# Crystals and Mirror Constructions for Quotients

by

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## Abstract

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This thesis develops a new approach to computing the quantum cohomology of symplectic reductions of partial flag varieties $X$, known as weight varieties. Motivated by a conjecture of Teleman [125], we use a mirror family Landau-Ginzburg model $(M_P, f_P)$ of $X$ introduced by Rietsch [115] to give a conjectural explicit description of the quantum cohomology of weight varieties. We specialise to the class of polygon spaces $\mathcal{P}_{r,n}$: these are symplectic reductions of the complex Grassmannian of 2-planes $\mathrm{Gr}_{\mathbb{C}}(2, n)$ by the maximal torus action. Polygon spaces in low rank have been classified and the quantum cohomology of these varieties is known. As a result, we are able to verify our conjectural description explicitly.

In addition, we investigate the appearance of combinatorial structures in representation theory in the mirror symmetry of complete flag varieties. We show that, on the $B$-model side, the extended string cone $\underline{C}_{\mathbf{i}}$ introduced by Caldero [24] to define toric degenerations on the $A$-model can be recovered via a discretisation process known as *tropicalisation*. Specifically, using a non-standard parameterisation of $M_B$ we recover the precise inequalities defining $\underline{C}_{\mathbf{i}}$. This provides an explicit approach to results previously obtained by Berenstein-Kazhdan [13].

For my Mum.

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# Chapter 1

# Introduction

## 1.1 Background

The phenomenon of mirror symmetry was first observed in the Hodge numbers of pairs of Calabi-Yau manifolds in the late 1980s by Greene-Plesser [55] and Candelas-Lynker-Schimmrigk [25]. In taking a Calabi-Yau resolution of the quotient of the smooth quintic three-fold $X \subseteq \mathbb{P}^4$ by the natural action of $\mathbb{Z}_5^5$, Greene-Plesser provided one of the first mirror constructions, constructing a family of mirrors $\{M_\omega\}$ to $X$. The subsequent calculation, by Candelas-de la Ossa-Greene-Parkes [26], of enumerative invariants of $X$ by period calculations on the mirror family $\{M_\omega\}$ stunned the algebraic geometry community and hinted at a remarkable connection between mirror pairs.

In this section we recall certain aspects of the development of mirror symmetry since this original contribution.

### Mirror constructions, mirror conjectures and quantum cohomology

Batyrev [6] generalised the Greene-Plesser construction, providing a general framework to construct mirror candidates for Calabi-Yau hypersurfaces in toric varieties. These methods were later extended to Calabi-Yau complete intersections in toric varieties [8] . To construct a mirror candidate of Calabi-Yau hypersurface in a (Fano, Gorenstein) toric variety, Batyrev used the moment polytope of the toric variety. He showed that the toric variety is Fano if and only if its moment polytope is reflexive, In this case, the dual polytope is also reflexive and corresponds to a Fano toric variety. The mirror candidates are then constructed from data attached to the dual reflexive polytope.

Givental [48] proposed an extension of mirror symmetry to Fano manifolds, conjecturing that the mirror to a Fano manifold $X$ ($A$-model) is a Landau-Ginzburg model $(M, f)$ ($B$-model), where $M \to T$ is a smooth family of varieties with quasi-affine total space and $f : M \to \mathbb{C}$ is a nonconstant holomorphic function called the *superpotential*. Givental's *mirror conjecture* states an equivalence between the quantum cohomology $D$-module of $X$ and the $D$-module generated by certain oscillatory integrals $\int_{\Gamma_t \subseteq M_t} \exp(f_t/\hbar)\omega_t$ associated with a family $(M_t, f_t, \omega_t)_{t\in T}$. Here $(M_t, f_t, \omega_t)_{t\in T}$ is the data of non-vanishing top forms $\omega_t$ on the fibres of the family $M \to T$, $f_t$ is the restriction of $f$ to each fibre, and $\Gamma_t$ is an appropriate family of Morse-theoretic middle dimensional cycles of $Re(f_t)$. In this setting, mirror symmetry for the pair $(X, (M, f))$ predicts an isomorphism

$$qH^*(X) \cong \mathrm{Jac}(f) := \mathbb{C}[M]/(\partial f)$$

between the (small) quantum cohomology algebra $qH^*(X)$ of $X$ and the Jacobian ring of $f$. The quantum structure is given by variation in the family. In particular, homogeneous spaces for compact, connected Lie groups should exhibit mirror-symmetric phenomena.

In the case of complete flag manifolds $\mathrm{SL}_{n+1}(\mathbb{C})/B$, Givental verified the mirror conjecture by considering a "2-dimensional Toda lattice" [49]. Starting from a *(complete) Gelfand-Tsetlin quiver* having $(n+1)(n+2)/2$ vertices,

$$\begin{tikzcd}
\bullet \arrow[d] & & & & \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & & & \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & & \\
{} \arrow[d] & \cdots \arrow[d] & \cdots \arrow[r] & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Givental constructs a trivial family $Y_t$, $t \in (\mathbb{C}^\times)^n$, with each $Y_t$ isomorphic to an $n(n+1)/2$-dimensional complex algebraic torus. The superpotential and volume forms are constructed from the combinatorial data of the quiver. The relation with the Toda lattice was later exploited to provide presentations of the quantum cohomology for complete flag manifolds $G/B$ ([50] in type $A$; [83] in general type).

Givental's construction and mirror conjectures are generalised by Batyrev-Ciocan-Fontanine-Kim-van Straten (BCKS) in [10] (see also [9]) to provide a conjectural mirror family to complete intersections in partial flag manifolds $\mathrm{SL}_{n+1}(\mathbb{C})/P$. The initial input is now a (degenerate) Gelfand-Tsetlin quiver corresponding to the stabiliser $P$ of a fixed partial flag in $\mathbb{C}^{n+1}$. From this data is constructed a toric degeneration of $\mathrm{SL}_{n+1}(\mathbb{C})/P$ to a (in general, singular) Gorenstein Fano toric variety $V_P$ (see also [124], [54]). The conjectural mirror family to $\mathrm{SL}_{n+1}(\mathbb{C})/P$ is now a small toric desingularisation $\hat{V}_P$ of $V_P^*$ [7], where $V_P^*$ is the toric variety whose moment polytope is dual to that of $V_P$. In certain cases, the generalised GKZ hypergeometric series ([46]) of the toric variety $\hat{V}_P$ is seen to give a solution to the quantum $D$-module of $\mathrm{SL}_{n+1}(\mathbb{C})/P$ [10, Section 5].

The theory of *standard monomial bases*, due to Lakshmibai-Musili-Seshadri [92], provides a monomial basis for spaces of sections of projective embeddings of partial flag varieties $G/P$. The explicit nature of these bases has led to consequences in geometry and representation theory: for example, effective determinations of the singular locus of Schubert varieties, and generalisations of the Littlewood-Richardson rule [92, Chapter 13]. In addition, Gonciulea-Lakshmibai [54] use the standard monomial basis to construct toric degenerations of $G/B$ and Schubert varieties in miniscule $G/P$.

For $G$ an arbitrary connected semisimple, simply-connected complex algebraic group, a generalisation was given by Caldero [24] who, for every reduced expression $\mathbf{i}$ of the longest element $w_0$ of the Weyl group of $G$, obtained toric degenerations for all Schubert varieties in $G/B$. The key tool used by Caldero is the specialisation at $q = 1$ of (the dual of) *Lusztig's canonical basis* [99] for the upper/lower part of the quantised universal enveloping algebra $U_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ of $G$. A key feature of his work is the construction of a lattice-semigroup whose points parameterise bases of representations of $G$, the *string cone lattice semigroup*, and a lattice-semigroup parameterising a weight basis of the coordinate ring of the base affine space known as the *extended string cone*. Alexeev-Brion [1] later determine conditions for the central toric fibre of Caldero's degeneration to be Gorenstein Fano, with a view to obtaining a mirror family construction similar to BCKS.

