Name ____

Mathematics 381 Fall 2007

Midterm

Thursday, 18 October 2007

- This is a closed-book, closed-note, closed-internet, closed-friend exam. You may bring use one 8.5' x 11" two-sided sheet of notes and a calculator. You may also use the standard normal table that has been provided in class.
- All work must be your own. You may not give or receive any kind of aid, either verbally, visually, or otherwise, during this exam. No other sources may be consulted, except as specified above.
- **The exam has 80 possible points. There are 6 questions and 8 pages, including this cover page. You have 2 hours to complete the exam so plan your time accordingly. I have included the possible points next to each problem.**
- **The last question (the question on page 8) is extra credit – you do not have to attempt it.**
- Some questions are more difficult than others, and the questions may not be in order of difficulty. Don't spend too much time on any one question; if you get stuck, go on and try another part.
- Whenever possible, show your work and explain your reasoning. In case you make a mistake, I can more easily give you partial credit if you explain your steps.
- Some parts of a question may require the answer to an earlier part of the question. If you can't solve the earlier part, you can still receive partial credit for the latter parts: make up a reasonable answer for the earlier part and use that in solving for the latter parts.

Question 1 (12 points total)

There are five red chips and 3 blue chips in a bowl. The red chips are numbered 1,2,3,4,5, respectively, and the blue chips are numbered 1,2,3, respectively.

A. (4 points) What is the probability that two chips drawn without replacement are the same color?

$$P(\text{same color}) = P(\text{both blue}) + P(\text{both red})$$
$$= \frac{3}{8}\frac{2}{7} + \frac{5}{8}\frac{4}{7} = \frac{13}{28} = 0.464$$

B. (4 points) What is probability that two chips drawn without replacement have the same number?

$$P(\text{same number}) = P(\text{both 1s}) + P(\text{both 2s}) + P(\text{both 3s})$$
$$= \left(\frac{2}{8}\frac{1}{7}\right)3 = \frac{3}{28} = 0.107$$

C. (6 points) What is the probability that two chips drawn without replacement have the same color or the same number?

Since two chips can't be both the same color and same number, the addition rule gives:

$$P(\text{same color or same number}) = P(\text{same color}) + P(\text{same number})$$
$$= \frac{13}{28} + \frac{3}{28} = \frac{4}{7} = 0.571$$

Question 2 (12 points total)

For each of the following scenarios, write the name of the distribution that should be used to model the phenomenon to determine the probability. You DO NOT need to calculate the probability.

A. A door-to-door salesperson is required to document five in-home visits each day. Suppose that she has a 30% chance of being invited into any given home, with each home representing an independent trial. What is the probability that she requires fewer than eight houses to achieve her fifth success?

Negative binomial

B. A corporate board contains 12 members. The board decides to create a five-person Committee to Hide Corporate Debt. Suppose four members of the board are accountants. What is the probability that the Committee will contain two accountants and three non-accountants?

Hypergeometric

C. A jury of 12 persons from a jury pool of 100 is to be selected. The jury pool consisted of 50 men and 50 women. What is the probability that no more than 3 women will be selected for this jury?

Hypergeometric

D. Suppose that on-the-job injuries in a textile mill occur at the rate of 0.1 per day. What is the probability that 2 accidents occur in a 30-day period?

Poisson

Question 3 (8 points)

Accident records provide the following information to insurance companies. The probability that a driver is in an accident is 0.15. If an accident has occurred, the damage to the vehicle amounts to 20% of its market value with a probability of 0.75, to 60% of its market value with probability 0.13, and a total loss with probability 0.12. How much should the insurance company charge to insure a $18000 car so that they at least break even?

P(being in an accident and having a 20% loss) = P(20% loss | in accident)P(in accident)
= (0.75)(0.15) = 0.1125

P(being in an accident and having a 60% loss) = P(60% loss | in accident)P(in accident)
= (0.13)(0.15) = 0.0195

P(being in an accident and having a 100% loss) = P(100% loss | in accident)P(in accident)
= (0.12)(0.15) = 0.0180

Let X = amount of payout.

x	0	3600	10800	18000
p	0.850	0.1125	0.0195	0.0180

$$E[X] = 0(.85) + 3600(.1125) + 10800(.0195) + 18000(.0180)$$
$$= \$939.50$$

This is what the insurance company needs to charge to break even.

Question 4 (15 points total)

Let X be a random variable having a gamma distribution with $\alpha = \beta = 2$ for positive values of x. Find the following quantities.

