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Abstractly Planar Spatial Graphs

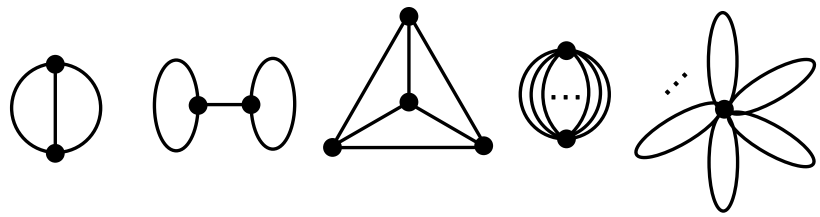

FIGURE 1: From left to right, we have the abstract graph type of θ-graphs, handcuff graphs, the tetrahedral graph, θ_n-graphs, and bouquets.

Beginning courses in graph theory prove many wonderful theorems about planar graphs. An even more wonderful theory arises when we put planar graphs (which we'll henceforth refer to as **abstractly planar** graphs) into 3-dimensional space. One way of doing this is to choose an embedding of an abstractly planar graph G in the sphere S^2 and then include S^2 into the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ (tamely). As an abstractly planar graph may have several planar embeddings, we may wonder if we can end up with different embeddings in S^3. It turns out, however, that in S^3 all planar embeddings give rise to **equivalent** (that is, ambient isotopic) spatial graphs [17]. Such spatial graphs are called **trivial**. A non-trivial spatial graph is **knotted**. Are there knotted embeddings? Yes there are!

Some of the most important spatial graphs have very few vertices and edges, and are thus abstractly planar. Important classes include spatial θ-graphs, θ_n-graphs, handcuff graphs, bouquets, and the tetrahedral graph. We depict the abstract graph type for these graphs in Figure 1. Dylan Thurston [29] has shown that the trivial θ-graph, the trivial tetrahedron graph, the unknot, and minimally twisted Möbius bands, together with a small collection of operations, are enough to generate all (framed) trivalent spatial graphs. Handcuff graphs and θ-graphs both show up in the theory of tunnel number one knots and links [4]. The theory of bouquets with n-petals is closely related to the theory of tangles with $2n$-arcs, as can be seen by looking at the exterior of a vertex. The hyperbolic spatial graphs of lowest volume [8] also fall into these classes. Spatial θ and θ_n graphs have received the most attention in the literature, however.

One reason spatial θ-graphs are so prevalent is that we can create one by attaching an arc to a knot so that the endpoints of the arc are distinct points on the knot. This construction arises naturally in knot theory, where the arc may record some information about the knot. Typical examples include knot tunnels or an arc which records the location of some crossing change, as in the first two diagrams of Figure 2. Attaching an arc to a knot K to produce a spatial θ-graph G guarantees that K is a cycle in G; that is, a **constituent knot**. What can we say about the other constituent knots? Perhaps suprisingly, Kinoshita [14] showed that, given three knots, there is a spatial θ-graph whose three constituent knots are precisely the three given knots. An example

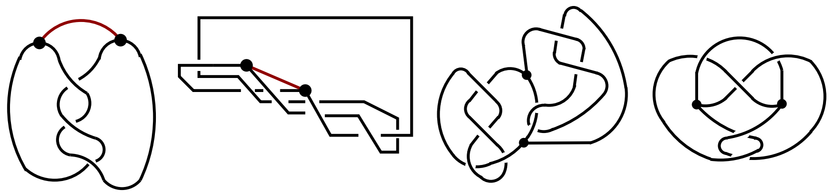

FIGURE 2: Four examples of spatial θ-graphs. From left to right: a trefoil knot with tunnel (in red); a knot with an arc (in red) marking the location of a crossing change; a θ-graph whose every constituent knot is a figure 8 knot; the Kinoshita graph.

of a θ-graph whose three constituent knots are all the Figure 8 knot is shown in Figure 2. Kinoshita's construction can be applied recursively to construct θ_n-graphs whose constituent knots are specified beforehand.

The trivial θ-graph has every cycle an unknot. Are there other θ-graphs with this property? A spatial graph having the property that no collection of disjoint cycles is a non-trivial link is a **ravel**. Similarly, a spatial graph which has the property that every proper subgraph is trivial has the **Brunnian property**. A knotted graph with the Brunnian property is **Brunnian**, **almost unknotted**, or **minimally knotted**. A θ-graph is Brunnian if and only if it is a ravel; but the same is not necessarily true for other graphs.