Rietsch [115] describes a Lie-theoretic construction of a mirror family $(M_P^t, f_P^t, \omega_t)$ to the flag variety $G/P$. Here $G$ is connected semisimple, simply-connected complex algebraic group, $P \subseteq G$ a parabolic subgroup; fix a maximal torus $T \subseteq G$. The remarkable feature is that the family $M_P$ is a subvariety of a Borel containing the dual torus ${}^LT$ inside the Langlands dual ${}^LG$, with base $Z(L_{{}^LP})$ being the centre of a Levi subgroup $L_{{}^LP} \subseteq {}^LP$ inside the dual ${}^LP$ of $P$. Building on the unpublished work of Dale Peterson, Rietsch gives an isomorphism

$$qH^*(G/P) \cong \mathrm{Jac}(f_P),$$

and an extension to the $T$-equivariant setting. Specific ($T$-equivariant) mirror conjectures for $G/P$ are stated, extending the previously formulated conjectures of Givental and BCKS in type $A$.

The $T$-equivariant Rietsch mirror conjectures are verified for complete flag varieties $G/B$ by Lam [93] (see earlier work of Rietsch [116] for the non-equivariant case) and, recently, for miniscule flag varieties $G/P$ by Lam-Templier [94]. An essential feature of these works is Berenstein-Kazdan's notion of *geometric crystal* [12], [13]. Incredibly, the mirror family $(M_P, f_P, \omega_t)$ proposed by Rietsch is exactly equal to the decorated geometric crystals studied by Berenstein-Kazhdan; moreover, it appears that the introduction of this central object was unbeknownest to either author(s). Originally introduced as a tool to understand $W$-invariant $\gamma$-functions appearing in the local Langlands program [20], geometric crystals associated to $G$ are birational models of the *Kashiwara crystals* [77] associated to the Langlands dual ${}^LG$. Kashiwara crystals are combinatorial models of Kashwiara's crystal bases [76], which are specialisations of Lusztig's canonical base at $q = 0$. The recovery of the Kashiwara crystal from the geometric crystal is via the process of *tropicalisation*.

## Homological mirror symmetry

There have been proposed two *intrinsic* approaches to mirror symmetry: Kontsevich's program of *homological mirror symmetry* and the geometric approach proposed by Strominger-Yau-Zaslow (known as the *SYZ conjecture*).

At his 1994 ICM address, Kontsevich proposed the following conjecture:

**Conjecture** (Homological Mirror Symmetry Conjecture [88])**.** For a mirror pair of Calabi-Yau manifolds $(X, M)$, (some enhanced version of) the Fukaya category $\mathfrak{F}(X)$ of $X$ [40, 41] is equivalent to the derived category of coherent sheaves on $M$. The same statement holds with the roles of $X$ and $M$ swapped.

Kontsevich's program of homological mirror symmetry (HMS) highlights a profound connection between the symplectic geometry of a Calabi-Yau manifold $X$ and the complex geometry of its mirror $M$. A consequence of the homological mirror symmetry conjecture would be the Givental-BCKS-Rietsch mirror conjectures relating quantum cohomology $D$-modules with oscillatory integrals and identifications of quantum cohomology with Jacobian rings of superpotentials. In particular, obtaining an identification of quantum cohomology with the Jacobian ring of the superpotential for the mirror provides a first order approximation to any conjecture in homological mirror symmetry.

In his 2014 ICM address, Teleman [125] described a conjectural mirror construction for symplectic reductions $M /\!\!/ G$, with $G$ a compact, connected Lie group and $M$ a compact Hamiltonian $G$-space. This construction is a consequence of a new program of topological actions of $G$ on Fukaya categories arising from Hamiltonian $G$-spaces and gauging topological quantum field theories. When $M = G/L$ is a coadjoint orbit considered as a Hamiltonian $T$-space, for $T \subseteq G$ a maximal torus, Teleman conjectured the following:

**Conjecture A** (Teleman, [125])**.** Let $\nu$ be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian $T$-action. Let $t \in Z({}^L L_{\mathbb{C}})$ denote the symplectic structure on $G/L$. Then, the Fukaya category of the symplectic reduction $(G/L) /\!\!/ T(\nu)$ can be computed as the category $\mathrm{Hom}(S_\nu, \Lambda(t))$, where $S_\nu$ is the cotangent fibre over $\exp(\nu)$ considered as an element in ${}^L T$ by duality.

Identify $G/L$ with $G_{\mathbb{C}}/P$, for some parabolic subgroup of the complexification $G_{\mathbb{C}}$ of $G$. Let $(M_P, f_P)$ be the mirror family introduced by Rietsch. Then, $M_P$ is a subgroup of a Borel containing the dual torus ${}^L T_{\mathbb{C}}$ inside the Langlands dual ${}^L G_{\mathbb{C}}$. A first approximation to the veracity of Teleman's conjecture would be the following consequence for quantum cohomology:

**Conjecture B** (Teleman, [125])**.** Let $\nu$ be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian $T$-action. Let $t \in Z({}^L L_{\mathbb{C}})$ denote the symplectic structure on $G/L$. Then, the quantum cohomology of the symplectic reduction $(G/L) /\!\!/ T(\nu)$ can be computed as the Jacobian ring of the restriction of the $T$-equivariant superpotential to a generic fibre of the canonical quotient homomorphism $e : M_P \to {}^L T_{\mathbb{C}}$. The quantum structure comes from the variation of $t \in Z({}^L L_{\mathbb{C}})$.

Moreover, if $G$ has nontrivial (finite) centre $Z$, then the number of critical points appears with multiplicity $|Z|$.

Let $G$ be a compact, connected Lie group, $T \subseteq G$ a maximal torus. The symplectic reductions of coadjoint orbits (with respect to the Hamiltonian $T$-action) are known as *weight varieties* and were initially studied in [86]. Weight varieties should be considered as geometric analogues of weight spaces of irreducible representations. Indeed, if $\lambda$ is a dominant weight and $(G/T, \mathcal{L}_\lambda)$ is the complete flag variety together with a polarisation $\mathcal{L}_\lambda$ such that $H^0(G/T, \mathcal{L}_\lambda) \cong V(\lambda)^*$, the unique irreducible representation of $G$ having lowest weight $-\lambda$, then the weight variety $(G/T) /\!\!/ T(\nu)$ inherits a polarisation $\mathcal{L}_{\lambda,\nu}$ and $\dim H^0((G/T) /\!\!/ T(\nu), \mathcal{L}_{\lambda,\nu}) = \dim V(\lambda)^*_\nu$, where $V(\lambda)^*_\nu$ is the $\nu$-weight space of $V(\lambda)^*$.

The (co)homology of weight varieties has been investigated and computed by several authors ([64],[37], [51], [53],[52]). For certain weight varieties that can be explicitly identified, the quantum cohomology has been computed (for example [31]). However, a general framework for computations of the quantum cohomology of weight varieties (in the spirit of Rietsch, say) have yet to be obtained. One aim of this thesis is to develop an approach to address this problem.

