$ is prime and hyperbolic.
III. Sutured Manifold Theory
Theorem: Two Brunnian graphs are neighborhood isotopic if and only if they are ambient isotopic.
III. Sutured Manifold Theory

Motivation

Theorem: A Brunnian handlebody has a unique Brunnian spine.

Theorem: Two Brunnian graphs are neighborhood isotopic if and only if they are ambient isotopic.
III. Sutured Manifold Theory

Definitions

\[ (N, \gamma \cup b) \]

\[ (N[b], \gamma) \]
Theorem (T. 2011): Suppose that \((N, \gamma \cup b)\) is taut and that \(b\) is an annulus component (s.t. ...). Let \(Q \subset N\) be a surface (s.t. ...). Then one of the following holds:

1. \(Q\) has a compressing or \(b\)-boundary compressing disc.
2. \((N[b], \beta)\) has a (lens space, core) summand.
3. \((N[b], \gamma)\) is taut and \(\beta\) can be isotoped to be disjoint from a Thurston-norm minimizing surface for \(N[b]\).
4. \(-2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b|\)
Theorem (T. 2011): Brunnian handlebodies have unique Brunnian spines.

\( N = \) exterior of handlebody

\( N[b] = \) solid torus

Brunnian spine
Theorem (T. 2011): Brunnian handlebodies have unique Brunnian spines.

$\mathcal{N} = \text{exterior of handlebody}$

$\mathcal{N}[b] = \text{solid torus}$

Brunnian spine

disc dual to another Brunnian spine

$\mathcal{N}[a] = \text{solid torus}$
Theorem (T. 2011): Brunnian handlebodies have unique Brunnian spines.

\[ N = \text{exterior of handlebody} \]
\[ N[b] = \text{solid torus} \]
\[ N[a] = \text{solid torus} \]
\[ Q = \text{meridian disc} \]
\[ Q = Q \cap N \]
III. Sutured Manifold Theory

Theorems

\[ Q = \text{meridian disc} \]
\[ Q = Q \cap \mathcal{N} \]
III. Sutured Manifold Theory

Theorems

\[ |\partial Q \cap \gamma| \leq |Q \cap b| \]

\[ Q = \text{meridian disc} \]

\[ Q = Q \cap N \]
III. Sutured Manifold Theory

Theorems

$Q = $ meridian disc

$Q = Q \cap N$

$|\partial Q \cap \gamma| \leq |Q \cap b|$

$|\partial_a Q \cap \gamma| = |\partial_a Q| |a \cap \gamma|$
III. Sutured Manifold Theory
Theorems

\[ Q = \text{meridian disc} \]
\[ Q = Q \cap N \]

\[ |\partial Q \cap \gamma| \leq |Q \cap b| \]
\[ |\partial_a Q \cap \gamma| = |\partial_a Q| |a \cap \gamma| \leq |\partial_a Q| (|a \cap b| - 2) \]
III. Sutured Manifold Theory

Theorems

\[ Q = \text{meridian disc} \]
\[ Q = Q \cap N \]

\[ |\partial Q \cap \gamma| \leq |Q \cap b| \]

\[ |\partial_a Q \cap \gamma| = |\partial_a Q| |a \cap \gamma| \leq |\partial_a Q| (|a \cap b| - 2) = |\partial_a Q \cap b| - 2 \]
Theorem (T.): Suppose that $(\mathcal{N}, \gamma \cup b)$ is taut and that $b$ is an annulus component (s.t. ...). Let $Q \subset \mathcal{N}$ be a surface (s.t. ...). Then one of:

1) $Q$ has a compressing or $b$-boundary compressing disc.

2) $(\mathcal{N}[b], \beta)$ has a (lens space, core) summand.

3) $(\mathcal{N}[b], \gamma)$ is taut and $\beta$ can be isotoped to be disjoint from a Thurston-norm minimizing surface for $\mathcal{N}[b]$.

4) $-2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b|$
Theorem (T.): Suppose that \((\mathcal{N}, \gamma \cup b)\) is taut and that \(b\) is an annulus component \((\text{s.t.} \ldots)\). Let \(Q \subset \mathcal{N}\) be a surface \((\text{s.t.} \ldots)\). Then one of:

1. \(Q\) has a compressing or \(b\)-boundary compressing disc.
2. \((\mathcal{N}[b], \beta)\) has a (lens space, core) summand.
3. \((\mathcal{N}[b], \gamma)\) is taut and \(\beta\) can be isotoped to be disjoint from a Thurston-norm minimizing surface for \(\mathcal{N}[b]\).
4. \(-2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b|\)

choose \(Q\) correctly
Theorem (T.): Suppose that $\langle N, \gamma \cup b \rangle$ is taut and that $b$ is an annulus component (s.t. ...). Let $Q \subset N$ be a surface (s.t. ...). Then one of:

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4. \(-2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b|\)

Brunnian spine

is non-planar

in \( S^3 \)

choose \( Q \) correctly
Theorem (T.): Suppose that \((\mathcal{N}, \gamma U b)\) is taut and that \(b\) is an annulus component (s.t. ...). Let \(Q \subset \mathcal{N}\) be a surface (s.t. ...). Then one of:

\[
-2\chi(Q) + |\partial Q \cap (\gamma U b)| \geq 2 |\partial Q \cap b|
\]
Theorem (T.): Suppose that \((N, \gamma \cup b)\) is taut and that \(b\) is an annulus component \((s.t. \ldots)\). Let \(Q \subset N\) be a surface \((s.t. \ldots)\). Then one of:

\[
(4) \quad -2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b|
\]

\[
-2(1 - |\partial_a Q|) + |\partial Q \cap \gamma| \geq |\partial Q \cap b| + |\partial_a Q|(|a \cap b| - |a \cap \gamma|)
\]
**Theorem (T.):** Suppose that $\mathcal{N}_\gamma \cup b$ is taut and that $b$ is an annulus component (s.t. ...). Let $Q \subset \mathcal{N}$ be a surface (s.t. ...). Then one of:

\[(4) \quad -2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b| \]

\[-2(1 - |\partial_a Q|) + |\partial Q \cap \gamma| \geq |\partial Q \cap b| + |\partial_a Q| (|a \cap b| - |a \cap \gamma|)\]

\[-2 + 2 |\partial_a Q| \quad \geq \quad |\partial_a Q| (|a \cap b| - |a \cap \gamma|)\]
Theorem (T.): Suppose that \((\mathcal{N}, \gamma \cup b)\) is taut and that \(b\) is an annulus component (s.t. ...). Let \(Q \subset \mathcal{N}\) be a surface (s.t. ...). Then one of:

\[
\begin{align*}
\text{(4) } & -2\chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b| \\
& -2(1 - |\partial_a Q|) + |\partial Q \cap \gamma| \geq |\partial Q \cap b| + |\partial_a Q|(|a \cap b| - |a \cap \gamma|) \\
& -2 + 2 |\partial_a Q| \geq |\partial_a Q|(|a \cap b| - |a \cap \gamma|) \\
& -2 \geq |\partial_a Q|(|a \cap b| - |a \cap \gamma| - 2)
\end{align*}
\]
Theorem (T.): Suppose that \((\mathcal{N}, \gamma \cup b)\) is taut and that \(b\) is an annulus component \((s.t. \ldots)\). Let \(Q \subset \mathcal{N}\) be a surface \((s.t. \ldots)\). Then one of:

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\text{(4) } -2 \chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b|
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-2(1 - |\partial_a Q|) + |\partial Q \cap \gamma| \geq |\partial Q \cap b| + |\partial_a Q| (|a \cap b| - |a \cap \gamma|)
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-2 + 2 |\partial_a Q| \geq |\partial_a Q| (|a \cap b| - |a \cap \gamma|)
\]

\[
-2 \geq |\partial_a Q| (|a \cap b| - |a \cap \gamma| - 2)
\]

\[
-2 \geq 0
\]
Theorem (T.): Suppose that $(N, \gamma \cup b)$ is taut and that $b$ is an annulus component (s.t. ...). Let $Q \subset N$ be a surface (s.t. ...). Then one of:

\[
\begin{align*}
(4) \ -2 \chi(Q) + |\partial Q \cap (\gamma \cup b)| \geq 2 |\partial Q \cap b| & \geq |\partial Q \cap b| + |\partial a Q| (|a \cap b| - |a \cap \gamma|) \\
-2(1 - |\partial a Q|) + |\partial Q \cap \gamma| \geq |\partial Q \cap b| + |\partial a Q| (|a \cap b| - |a \cap \gamma|) \\
-2 + 2 |\partial a Q| & \geq |\partial a Q| (|a \cap b| - |a \cap \gamma|) \\
-2 & \geq |\partial a Q| (|a \cap b| - |a \cap \gamma| - 2) \\
-2 \geq 0
\end{align*}
\]
Theorem (T.): Suppose that \((N, \gamma \cup b)\) is taut and that \(b\) is an annulus component (s.t. ...). Let \(Q \subset N\) be a surface (s.t. ...). Then one of:

Q.E.D.
Questions/Problems:

1. Characterize the braids resulting in Brunnian graphs. Which have c-disc number 2? 3?

2. Which braids give rise to isotopic graphs?

3. Find a normal form for graphs of handle number 2 and 3.

4. Construct prime, atoroidal Brunnian graphs of arbitrarily c-disc number.

5. Do Brunnian graphs have unique exteriors?

6. Characterize the constituent knots of Brunnian handlebodies.