Kinoshita [13] also provided the first example of a Brunnian θ-graph, now named after him. It is the rightmost diagram in Figure 2. Wolcott [30] later generalized this construction to a family now known as the Kinoshita-Wolcott graphs. More examples of Brunnian θ-graphs are given in [16] and [10]. Suzuki [24] generalized Kinoshita's construction to θ_n graphs. Every abstractly planar graph without degree zero and degree one vertices has a Brunnian spatial embedding [12, 32]. Ravels are of interest to chemists [3]; Flapan and Miller [6] have constructed many examples.

How can we be sure that Kinoshita's graph really is knotted, or indeed that any given spatial embedding of an abstractly planar graph really is knotted?

An equivalence between spatial graphs takes the constituent knots of one to the constituent knots of the other (see [11]). Thus, if one spatial graph has a constituent knot K and another has no constituent knot of the same knot type, the graphs can't be equivalent. This doesn't help us show minimally knotted spatial graphs are knotted, though; we need other tools. As always in knot theory, we might ask for an invariant and there are some very nice invariants available. In general, Brunnian graphs and ravels provide good tests for the strength of invariants of spatial graphs. The three most popular are the Yamada polynomial [33], Litherland's version of the Alexander polynomial [15], and Thompson's polynomial invariant [28]. This last polynomial is defined recursively, but is zero if and only if the graph is trivial. It is based on an earlier algorithm of Scharlemann and Thompson [22] for determining if a spatial graph is unknotted. Their results were also adapted by Wu [31], who showed that a spatial graph is unknotted if and only if each cycle bounds a disc disjoint from the rest of the graph.

We can also turn to other tools from topology and algebra. Kinoshita and Suzuki used Alexander ideals to prove the non-triviality of their Brunnian graphs. Though, McAtee, Silver, and Williams [19] point out that Suzuki's proof contains an error. The first complete proof of their non-triviality is likely given by Scharlemann [21], using topological techniques stemming from the braid groups. In [20, Example 22], the topology of surfaces containing the spatial graph is used to prove the Kinoshita graph is knotted, and in [19], quandle colorings are used. In [10], a combination of handlebody theory and rational tangles are applied to an infinite family of θ-graphs. One popular and beautifully simple approach for θ_n-graphs is to use branched covers. Livingston [16] uses these to prove Suzuki's graphs are non-trivial and Calcut and Metcalf-Burton [2] use them to show Kinoshita's graph is prime, in a sense which we now explore.

Whenever mathematicians are introduced to some new mathematical object, we want to be able to create more of them and to understand how the object fits into the larger context of known mathematics. For the remainder, we take up the question of creating new spatial graphs from old ones and understanding how spatial graphs are related to knot theory and 3-manifold theory.

Throughout the study of manifolds, the connected sum is one of the most important methods of combining two manifolds. Classical results show that if the summands are oriented and connected, there's a unique way of summing two manifolds. For spatial graphs, on the other hand, there are potentially many more options. For starters, we have a choice of where to perform the sum. Given two spatial graphs G_1 and G_2 in distinct copies of S^3, we make the choice by picking **summing points** p_1 and p_2 in each copy of S^3 for $i = 1, 2$. We can pick the points both to be disjoint from the graphs, we can pick them to be contained in the interiors of edges in the graphs, or we can pick them both to be vertices of the graphs having the same degree. We'll denote the result by $G_1 \#_k G_2$ where $k = 0$ if the points are disjoint from the graphs; $k = 2$ if the points are interior to edges; and, otherwise, k is the degree of the vertices[1]. Figure 3 depicts the the case when $k = 0$ (the **distant sum**), $k = 2$ (the **connected sum**), and $k = 3$ (**the trivalent vertex sum**). Even for a fixed k, in general, $G_1 \#_k G_2$ is not uniquely defined. Summing operations are associative. If G is equivalent to $G_1 \#_k G_2$ and neither is a trivial θ_k-graph, then we say that G is **k-composite**. If G is neither trivial nor k-composite, it is **k-prime**.

For simplicity, let's consider only spatial θ-graphs. We also assume our θ-graphs are oriented. To orient a θ-graph, choose one vertex as source, one vertex as sink, and color the edges red, blue, and green. We then restrict $\#_3$ so

[1] This creates some ambiguity when $k = 2$, but we will ignore this.