An important class of weight varieties are quotients of $\mathrm{Gr}_{\mathbb{C}}(2, n)$. These symplectic reductions have a moduli intepretation as the moduli of spatial $n$-gons with fixed side-lengths $r \in \mathbb{R}^n_{>0}$, called *polygon spaces* $\mathcal{P}_{r,n}$. Polygon spaces are related to the moduli space $\overline{M}_{0,n}$ of stable $n$-pointed rational curves ([81], [37], [85]) and the moduli space of flat connections on a punctured sphere [86]. Examples of polygon spaces include $\mathbb{P}^{n-3}_{\mathbb{C}}$, $(\mathbb{P}^1_{\mathbb{C}})^{n-3}$ and blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ at $0, 1, 2, 3, 4$ points [38].

## SYZ conjecture

In [123], Strominger-Yau-Zaslow interpret mirror symmetry in terms of $T$-*duality* in string theory.

**Conjecture** (SYZ Conjecture [123])**.** If $X$ and $M$ are mirror pairs of Calabi-Yau $n$-folds, then there exist fibrations $g : X \to B$ and $g' : M \to B$, whose fibres are special Lagrangian, with general fibre an $n$-torus. Furthermore, these fibrations are dual in the sense that, canonically, $X_b \cong H^1(M_b, \mathbb{R}/\mathbb{Z})$ and $M_b \cong H^1(X_b, \mathbb{R}/\mathbb{Z})$, whenever the fibres $X_b$ and $M_b$ are non-singular tori.

The SYZ conjecture proposes an approach for a geometric construction of the mirror of a Calabi-Yau manifold $X$: once a Lagrangian torus fibration $X \to B$ of $X$ has been obtained, attempt to build $M$ by dualising the toric fibres [57], [67]. In the Fano setting, Auroux [4] extended the SYZ-conjecture:

**Conjecture** ([4, Conjecture 1.1 ])**.** Let $X$ be a compact Kähler manifold, $D \subseteq X$ an anticanonical divisor, $\Omega$ a holomorphic volume form defined over $X \setminus D$. Then, the mirror LG-model $(M, f)$ can be constructed as a moduli space of special Lagrangian tori in $X \setminus D$ equipped with flat U(1)-connections, with superpotential $f : M \to \mathbb{C}$ given by Fukaya-Oh-Ohta-Ono's $m_0$ obstruction to Floer homology [39].

One method of constructing Lagrangian torus fibration of a variety $X$ is via integrable systems on (a dense subset of) $X$. The notion of a toric degeneration of an integrable system on a projective manifold was introduced by Nishinou-Nohara-Ueda [112] (see also [62]): roughly, this is a toric degeneration of $X$ such that the integrable can be transported to an integrable system on the toric limit. If the toric limit is Fano and admits a small resolution then the authors compute Floer-theoretic potential functions for $X$, using deep results of [39].

By degenerating the Gelfand-Tsetlin integrable system [61] on the complete flag variety $\mathrm{SL}_{n+1}(\mathbb{C})/B$ and making explicit computations of holomorphic disks, Nishinou-Nohara-Ueda compute the potential function. In this way, they recover the superpotential introduced by Givental using the Gelfand-Tsetlin quiver.

In later work, Nohara-Ueda [113] construct a family of integrable systems $\Psi_\Gamma$ on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ parameterised by triangulations $\Gamma$ of a fixed convex planar $n$-gon $\Pi$ (the *reference polygon*). For any triangulation $\Gamma$ of $\Pi$, the integrable system $\Psi_\Gamma$ admits a toric degeneration (originally determined in [121]) and, if the Kahler structure on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ represents the first Chern class, then the toric limit of $\Psi_\Gamma$ is Gorenstein Fano and admits a small resolution. This allows them to compute the potential function associated to $\mathrm{Gr}_{\mathbb{C}}(2, n)$. Moreover, for a certain triangulation they recover the superpotential constructed by physical considerations in [35].

The integrable system $\Psi_\Gamma$ is invariant with respect to the natural torus action on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ and induces an integrable system $\Phi_\Gamma$ on the polygon spaces $\mathcal{P}_{r,n}$. Moreover, the toric degenerations of $\Psi_\Gamma$ induces a toric degeneration of $\Phi_\Gamma$. The moment polytope of the central limit of $\Phi_\Gamma$ is realised as the intersection of the moment polytope of $\Psi_\Gamma$ with an affine subspace.

## Thesis results

This thesis is comprised of two parts.

In the first part, we develop a new approach to computing the quantum cohomology rings of symplectic reductions of partial flag varieties $X$, also known as *weight varieties*. Motivated by a conjecture of Teleman [125], we use a mirror family $(M_p, f_P)$ of $X$ introduced by Rietsch [115] to give a conjectural explicit presentation of the quantum cohomology of weight varieties. We determine explicit expressions for the superpotential $f_P$ with respect to a family of parameterisations of $M_P$, originally studied by Lusztig, Fomin-Zelevinsky in the context of total positivity of reductive groups.

In order to test our conjectural description, we specialise our focus to the class of *polygon spaces* $\mathcal{P}_{r,n}$ in type $A$. The polygon space $\mathcal{P}_{r,n}$ is a symplectic quotient of the Grassmannian of 2-planes in $\mathbb{C}^n$, and has a modul interpretation as the moduli space of spatial $n$-gons with fixed consecutive side length given by $r \in \mathbb{R}^n_{>0}$.

Polygon spaces of low dimension have been classified: the moduli space of 4-gons $\mathcal{P}_{r,4}$ is diffeomorphic to $\mathbb{P}^1_{\mathbb{C}}$ (independent of $r$); the moduli space of 5-gons $\mathcal{P}_{r,5}$ is a rational surface diffeomorphic to either $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^2_{\mathbb{C}}$ or the Del Pezzo surface obtained by blowing up $\mathbb{P}^2_{\mathbb{C}}$ at $1, 2, 3, 4$ points.

We obtain the following results.

**Theorem 3.4.15.** Let $X = \mathrm{Gr}_{\mathbb{C}}(2,4) = \mathrm{SL}_4(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, $(M_P, F_P)$ the Rietsch mirror family. Let $e : M_P \to {}^LT$ be the equivariant structure map. Let $\mathcal{P}_{r,4}$, $r \in \mathbb{Z}^4_{>0}$, be the space of 4-gons realised as the symplectic reduction of $X$. Then, the quantum cohomology of $\mathcal{P}_{r,4}$ can be computed as the Jacobian ring of the restriction of $f_P$ to a generic fibre of $e$.

**Theorem 3.4.16.** Let $X = \mathrm{Gr}_{\mathbb{C}}(2,5) = \mathrm{SL}_4(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, $(M_P, F_P)$ the Rietsch mirror family. Let $e : M_P \to {}^LT$ be the equivariant structure map. Let $\mathcal{P}_{r,5}$, $r \in \mathbb{Z}^4_{>0}$, be the space of 4-gons realised as the symplectic reduction of $X$. Let $r \in \{(1,1,1,1,2), (1,2,2,3)\}$. Then, for the quantum cohomology of $\mathcal{P}_{r,5}$ can be computed as the Jacobian ring of the restriction of $f_P$ to a generic fibre of $e$.

In the second part of this thesis, we investigate the appearance of combinatorial structures in representation theory, known as *Kashiwara crystals*, in the mirror symmetry of partial flag varieties. We show that, on the $B$-model side of mirror symmetry for the complete flag variety, the extended string cone introduced by Caldero to define a family of toric degenerations on the $A$-model side, and later used by Alexeev-Brion [1] in the context of mirror symmetry, can be recovered via a discretisation process known as *tropicalisation*. Specifically, using a non-standard parameterisation of $M_P$ we explicitly recover the extended string cone via tropicalisation.

