MA381 – Fall 2007 **Quiz 3**

Quiz Policies: This is a take home quiz. You are allowed to use the textbook and your notes, but are not allowed to discuss this quiz with anyone or use any other means of assistance. You may ask me for clarification of any of the questions. Remember, cheating is bad – don't do it. If you do cheat, you will get a zero on this quiz and, even worse, I will think ill of you. This is due at the *beginning* of class on the day indicated.

1. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute
   a. P(X > 5)
   b. P(4 < X < 16)
   c. P(X < 8)
   d. Find the value of C such that P(X < C) = 0.25.

$$\begin{aligned} A: P(X>5) &= 1-P(X<5) \\ &= 1-P\left(Z<\frac{5-10}{6}\right) \\ &= 1-P(Z<-0.83) \\ &= 0.7977 \end{aligned}$$

$$\begin{aligned} B: P(4<X<16) &= P\left(\frac{4-10}{6}<Z<\frac{16-10}{6}\right) \\ &= P(-1<Z<1) = P(Z<1)-P(Z<-1) \\ &= 0.6827 \end{aligned}$$

$$\begin{aligned} C: P(X<8) &= P\left(Z<\frac{8-10}{6}\right) \\ &= P(Z<-0.33) = 0.3695 \end{aligned}$$

$$\begin{aligned} &D: P(X<C) = 0.25 \\ &\quad P(Z<-0.67) = 0.25 \\ &\quad -0.67 = \frac{C-10}{6} \\ &\quad C = 6 \end{aligned}$$

2. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10 inches.

$$E[\text{points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$$

3. The number of years a radio functions is exponentially distributed with parameter β = 8. If Jones buys a used radio, what is the probability that it will function an additional 8 years?

Let X denote the amount of time the radio functions.

$$P(X > 8) = \int_8^\infty \frac{1}{8} e^{-y/8}\, dy = -e^{-y/8}\Big|_8^\infty = 0 - (-e^{-1}) = 0.368$$

4. The time (in hours) required to repair a machine has a gamma distribution with parameters α = 1 and β = 2.

   a. What is the probability that a repair time exceeds 2 hours?
   b. What is the probability that a repair time will take at least 10 hours, given that its duration exceeds 9 hours?

Let X denote repair time. Note that if X is a Gamma(1,2),
it's just an exponential with $\beta = 2$.

$$A: P(X > 2) = \int_2^\infty \frac{1}{2} e^{-y/2}\, dy = -e^{-y/2}\Big|_2^\infty = 0 - (-e^{-1}) = 0.368$$

$$B: P(X > 10 \mid X > 9) = \frac{P(X > 10)}{P(X > 9)} = \frac{\int_{10}^\infty \frac{1}{2} e^{-y/2}\, dy}{\int_9^\infty \frac{1}{2} e^{-y/2}\, dy}$$

$$= \frac{-e^{-y/2}\Big|_{10}^\infty}{-e^{-y/2}\Big|_9^\infty} = \frac{0 - (-e^{-5})}{0 - (-e^{-4.5})} = e^{-.5} = 0.607$$

Note that the exponential is also memoryless!

5. If X is a beta random variable with parameters α and β, show that

$$E(X)=\frac{\alpha}{\alpha+\beta}$$

$$Var(X)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$E(X)=\int_{-\infty}^{\infty} x\left(\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\right)dx$$

$$=\int_{0}^{1} x\left(\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\right)dx$$

$$=\frac{1}{B(\alpha,\beta)}\int_{0}^{1} x^{\alpha}(1-x)^{\beta-1}dx$$

$$=\frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \text{ since } \alpha > 0 \text{ inplies } \alpha + 1 > 0$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}=\frac{\alpha}{\alpha+\beta}$$

Similarly,

$$E(X^2)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{1} x^{\alpha+1}(1-x)^{\beta-1}dy$$

$$=\frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+2)}$$

$$=\frac{(\alpha+1)(\alpha)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$Var(X)=E(X^2)-[E(X)]^2$$

$$=\frac{(\alpha+1)(\alpha)}{(\alpha+\beta)(\alpha+\beta+1)}-\frac{\alpha^2}{(\alpha+\beta)^2}$$

$$=\frac{\alpha(\alpha+1)(\alpha+\beta)-\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$