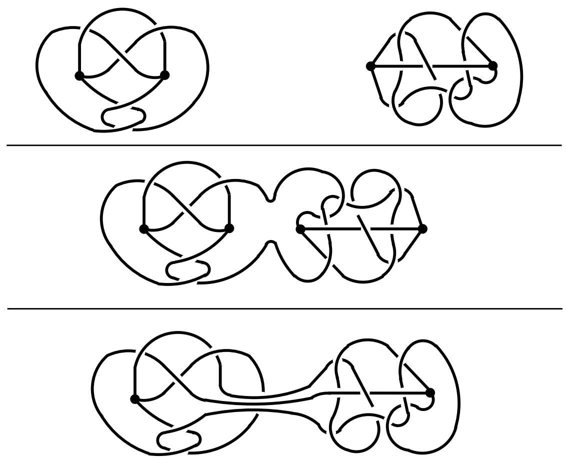

FIGURE 3: From top to bottom we have the distant sum, the connected sum, and the trivalent vertex sum of two spatial θ-graphs.

that, when forming $M_1\#_3M_2$, the summing point p_1 is the sink vertex of G_1, the summing point p_2 is the source vertex of G_2, and the gluing map takes red, blue, and green endpoints to red, blue and green endpoints respectively. Under the operation $\#_3$, the set $\mathbb{G}$ of oriented θ-graphs in S^3 is particularly rich. The operation $\#_3$ is well-defined [30] and it makes the set $\mathbb{G}$ into a semigroup with the trivial graph as the identity. The center of the semigroup consists of the θ-graphs which are a connected sum of a trivial θ-graph and a knot. Elements of $\mathbb{G}$ have a 3-prime factorizations which are unique up to summing with elements of the center [18]. Conjecturally [9], (3-manifold, graph) pairs more generally have unique factorizations.

For θ-graphs, the property of being Brunnian also persists under $\#_k$. Indeed, for $G_1, G_2 \in \mathbb{M}$, every trivalent vertex sum, $G_1\#_3G_2$ is Brunnian if and only if G_1 and G_2 are both Brunnian. (Exercise!) For general spatial graphs, the property of being Brunnian may not persist under trivalent vertex sum. (Another exercise!) The property of being a ravel does persist under trivalent vertex sum. However, for $k \geq 4$, the property of being a ravel need not persist under $\#_k$, as there are knots with essential tangle decompositions; such knots result from summing bouquets. Even for θ_k graphs, we may choose the gluing homeomorphism to be a complicated element of the mapping class group of the punctured sphere, in which case we may end up with knotted cycles after performing the sum.

How can we construct infinitely many *3-prime* Brunnian θ-graphs? The Kinoshita and Kinoshita-Wolcott graphs are 3-prime [15, 2], as are the Brunnian θ-graphs found in [10] (see [27] for an indication of how this might be proved). Here is a very general method (essentially found in [23]) which likely produces arbitrarily complicated Brunnian θ-graphs, most of which are prob-

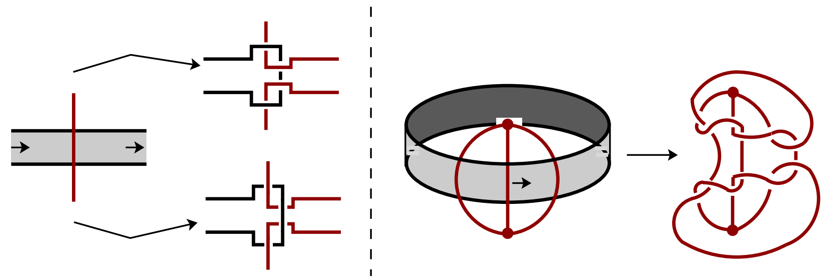

FIGURE 4: The left side shows how to form $G(B)$. W e take the union of G and ∂B, but wherever an edge of G intersects B as on the left, we replace it with either of the pictured two "belt buckles." On the right, we see that using the indicated belt to buckle the trivial theta-graph produces the trivalent vertex sum of the Kinoshita graph with its mirror image.

ably 3-prime. For a spatial graph $G \subset S^3$, a new spatial graph $G(B)$, called a **buckling** of G, is determined by a choice of oriented annulus B, called a **belt**, intersecting G in interval fibers. We create $G(B)$ as follows. At each intersection arc α between B and G we replace a neighborhood of α in S^3 with a **belt buckle** as on the left of Figure 4 and include the remaining portions of ∂B as part of $G(B)$. The right of Figure 4 shows how a certain buckling of the trivial θ-graph produces the trivalent vertex sum of the Kinoshita graph with its mirror image.