**Theorem 4.4.5.** Let $G$ be a reductive complex algebraic group, $B \subseteq G$ a Borel subgroup. Let $(M_B, f_B)$ be the Rietsch mirror family to ${}^LG/{}^LB$. Then, for every $\mathbf{i}$, a reduced expression of the longest element $w_0$ of the Weyl group of $G$, there exists a parameterisation $j_{\mathbf{i}}$ of a dense open subset of $M_P$ with respect to which the tropical locus $\{\mathrm{Trop}(f_B) \geq 0\}$ is precisely the extended string cone $\underline{C}_{\mathbf{i}}$. Moreover, the $\lambda$-inequalities defining $\underline{C}_{\mathbf{i}}$ are explicitly recovered.

We conclude with an observation on how crystal structure on the $B$-model side controls aspects of integrable systems appearing on the $A$-model side.

## 1.2 Outline

The structure of this thesis is as follows: in Chapter 2 we introduce the background required from symplectic geometry. In Section 2.1 we discuss the general setting of Hamiltonian $G$-spaces, for $G$ a compact, connected Lie group, and introduce the moment map. For Hamiltonian $T$-spaces, with $T$ a compact torus, we see that the moment polytope admits internal structure, decomposing into chambers and walls. In Section 2.2 we introduce, following Marsden-Weinstein-Meyer, the symplectic reduction of a Hamiltonian $G$-space. In this section we describe the important example of polygon spaces $\mathcal{P}_{r,n}$. In Section 2.3 we discuss the symplectic geometry of coadjoint orbits and show that they are Hamiltion $T$-spaces for the coadjoint action of $T$. We describe the wall structure on their moment polytopes in terms of root-theoretic data. In Section 2.4 we provide some examples of the symplectic reduction of coadjoint orbits (so-called weight varieties). We finish the chapter with a brief discussion on the well-known connection between the symplectic reduction and $GIT$ quotients.

In Chapter 3 we develop a new approach to computing the quantum cohomology of weight varieties. In Section 3.1 we introduce the Landau-Ginzburg model $(M_P, f_P)$ first proposed by Rietsch, and discuss its connection to computing ($T$-equivariant) quantum cohomology of partial flag varieties. In Section 3.2 we present Teleman's conjectural mirror construction for the quantum cohomology of weight varieties. Our conjectural description of the quantum cohomology of weight varieties is given in Conjecture 3.2.7. In Section 3.3 we obtain explicit expressions for the superpotential $f_P$, which will be essential in verifying Conjecture 3.2.7. Section 3.4 specialises to the type $A$ setting and we make new quantum cohomology computations for polygon spaces $\mathcal{P}_{r,n}$ of low rank, thereby verifying Conjecture 3.2.7 in this setting. We conclude this chapter with an outline of future directions of research.

Chapter 4 is an investigation into the appearance of representation-theoretic structures in the mirror symmetry for partial flag varieties. In Section 4.1, we recall background from the theory of quantised universal enveloping algebras. Section 4.2 introduces Lusztig's canonical basis $\mathcal{B}$ and its consequences for representation theory. In particular, we give a brief account of the role of $\mathcal{B}$ in determining combinatorial tensor product multiplicity formulae. We define several parameterisations of $\mathcal{B}$ including the the family of string parameterisations due to Littelmann. We conclude this section by introducing the extended string cone $\underline{C}_{\mathbf{i}}$ and the $\lambda$-inequalities that define it. In Section 4.3, we give a brief account of Kashiwara's theory of crystals and their geometric counterparts developed by Berenstein-Kazhdan. In this section we develop the tool of tropicalisation, realised as a functor from a certain class of varieties to Set. Section 4.4 introduces a non-standard parameterisation of the Rietsch mirror $(M_B, f_B)$, and we state and prove our main result Theorem 4.4.5. We conclude with a discussion illuminating intriguing similarities between the hierarchy of a family of integrable systems on the $A$-model side (introduced in [113]) and the crystal structure obtained in Theorem 4.4.5 (on the $B$-model side).

## 1.3 Notation

In this preliminary section we introduce the conventions and definitions we adopt throughout this thesis.

Let $\mathbb{P}$ be a monoid, $A$ some nonempty set. We write $\mathbb{P}A$ for the $\mathbb{P}$-span of $A$ (i.e. the free $\mathbb{P}$-module generated by $A$ if $A$ is not a subset of some $\mathbb{P}$-module). If $G$ is a group then $Z(G)$ will denote the centre of $G$.

We introduce our conventions for Lie theoretic objects, for further details see [122]. Let $G$ be a complex reductive algebraic group; unless otherwise stated $G$ will be assumed connected. We fix a choice of maximal torus $T \subseteq G$, a Borel subgroup $B_+ \subseteq G$ containing $T$, and opposite Borel subgroup $B_-$ so that $B_- \cap B_+ = T$. We write $N_\pm$ for the unipotent radical of $B_\pm$. A parabolic subgroup $P \subseteq G$ admits a Levi decomposition $P = L_P N_P$, where $N_P$ is the unipotent radical, $L_P$ is reductive and $N_P \cap L_P = \{e\}$. We write $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}_\pm, \mathfrak{n}_\pm, \mathfrak{p}$ for the corresponding Lie algebras. We denote the Weyl group $W = N_G(T)/T$. For $w \in W$, $t \in T$, we will sometimes write $t^w = wtw^{-1}$.

The above choices are uniquely determined (up to isomorphism) by the root datum $\Psi(G) = (X, R, X^\vee, R^\vee)$ associated to the pair $(G, T)$. Here

$$X \coloneqq \mathrm{Hom}(T, \mathbb{G}_m), \quad X^\vee \coloneqq \mathrm{Hom}(\mathbb{G}_m, T)$$

and $R \subseteq X$ is the set of roots relative to $T$; $R^\vee \subseteq X^\vee$ is the corresponding set of coroots. We will interchangeably refer to elements of $X$ (resp. $X^\vee$) as weights or characters (resp. coweights or cocharacters). There is a canonical pairing between $X$ and $X^\vee$

$$\begin{array}{rccc} \langle , \rangle : & X \times X^\vee & \longrightarrow & \mathbb{Z} \\ & (\lambda, \mu^\vee) & \longmapsto & \langle \lambda, \mu^\vee \rangle \end{array}$$

defined by $(\lambda \circ \mu^\vee)(z) = z^{\langle \lambda, \mu^\vee \rangle}$. With respect to this pairing there are canonical identifications

$$X \cong \mathrm{Hom}(X^\vee, \mathbb{Z}), \quad X^\vee \cong \mathrm{Hom}(X, \mathbb{Z}).$$

Denote the root lattice $Q \coloneqq \mathbb{Z}R$, and the coroot lattice $Q^\vee \coloneqq \mathbb{Z}R^\vee$. We define the lattice of integral weights $\Pi \subseteq X_\mathbb{Q} \coloneqq \mathbb{Q} \otimes X$ to be the lattice

$$\Pi = \{\lambda \in X_\mathbb{Q} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ \alpha^\vee \in R^\vee\}.$$

We write $w(\lambda)$, or simply $w\lambda$, (resp. $w(\lambda^\vee)$) for the action of $w \in W$ on $\lambda \in X$ (resp. $\lambda^\vee \in X^\vee$); this descends to an action on $Q$ (resp. $Q^\vee$) preserving $R$ (resp. $R^\vee$).

