Exam #2

1. True or False?

T  F  If the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) span \( \mathbb{R}^3 \), then the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) must form a basis of \( \mathbb{R}^3 \).

T  F  If the rank of a \( 7 \times 10 \) matrix \( A \) is 4, then the kernel of \( A \) must be six-dimensional.

T  F  If \( V \) is the set of all \( 2 \times 2 \) matrices \( A \) such that the vector \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is in the image of \( A \), then \( V \) is a subspace of \( \mathbb{R}^{2 \times 2} \).

T  F  For every subspace \( V \) of \( \mathbb{R}^4 \) there exists a \( 4 \times 4 \) matrix \( A \) such that \( V = \text{im}(A) \).

T  F  There exists a noninvertible \( 2 \times 2 \) matrix \( A \) that is similar to \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

2. Are the functions below isomorphisms? You will earn 2 points for each correct answer, and 1 point if you don’t answer. No explanation is needed. We are told that one (and only one) of these functions fails to be linear.

Yes  No  \( T(A) = SAS^{-1}, \) where \( S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \), from \( \mathbb{R}^{2 \times 2} \) to \( \mathbb{R}^{2 \times 2} \).

Yes  No  \( T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x + 6y \\ 6x + 7y \\ 7x + 8y \\ 8x + 9y \end{bmatrix} \) from \( \mathbb{R}^3 \) to \( \mathbb{R}^{2 \times 2} \).

Yes  No  \( T(f(x)) = f(x) + 3 \) from \( P_2 \) to \( P_2 \).

Yes  No  \( T(f(x)) = f(0) + f(1)x + f(2)x^2 \) from \( P_2 \) to \( P_2 \).

Yes  No  \( T(f(x)) = (x - 1)f(x) \) from \( P \) to \( P \).

3. Find a basis of the subspace \( V \) of \( P_3 \) consisting of all polynomials \( f(x) \) with \( f(1) = f(2) \). Find the dimension of \( V \).
4. If \( b \neq 0 \), find the matrix \( B \) of the linear transformation \( T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x} \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) with respect to the basis \( \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \). Express the entries in the second column of \( B \) in terms of the determinant of \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and the trace of \( A \) (the trace is the sum of the diagonal entries, \( a + d \)).

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5. Let \( V \) be the span of the matrices \( I_2 \) and \( A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \) in \( \mathbb{R}^{2 \times 2} \). Consider the linear transformation \( T(M) = AM \) from \( V \) to \( V \).

a. Compute \( A^2 \). Write your answer as a scalar multiple of matrix \( A \).

b. Find the matrix \( B \) of \( T \) with respect to the basis \( \mathcal{B} = I_2, A \). Use the commutative diagram below.

\[
M = c_1 I_2 + c_2 A \quad \xrightarrow{T} \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \xrightarrow{B}
\]

c. Find a basis of the image of \( T \)

d. Find a basis of the kernel of \( T \)
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1. True or False?
   
   a. F  As a counter example, consider \( \vec{v}_1 = \vec{e}_1, \quad \vec{v}_2 = \vec{e}_2, \quad \vec{v}_3 = \vec{0}, \quad \vec{v}_4 = \vec{e}_3 \)
   
   b. T  \( \dim(\ker A) = \# \text{ columns} \) \( - \) \( \rank A \) \( = 10 - 4 = 6 \)
   
   c. F  The zero matrix \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) isn't in \( V \).
   
   d. T  Pick a basis \( \vec{v}_1, \ldots, \vec{v}_m \) of \( V \). Make \( \vec{v}_1, \ldots, \vec{v}_m \) the first \( m \) columns of \( A \), with the remaining columns (if any) all being \( \vec{0} \) (or otherwise dependent on the \( \vec{v}_i \)).
   
   e. F  The matrix \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) is invertible, and any matrix that is similar to an invertible matrix is invertible as well.

2. Are the functions below isomorphisms?
   
   a. Yes  The inverse is \( A = S^{-1}BS \)
   
   b. No  The dimensions of domain and codomain aren't equal.
   
   c. No  That's the nonlinear one; note that \( T(0) = 3 \)
   
   d. Yes  The kernel is 0, since the only polynomial \( f(x) \) in \( P_2 \) with \( f(0) = f(1) = f(2) = 0 \) is the zero polynomial.
   
   e. No  The image isn't all of \( P \), but \( \text{Im}(T) = \{ g \text{ in } P : g(1) = 0 \} \).

3. We are looking for the polynomials \( f(x) = a + bx + cx^2 + dx^3 \) such that \( f(1) = f(2) \), or, \( a + b + c + d = a + 2b + 4c + 8d \), or \( b + 3c + 7d = 0 \), or \( b = -3c - 7d \). These polynomials are of the form \( f(x) = a + (-3c - 7d)x + cx^2 + dx^3 = a \cdot 1 + c(x^2 - 3x) + d(x^3 - 7x) \), so that \( a, x^2 - 3x, x^3 - 7x \) is a basis of \( V \), and \( \dim(V) = 3 \).

4. With \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( S = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix} \), we have \( B = S^{-1}AS = \frac{1}{b} \begin{bmatrix} -d & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix} \)

   \( = \begin{bmatrix} 0 & bc - ad \\ 1 & a + d \end{bmatrix} = \begin{bmatrix} 0 & -\det(A) \\ 1 & \text{trace}(A) \end{bmatrix} \)
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5. a. \( A^2 = \begin{bmatrix} 7 & 14 \\ 21 & 42 \end{bmatrix} = 7A \)

b. \( T \)

\[ M = c_1 I_2 + c_2 A \]

\[ T(M) = AM = c_1 A + c_2 A^2 = (c_1 + 7c_2)A \]

\[ T \]

\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \]

\[ B \]

\[ \begin{bmatrix} 0 \\ c_1 + 7c_2 \end{bmatrix} \]

Thus \( B = \begin{bmatrix} 0 & 0 \\ 1 & 7 \end{bmatrix} \)

c. A basis of the image of \( B \) is \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), and a basis of the image of \( T \) is \( A \).

d. A basis of the kernel of \( B \) is \( \begin{bmatrix} 7 \\ -1 \end{bmatrix} \), and a basis of the kernel of \( T \) is \( 7I_2 - A = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix} \).