A. (10 points) $E(2X + X^2)$ and $V(2X + X^2)$.

$$m(t) = (1-\beta t)^{-\alpha} = (1-2t)^{-2}$$
$$m'(t) = 4(1-2t)^{-3} \rightarrow E[X] = 4$$
$$m''(t) = 24(1-2t)^{-4} \rightarrow E[X^2] = 24$$
$$m'''(t) = 192(1-2t)^{-5} \rightarrow E[X^3] = 192$$
$$m^{(4)}(t) = 1920(1-2t)^{-6} \rightarrow E[X^4] = 1920$$

Thus,

$$E(2X + X^2) = 2E[X] + E[X^2] = 2(4) + 24$$
$$= 32$$

Let $C = 2X + X^2$

$$V(C) = E[C^2] - [E[C]]^2$$
$$E[C^2] = E[(2X + X^2)(2X + X^2)] = E[4X^2 + 4X^3 + X^4]$$
$$= 4E[X] + 4E[X^3] + E[X^4]$$
$$= 4(24) + 4(192) + 1920 = 2784$$
$$V[C] = 2784 - 32^2 = 1760$$

B. (5 points) The values of X that bound at least three-quarters of the distribution of $2X + X^2$. Using Tchebysheff's theorem with k = 2, we know that 75% of the data falls within $\mu \pm 2\sigma = 32 \pm 2\sqrt{1760}$. This gives an interval of (-51,9, 115.9). However, $2X + X^2$ must be positive, so the interval is (0, 115.9).

Question 5 (10 points)

The first quartile is defined as the point in the distribution of X such that F(X) = 0.25. Similarly, the third quartile is defined as the point in the distribution of X such that F(X) = 0.75. The distance between these two points is defined as the inter-quartile range. Which distribution has a larger interquartile range, a normal distribution with $\mu = 0$, and $\sigma = 1$, or an exponential distribution with $\beta = 4$? Show your work.

<u>For the normal</u>

Using z-table, the points that describe F(Q1) = .25 and F(Q3) = .75 are given by Q1 = -0.67 and Q2 = 0.67. This the interquartile range is 0.67 – (-0.67) = 1.34.

<u>For the exponential(4)</u>

$$\int_0^{Q1} \frac{1}{4} e^{-y/4} dy = -e^{-y/4} \Big|_0^{Q1} = 1 - e^{-Q1/4} = .25$$

$$\Rightarrow Q1 = -4\ln(0.75) = 1.15$$

$$\int_0^{Q3} \frac{1}{4} e^{-y/4} dy = -e^{-y/4} \Big|_0^{Q3} = 1 - e^{-Q3/4} = .75$$

$$\Rightarrow Q3 = -4\ln(0.25) = 5.55$$

So interquartile range is $5.55 - 1.15 = 4.40$.

So the exp(4) has a larger IQR.

Question 6 (18 points total)

Let X have the density function,

$$f(x) = \begin{cases} cx^2 e^{-x/4}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

A. (6 points) Find the value of *c* that makes this a valid density function.

Since this is a gamma(3,4),

$$c = \frac{1}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{4^3 \Gamma(3)} = \frac{1}{64(2)} = \frac{1}{128}$$

B. (6 points) Find the moment generating function of this density if it exists.

$$m(t) = (1-4t)^{-3}$$

C. (6 points) Use the moment generating function you found in B to calculate the mean and variance of X.

$$m'(t) = 12(1-4t)^{-4} \rightarrow E(X) = 12$$
$$m''(t) = 192(1-4t)^{-5} \rightarrow E(X^2) = 192$$
$$E(X) = 12$$
$$V(X) = 192 - 12^2 = 48$$

Extra Credit:

(5 points)

An urn contains n red and m black balls. They are withdrawn one at a time until a total of r, $r \leq n$, red balls have been withdrawn. Find the probability that a total of k balls are withdrawn. Be sure to specify the possible values that k can take on.

The first k-1 draws follow a hypergeometric distribution.
However, all k draws do not since we are fixing the last draw
to be a red ball.

So, the probability of drawing r-1 red balls in k-1 draws
is given by,

$$\frac{\binom{n}{r-1}\binom{m}{(k-1)-(r-1)}}{\binom{n+m}{k-1}}$$

The probability of drawing a red ball on the kth try
is, $\frac{n-(r-1)}{(n+m)-(k-1)}$.

Thus the final probability is given by,

$$\frac{\binom{n}{r-1}\binom{m}{(k-1)-(r-1)}}{\binom{n+m}{k-1}}\frac{n-(r-1)}{(n+m)-(k-1)}; \quad k = r, r+1, r+2, \ldots, r+m$$