The choice of Borel $B_+$ induces a choice of positive roots $R^+ \subseteq R$ and simple roots $S \subseteq R^+$. We write $S^\vee \subseteq X^\vee$ for the simple coroots. We write $Q_{\geq 0} \coloneqq \mathbb{Z}_{\geq 0} R^+ = \mathbb{Z}_{\geq 0} S$, and $Q_{\leq} \coloneqq \mathbb{Z}_{\geq 0} R^- = -\mathbb{Z}_{\geq 0} S$, with analogous definitions for $Q^\vee_{\geq 0}$, $Q^\vee_{\leq 0}$. The monoid of dominant weights is

$$X_+ \coloneqq \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0,\ \alpha^\vee \in S^\vee\},$$

with an analogous definition for the monoid of dominant coweights $X_+^\vee$. A weight $\lambda \in X$ is antidominant if $w_0(\lambda) \in X_+$ (see below for the definition of $w_0$); there is an analogous definition of antidominant coweight. We denote the monoid of dominant weights (resp. dominant coweights) $X_-$ (resp. $X_-^\vee$).

If $G$ is a reductive complex algebraic group with root datum $(X, R, X^\vee, R^\vee)$, then we call $(\Pi, S, \Pi^\vee, S^\vee)$ the associated Cartan datum.

There is a partial ordering on $X$ (resp. $X^\vee$) defined as follows:

$$\lambda \geq \mu \text{ (resp. } \lambda^\vee \geq \mu^\vee) \quad \Longleftrightarrow \quad \lambda - \mu \in Q_+ \text{ (resp. } \lambda^\vee - \mu^\vee \in Q_+^\vee).$$

There is a unique identification

$$\begin{array}{ccc} S & \longleftrightarrow & S^\vee \\ \alpha & \longleftrightarrow & \alpha^\vee \end{array}$$

such that $\langle \alpha, \alpha^\vee \rangle = 2$. Using this identification we index both $S$ and $S^\vee$ by the same set $I$, so that $S = \{\alpha_i\}_{i \in I}$ and $S^\vee = \{\alpha_i^\vee\}_{i \in I}$, where $\alpha_i^\vee = (\alpha_i)^\vee$. Define the fundamental weights $\varpi_i \in \Pi$, $i \in I$, to be the weights such that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$, for $i, j \in I$.

There is an involution $i \mapsto i^*$ on $I$, where $-w_0(\alpha_i) = \alpha_{i^*}$, for $i \in I$ (see below for definition of $w_0$). This is equivalent to $w_0 s_i w_0 = s_{i^*}$.

For $\alpha \in R^+$, we make a choice of corresponding root subgroup homomorphism

$$x_\alpha : \ \mathbb{A}^1 \longrightarrow N_+$$

satisfying

$$t x_\alpha(c) = x_\alpha(\alpha(t)c)t, \quad t \in T.$$

We write $x_i \coloneqq x_{\alpha_i}$ and $y_i \coloneqq x_{-\alpha_i}$, for $i \in I$. If we totally order $R^+ = \{\beta_1, \dots, \beta_m\}$ then there is an isomorphism of varieties $\prod_{j=1}^m x_{\beta_i} : \mathbb{A}^m \to N_+$. It is well-known that $G$ is generated by $T$ and $\operatorname{im} x_\alpha$, $\alpha \in S \cup -S$.

If $P \supseteq B_+$ then there is a unique subset $J = J(P) \subseteq I$ such that $P$ is generated by $B_+$ and $\operatorname{im} y_j$, $j \in J$. We write $P = P_J$ if we want to make $J$ explicit; $P$ is called a standard parabolic subgroup. The Levi subgroup $L_P$ is generated by $T$ and $\operatorname{im} x_j$, $\operatorname{im} y_j$, $j \in J$. We write $W_P$ for the Weyl group of the pair $(L_P, T)$. If $P = P_J$ then $W_P$ is identified with the subgroup of $W$ generated by $s_j$, $j \in J$. Write $W^P \subseteq W$ for the set of minimal length coset representatives of $W/W_P$. The centre $Z(L_P)$ is a subgroup of $T$ equal to $T^{W_P}$, the elements in $T$ fixed by $W_P$

The Weyl group $W$ is generated by reflections $s_i$, $i \in I$, subject to the standard Coxeter relations

$$s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1,$$

where $m_{ij} = 2, 3, 4$ or $6$ whenever, respectively, $\langle \alpha_i, \alpha_j^\vee \rangle = 0, 1, 2, 3$. The latter relations are called the braid relations. If $w = s_{i_1} \cdots s_{i_r}$, with $r$ minimal, then we define $\ell(w) = r$, the length of $w$. For such a presentation of $w$ we call the sequence $(i_1, \dots, i_r)$ a reduced expression of $w$. The set of all reduced expressions of $w$ will be denoted $R(w)$. There is a unique element $w_0 \in W$, with $w_0^2 = e \in W$, having maximal length $\ell(w_0) = \dim N_+ = |R^+|$. We write $W_P$ for the Weyl group of the pair $(L_P, T)$. If $P = P_J$ then $W_P$ is identified with the subgroup of $W$ generated by $s_j$, $j \in J$. Let $w_0^P \in W_P$ be the longest element. Define $w_P^{-1} \in W$ to be the longest element of $W^P$.

A result of Matsumoto, Tits (see [18]) shows that any two reduced expressions are related by braid relations. Define

$$\overline{s}_i \coloneqq x_i(-1)y_i(1)x_i(-1), \quad i \in I.$$

Then, $\overline{s}_i \in N_G(T)$ and is a representative of $s_i \in W$. The $\overline{s}_i$, $i \in I$, satisfy the braid relations so that the element

$$\overline{w} = \overline{s}_{i_1} \cdots \overline{s}_{i_r} \in N_G(T),$$

where $(i_1, \ldots, i_r) \in R(w)$, is a well-defined representative of $w \in W$. In particular, if $u, v \in W$ and $w = uv$, with $\ell(w) = \ell(u) + \ell(v)$, then $\overline{w} = \overline{uv}$. In general, the $\overline{s}_i$ do not satisfy $\overline{s}_i^2 = 1 \in G$, although we have $\overline{s}_i^2 = \alpha_i^\vee(-1)$. For the longest element $w_0 \in W$, $\overline{w}_0 B_\pm \overline{w}_0^{-1} = B_\mp$: in particular, $\overline{w}_0 N_\pm \overline{w}_0^{-1} = N_\mp$.

We will use the following involutive antiautomorphisms of $G$

(i) the transpose $g \mapsto g^T$, determined by

$$x_i(a)^T = y_i(a), \quad y_i(a)^T = x_i(a), \quad t^T = t, \quad i \in I, t \in T; \tag{1.3.1}$$

(ii) the positive inverse $g \mapsto g^\iota$, determined by

$$x_i(a)^\iota = x_i(a), \quad y_i(a)^\iota = y_i(a), \quad t^\iota = t^{-1}, \quad i \in I, t \in T. \tag{1.3.2}$$

These antiautomorphisms commute with each other and with the involutive antiautomorphism $g \mapsto g^{-1}$ of $G$. We have

$$\overline{w}^T = \overline{w}^{-1}, \quad \text{and} \quad \overline{w}^\iota = \overline{w^{-1}}.$$

For any $g = utv \in G_0 = N_- T N_+$ admitting Gauss decomposition, we define

$$\pi^-(g) = u, \quad \pi^0(g) = t, \quad \pi^+(g) = v, \quad \pi^{\leq 0}(g) = ut, \quad \pi^{\geq 0}(g) = tv. \tag{1.3.3}$$

Following [15, Section 6], we define the generalised minors $\Delta_{u\mu, v\mu}$, $\mu \in X_+$, $u, v \in W$, to be the regular functions on $G$ whose restriction to $\overline{u} G_0 \overline{v}^{-1}$ is given by

$$\Delta_{u\mu, v\mu}(g) \coloneqq \mu(\pi^0(\overline{u}^{-1} g \overline{v})).$$

When $G$ is type $A$, so that $W$ is identified with a group of permutations, the generalised minor $\Delta_{u\varpi_i, v\varpi_i}$ is the matrix minor with row set $I = \{u(1), \ldots, u(i)\}$ and column set $J = \{v(1), \ldots, v(i)\}$.