Not very much is known about how buckling affects a spatial graph. It is not difficult to see, however, that G and $G(B)$ are abstractly isomorphic and that if e is an edge of G intersecting the band B, then the subgraphs $G(B)-e$ and $G-e$ are equivalent. In particular, if G has the Brunnian property, and if B intersects every edge of G, then $G(B)$ also has the Brunnian property. It seems difficult to determine whether or not $G(B)$ is trivial. Nevertheless, we conjecture:

Conjecture 1 *There is no θ-graph G in S^3 such that there is a belt B intersecting all the edges of G with $G(B)$ either the trivial θ-graph or the Kinoshita graph.*

It seems plausible that the knot type of the core of the belt B is relevant to deciding, for a particular G, if $G(B)$ is non-trivial. In [10], there is an example of a buckling which produces a graph with the Brunnian property having an essential torus in its exterior (and, hence, is non-hyperbolic and non-trivial).

Finally, we consider the topology and geometry of the exteriors of $G \in \mathbb{G}$. The graph G may be **hyperbolic with parabolic meridians** [8]. This means that the complement of the edges in the exterior of the vertices of G supports a complete hyperbolic metric with annular cusps. It follows from work of Thurston (see [8, Corollary 2.5]) that $G \in \mathbb{G}$ is hyperbolic with parabolic meridians if and only if it is 2-prime and the exterior of G does not contain an

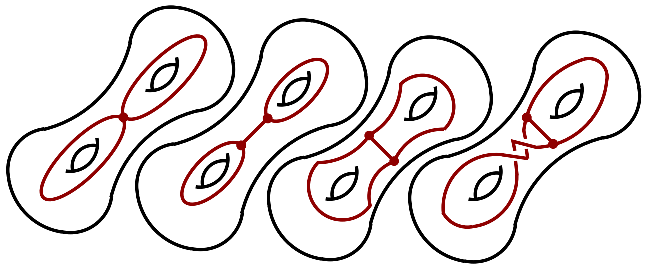

FIGURE 5: Four different spines for a genus 2 handlebody

essential torus. In particular, if $G \subset S^3$ is a Brunnian θ-graph, then (S^3, G) is hyperbolic with parabolic meridians whenever its exterior does not contain an essential torus. Thurston's work can also be used to show that if $G = G_1 \#_3 G_2$ is some trivalent vertex sum of non-trivial elements of $\mathbb{G}$, then G is hyperbolic with parabolic meridians if and only if both G_1 and G_2 are. This suggests that volume $\mathrm{vol}(G)$ is a particularly interesting invariant for $G \in \mathbb{G}$. We ask (based on [1]):

Question 1 *Suppose that $G_i \in \mathbb{G}$ for $i = 1, 2$ are both hyperbolic with parabolic meridians. How different can $\mathrm{vol}(G_1 \#_3 G_2)$ and $\mathrm{vol}(G_1) + \mathrm{vol}(G_2)$ be?*

Finally, we consider the uniqueness up to homeomorphism of the exterior of spatial graphs. A regular neighborhood of a spatial graph is called a **handlebody** and the graph is called a **spine** for the handlebody. Handlebodies may have many different spines. Indeed, if the genus of the handlebody is at least 2, it will have infinitely many spines. Figure 5 depicts four different spines for a genus 2 handlebody. We say that two spatial graphs G and G' are **neighborhood-equivalent** if they have isotopic regular neighborhoods. Neighborhood-equivalent spatial graphs have homeomorphic exteriors. In particular, spatial graphs are not determined by their complements, unlike knots in S^3 [7].

If two spatial graphs are neighborhood-equivalent, we can also ask how their constituent knots are related. For θ-graphs, this question was studied extensively in [25, 26], where it was connected to an operation on knots and 2-component links called **boring**. Rational tangle replacement on knots (an important operation in studying DNA, see e.g. [5]) is an example of boring. One interesting result (see [26, Theorem 6.5]) is that no two non-equivalent *Brunnian* θ-graphs are neighborhood equivalent. This suggests that Brunnian θ-graphs may be determined by their complements.

Conjecture 2 *If two Brunnian θ-graphs have homeomorphic exteriors, then they are equivalent.*

In general, the topology of Brunnian graphs (not necessarily, θ-graphs) is an area ripe for further study.

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