If $G$ is a reductive complex algebraic group with root datum $(X, R, X^\vee, R^\vee)$ then the Langlands dual group ${}^L G$ is the reductive complex algebraic group with dual root datum $(X^\vee, R^\vee, X, R)$. When referring to subgroups of the Langlands dual we will write ${}^L T$, ${}^L B_\pm$, ${}^L N_\pm$ etc. We will also write $X({}^L T)$ when referring to the weight lattice of the pair $({}^L G, {}^L T)$, with similar notation for the other objects defined above.

# Chapter 2

# Symplectic geometry of coadjoint orbits

In this chapter we introduce the necessary background from symplectic geometry and Hamiltonian actions of compact Lie groups. In Section 2.1 we introduce Hamiltonian $G$-spaces, where $G$ is a compact, connected Lie group. We introduce the additional data of the moment map and describe how the moment polytope admits a chamber structure. In Section 2.2 we recall the notion of symplectic reduction and indicate the construction of the modul space of spatial polygons. We also present the construction of the complex Grassmannian of 2-planes $\mathrm{Gr}_{\mathbb{C}}(2,n)$ via symplectic reduction. In Section 2.3 we consider in more detail a special case of Hamiltonian $G$-spaces, namely, the coadjoint orbits of $G$. We show that the chamber structure of the moment polytope can be obtained from the structure of the Weyl group and root system in $\mathfrak{g}$. Section 2.4 introduces the class of weight varieties: these are those symplectic manifolds that can be realised as reductions of coadjoint orbits. We close this section with an analysis of the chamber structure for the $\mathrm{Gr}_{\mathbb{C}}(2,n)$. Finally, in Section 2.5 we briefly discuss the relationship between symplectic reduction and GIT quotients in algebraic geometry, focusing mainly on the case of coadjoint orbits.

Most of the material in this chapter is standard and can be found in any graduate textbook on symplectic geometry, for example [3],[107]. The material concerning reduction of coadjoint orbits and the wall structure of moment polytopes can be found in [86], or [58].

## 2.1 Hamiltonian $G$-spaces

Let $G$ be a compact, connected Lie group, $(M,\omega)$ a symplectic manifold. We are interested in symplectic (left) actions of $G$ on $M$,

$$\begin{aligned} a: \quad G &\longrightarrow \mathrm{Symp}(M,\omega) \\ g &\longmapsto a_g \end{aligned}$$

where $\mathrm{Symp}(M,\omega)$ is the group of symplectomorphisms of $(M,\omega)$.

**Remark 2.1.1.** We will write $g \cdot m := a_g(m)$, $g \in G$, $m \in M$, whenever a group $G$ acts on a set $M$.

For each $X \in \mathfrak{g}$, we let $\underline{X}$ denote the infinitesimal action of $X$ on $M$ induced by $a$. This is the (unique) vector field on $M$ with flow $\{a_{\exp(-tX)}\}_{t\in\mathbb{R}}$. Explicitly, for each $m \in M$, we consider the orbit map

$$\begin{array}{rccc} \sigma_m : & G & \longrightarrow & M \\ & g & \longmapsto & g \cdot m \end{array}$$

Then, $\underline{X}_m := (d\sigma_m)_e(-X) \in T_mM$.

**Remark 2.1.2.**

1) The sign appearing in the definition of $\underline{X}$ ensures that $[\underline{X}, \underline{Y}] = \underline{[X, Y]}$.

2) We will refer to the vector field $\underline{X}$, $X \in \mathfrak{g}$, as a *fundamental vector field.*

For any $m \in M$, the tangent space at $m$ to the orbit $G \cdot m$ is spanned by the fundamental vector fields

$$T_m(G \cdot m) = \{\underline{X}_m \mid X \in \mathfrak{g}\}$$

**Definition 2.1.3.** The action $a : G \to \mathrm{Symp}(M, \omega)$ is *Hamiltonian* if, for every $X \in \mathfrak{g}$, there exists a function

$$\mu : M \longrightarrow \mathfrak{g}^*,$$

such that

1) for each $X \in \mathfrak{g}$, the function

$$\begin{array}{rccc} \mu^X : & M & \longrightarrow & \mathbb{R} \\ & m & \longmapsto & \langle \mu(m), X \rangle \end{array}$$

is a Hamiltonian function for the fundamental vector field $\underline{X}$, so that

$$d\mu^X = i_{\underline{X}}\omega.$$

2) $\mu$ is equivariant: for every $g \in G$, we have

$$\mu \circ a_g = \mathrm{Ad}^*(g) \circ \mu$$

Here $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ is the coadjoint action of $G$ on $\mathfrak{g}^*$.

We call the datum $(M, \omega, a, \mu)$ a *Hamiltonian $G$-space*, and $\mu$ is the *moment map.*

**Remark 2.1.4.** We will refer to a Hamiltonian $G$-space $(M, \omega, a, \mu)$ as $(M, \omega)$, the extra data of the action and choice of a moment map being implicit.

If $\mu$ is the moment map for a Hamiltonian $G$-space then, for $X \in \mathfrak{g}$, we have

$$\mu^X = H^X \circ \mu,$$

where $H^X : \mathfrak{g}^* \to \mathbb{R}$ is the linear 'evaluation at $X$' map. Infinitesimally, we obtain

$$i_{\underline{X}}\omega = d\mu^X = H^X \circ d\mu \implies \omega_m(\underline{X}_m, V) = \langle d\mu_m(V), X\rangle, \quad V \in T_m M. \tag{2.1.1}$$

Hence, $\ker d\mu_m$ is the $\omega$-complement of $T_m(G \cdot m)$, and the annihilator of $\operatorname{im} d\mu_m \subseteq \mathfrak{g}^*$ is

$$(\operatorname{im} d\mu_m)^\circ = \{X \in \mathfrak{g} \mid \underline{X}_m = 0\},$$

which can be identified with the Lie algebra of the stabiliser $G_m = \{g \in G \mid a_g(m) = m\}$.

**Lemma 2.1.5.** *$d\mu_m$ is surjective if and only if the stabiliser $G_m$ is discrete (hence, finite). In particular, $m \in M$ is a critical point of $\mu$ if and only if* $\dim G_m \geq 1$.

Let $H \subseteq G$ be a subgroup, and denote $(H)$ be the *type of $H$*: $(H)$ is the set of subgroups of $G$ that are conjugate to $H$. The *orbit-type stratification* of $M$ is the partition of $M$ into subsets

$$M_{(H)} = \{m \in M \mid G_m \in (H)\},$$

where $H \subseteq G$ is a subgroup. By the equivariant Darboux theorem [60], each subset $M_{(H)}$ is a union of $G$-invariant symplectic submanifolds of $M$ (not necessarily of the same dimension). Moreover, $M_{(H)}$ is a Hamiltonian $G$-space with moment map $\mu_{|M_{(H)}}$.

**Remark 2.1.6.** An example when the connected components have different dimensions is easily seen: consider the action of $S^1$ on $M = \mathbb{C}P^2$, where $e^{it} \cdot [t \cdot z_0 : z_1 : z_2]$, $e^{it} \in S^1$, then the fixed point set is $M_{(S^1)}$ consists of the point $[1 : 0 : 0]$ and the line at infinity $[0 : z_1 : z_2]$.

If $G$ is commutative then $(H) = \{H\}$, and $M$ partitions into a union of $G$-invariant submanifolds

$$M = \bigcup_{H \subseteq G} \operatorname{Fix}(H), \tag{2.1.2}$$

where

$$\operatorname{Fix}(H) = \{m \in M \mid h \cdot m = m, \text{ for every } h \in H\}.$$

When $M$ is compact the union in (2.1.2) is finite [3, Ch. 2]. Hence, by Lemma 2.1.5 we obtain the following result.

**Proposition 2.1.7.** *Let $G$ be a commutative compact, connected Lie group, and $(M, \omega)$ be a Hamiltonian $G$-space, with $\mu$ the corresponding moment map. The collection of critical points of $\mu$ is a union of Hamiltonian $G$-spaces of the form* $\operatorname{Fix}(H)$, *for $H \subseteq G$ a positive dimensional stabiliser of some point: if $M$ is compact then this union is finite. The critical values of $\mu$ are the images of these submanifolds.*

**Remark 2.1.8.** For the remainder of this section we assume that $G = T$ is a torus.

Let $(M, \omega)$ be a Hamiltonian $T$-space with moment map $\mu$.

**Theorem 2.1.9** (Atiyah, Guillemin-Sternberg [2, 59]). *Let $(M, \omega)$ be a Hamiltonian $T$-space with moment map $\mu : M \to \mathfrak{t}^*$. Assume that $M$ is compact. Then, the set of fixed points of the action is a finite union of connected symplectic submanifolds $C_1, \ldots, C_N$. Moreover, $\mu$ is constant on each of these components, $\mu(C_i) = \xi_i \in \mathfrak{t}^*$, and $\mu(M)$ is the convex hull of $\xi_1, \ldots, \xi_N$,*

$$\mu(M) = \left\{ \sum_{i=1}^{N} c_i \xi_i \;\middle|\; \sum_{i=1}^{N} c_i = 1, c_j \geq 0 \right\} \subseteq \mathfrak{t}^*$$

**Definition 2.1.10.** The *moment polytope* associated to a Hamiltonian $T$-space $(M, \omega)$ with moment map $\mu$ is the polytope $\mu(M) \subseteq \mathfrak{t}^*$.

**Remark 2.1.11.** For a Hamiltonian $T$-space $(M, \omega)$ with moment map $\mu$ we will write $\Delta_M \coloneqq \mu(M)$ for its moment polytope, or simply $\Delta$ if there is no risk of confusion.

Let $F_1, \ldots, F_r$ be the closures of the connected components $X_1, \ldots, X_r$ of the orbit-type strata from Proposition 2.1.7 with corresponding stabiliser subgroups $T_1, \ldots, T_r \subseteq T$. Each $F_j$ is a connected component of $\mathrm{Fix}(T_j)$, where $T_j$ is the stabiliser of a generic point in $F_j$, and $F_j$ is a Hamiltonian $T$-space with moment map $\mu_{|F_j}$.

Set $H_j = T/T_j$. The $T$-action has kernel $T_j$ and $F_j$ inherits an effective Hamiltonian $H_j$-action. The moment map $\mu_j : F_j \to \mathfrak{h}_j^*$ for this action is unique up to a constant, which we now specify.

For any $m \in F_j$, (2.1.1) shows that $\mathrm{im}\left(d\mu_{|F_j}\right)_m$ is the annihilator of $\mathfrak{t}_j$ inside $\mathfrak{t}^*$, so that $\mathrm{im}\left(d\mu_{|F_j}\right)_m = \mathfrak{h}_j^*$. In particular, $\mu(F_j)$ is some translate of $\mathfrak{h}_j^*$ inside $\mathfrak{t}^*$. For example, if $m_j \in F_j$ as a fixed point for the $T$-action then $\mu(F_j) \subseteq \mu(m_j) + \mathfrak{h}_j^*$. Hence, a moment map $\mu_j$ for the $H_j$-action on $F_j$ can be specified by requiring that $\mu(F_j)$ lands in $\mathfrak{h}_j^*$.

**Remark 2.1.12.** Applying Theorem 2.1.9 we obtain $\mu(F_j)$ is a convex polytope, for each $j$. By the above discussion, this convex polytope is a subset of the intersection $\Delta_M \cap (\xi_j + \mathfrak{h}_j^*)$, where $\xi_j$ is the image of some fixed point in $F_j$. Furthermore, if $F_i \subseteq F_j$ then $\mu(F_i) \subseteq \mu(F_j)$.

**Definition 2.1.13.** A *codimension-$k$ wall in $\Delta_M$* (or simply, a *wall in $\Delta_M$*) is the image of some $F_j$ in $\Delta_M$, where $\dim T_j = k$. We denote the set of all walls by $\mathcal{F}$.

A wall is *proper* if it has positive codimension.

An *internal wall* is a proper wall containing a point in the interior of $\Delta_M$. An *external wall* is any proper wall that is not internal.

**Definition 2.1.14.** A *chamber* is a connected component of the set

$$\Delta_M^\circ \coloneqq \Delta_M \setminus \bigcup_{\substack{d \in \mathcal{F} \\ d \text{ proper}}} d \tag{2.1.3}$$

**Remark 2.1.15.** By Proposition 2.1.7, $\Delta_M^\circ$ is precisely the set of regular values of $\mu$. Moreover, $\Delta$ decomposes into a union of polytopes: the interior of each polytope corresponds to a unique chamber.

## 2.2 Symplectic reduction

Let $G$ be a compact Lie group (not necessarily a torus) and $(M, \omega)$ a Hamiltonian $G$-space with moment map $\mu : M \to \mathfrak{g}^*$. Let $\lambda \in \Delta_M$. By equivariance of the moment map, the level set $\mu^{-1}(\lambda)$ is $G_\lambda$-invariant, where $G_\lambda$ is the stabiliser of $\lambda$ under the coadjoint action of $G$ on $\mathfrak{g}^*$. If $\lambda$ is a regular value then the action of $G_\lambda$ is locally free and the quotient $\mu^{-1}(\lambda)/G_\lambda$ admits, at worst, orbifold singularities.

Consider the diagram:

$$\begin{tikzcd} \mu^{-1}(\lambda) \arrow[r, hook, "i_\lambda"] \arrow[d, "p_\lambda"'] & M \\ \mu^{-1}(\lambda)/G_\lambda & \end{tikzcd}$$

Whenever the action on $\mu^{-1}(\lambda)$ is free, the quotient $M_\lambda$ admits a canonical manifold structure, and the quotient map $p_\lambda$ is a principal $G_\lambda$-bundle.

**Theorem 2.2.1** (Marsden-Weinstein [104], Meyer [109])**.** *Let $G$ be a compact Lie group and $(M, \omega)$ be a Hamiltonian $G$-space with moment map $\mu$. Assume that $\lambda \in \mathfrak{g}^*$ is a regular value of $\mu$. Then, the topological quotient $\mu^{-1}(\lambda)/G_\lambda$ is a symplectic orbifold of dimension $\dim M - 2\dim G_\lambda$, and there exists a unique symplectic form $\omega^{\mathrm{red}}$ on $\mu^{-1}(\lambda)/G_\lambda$ such that*

$$p_\lambda^* \omega^{\mathrm{red}} = i_\lambda^* \omega$$

*In particular, whenever $G$ acts freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ inherits the structure of a symplectic manifold.*

**Definition 2.2.2.** The symplectic orbifold $(\mu^{-1}(\lambda), \omega^{\mathrm{red}})$ is the *symplectic reduction of $M$ by $G_\lambda$ at $\lambda$*, and will be denoted $M /\!\!/ G_\lambda(\lambda)$.

We will need the following result.

**Proposition 2.2.3** (Reduction in stages)**.** *Let $G = H \times K$ be a compact, connected Lie group and $(M, \omega)$ be a Hamiltonian $G$-space with moment map $\mu$. Let $\mu_H$ and $\mu_K$ be the moment maps of the induced actions of $H$ and $K$ on $M$. Then, $\mu$ can be canonically identified with $\mu_H \times \mu_K$. For any regular value $\nu = (\alpha, \beta) \in \mathfrak{g}^* = \mathfrak{h}^* \times \mathfrak{k}^*$ so that $\alpha$ is a regular value of $\mu_H$ and $\beta$ is a regular value of $\mu_K$, the symplectic reduction $\mu^{-1}(\nu)/G$ is symplectomorphic to the symplectic reduction of the Hamiltonian $K$-space $\mu_H^{-1}(\alpha)/H$ at $\beta$. An analogous statement holds with the roles of $H$ and $K$ reversed.*

We finish this section with some examples of symplectic reduction that we will return to in Section 2.5.

**Example 2.2.4.** Let $(S^2, \omega)$ be the 2-sphere with its standard $SO(3)$-invariant symplectic form $\omega$ so that $\int_{S^2} \omega = 4\pi$. Let $n \geq 3$ and fix $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, a sequence of positive real numbers. The product manifold $(S^2)^n$ is given the symplectic form $\Omega_r = \sum_{i=1}^n r_i \omega_i$, where $\omega_i$ is the pull-back of $\omega$ along the $j^{th}$ projection. Points in $(S^2)^n$ can be identified with polygonal paths in $\mathbb{R}^3$ having consecutive edge-lengths $r_1, \ldots, r_n$. The natural action of $SO(3)$ provides a diagonal action on $((S^2)^n, \Omega_r)$ with moment map

$$\begin{array}{rccc} \mu_{r,n}: & (S^2)^n & \longrightarrow & \mathfrak{so}(3)^* \cong \mathbb{R}^3 \\ & (a_1, \ldots, a_n) & \longmapsto & \sum_{i=1}^n r_i a_i \end{array}$$

The set $\mu_{r,n}^{-1}(0)$ can be identified with polygons in $\mathbb{R}^3$ having consecutive side-lengths $r_1, \ldots, r_n$. The critical points of $\mu_{r,n}$ are the degenerate polygons: these are those polygons $P$ lying completely in a line. In particular, whenever $n$ is odd, there are no degenerate polygons in $\mu_{r,n}^{-1}(0)$.

When $n$ is odd, Theorem 2.2.1 implies that the symplectic reduction is a smooth $(2n-6)$-dimensional symplectic manifold.

**Definition 2.2.5.** The symplectic reduction of $(S^2)^n$ by SO(3) at 0 is called the *moduli space of spatial $n$-gons* $\mathcal{P}_{r,n}$ or, simply, a *polygon space*.

Polygon spaces have been studied intensively over the past couple of decades and have connections with the moduli space $\overline{M}_{0,n}$ of stable $n$-pointed rational curves ([81], [37], [85]) and the moduli space of flat connections on a punctured sphere [86]. We record the following examples of polygon spaces.

**Example 2.2.6.** (i) The simplest case when $n = 3$ is trivial as there is exactly one 3-gon in $\mathbb{R}^3$ with prescribed side-lengths, up to SO(3)-invariance. Hence, $\mathcal{P}_{r,n}$ is a point.

(ii) Let $n = 4$. In this case $\mathcal{P}_{r,n}$ is a 2-dimensional symplectic manifold diffeomorphic to $S^2$ (i.e. independent of $r$).

(iii) Let $n = 5$. Then, $\mathcal{P}_{r,n}$ is either $S^2 \times S^2$ or a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at $0, 1, 2, 3, 4$ points [37].

(iv) When $r = (1, 1 \ldots, 1, n, n, n)$, $\mathcal{P}_{r,n}$ is identified with $(\mathbb{P}^1_{\mathbb{C}})^{n-3}$. This can be seen by the following argument: consider $\mathcal{P}_{r,n}$ to be the moduli space of weighted configurations of points in $\mathbb{P}^1_{\mathbb{C}}$ [33]. Let $z_1, \ldots, z_n$ be such a configuration all lying in the same affine chart, which we take to be $\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$. Consider the cross-ratios

$$w_i = \frac{z_{n-2} - z_n}{z_n - z_{n-1}} \cdot \frac{z_i - z_{n-1}}{z_n - z_i}, \quad i = 1, \ldots, n-3.$$

The points $z_{n-2}, z_{n-1}, z_n$ never collide on the subset of semi-stable configurations implying the existence of a map

$$\mathcal{P}_{r,n} \longrightarrow (\mathbb{P}^1_{\mathbb{C}})^{n-3}$$

This map is an isomorphism.

(v) When $r = (2, 2, \ldots, 2, 2n-3)$, so that the polygons have one long side, $\mathcal{P}_{r,n}$ is diffeomorphic to $\mathbb{P}^{n-3}_{\mathbb{C}}$.

We recall the well-known construction of Grassmannians of $k$-planes in $\mathbb{C}^n$ via symplectic reduction.

**Example 2.2.7.** Consider $G = U(k)$ acting on the space of $n \times k$ complex matrices $\mathrm{Mat}_{n\times k}(\mathbb{C}) \cong \mathbb{C}^{kn}$: $k \cdot A = Ak^{-1}$. Consider $\mathrm{Mat}_{n\times k}(\mathbb{C})$ equipped with the standard symplectic form on complex affine space. Then, we have

$$\begin{array}{rccl} \mu: & \mathrm{Mat}_{n\times k}(\mathbb{C}) & \longrightarrow & \mathfrak{u}(k)^* \\ & A & \longmapsto & \mu(A) : x \mapsto \frac{i}{2}\operatorname{tr}(xAx^*) \end{array}$$

Using a $U(k)$-equivariant identification (via the Killing form, say), we identify $\mathfrak{u}(k)^* \cong \mathfrak{u}(k)$ and the moment map is

$$\mu(A) = \frac{i}{2}A^*A.$$

The point $y = \frac{i}{2}\mathbb{I}_k \in \mathfrak{u}(k)$ is fixed by the coadjoint action of $U(k)$ on $\mathfrak{u}(k)$ and

$$\mu^{-1}(y) = \{A \mid \mathrm{Mat}_{n\times k} \mid A^*A = \mathbb{I}_k\}.$$

This is the set of unitary $k$-frames in $\mathbb{C}^m$. Hence, the symplectic reduction $\mu^{-1}(y)/U(k)$ is the complex Grassmannian of $k$-planes in $\mathbb{C}^n$.

## 2.3 Coadjoint orbits

Let $G$ be a compact, connected Lie group, $T \subseteq G$ a maximal torus, $W$ the Weyl group for the pair $(G, T)$. Denote the Lie algebra of $G$ (resp. $T$) by $\mathfrak{g}$ (resp. $\mathfrak{t}$), and let $\mathfrak{g}^*$ (resp. $\mathfrak{t}^*$) by the dual vector space.

In this section we will consider the (co)adjoint actions of $G$ on $\mathfrak{g}$ and $\mathfrak{g}^*$. For each $X \in \mathfrak{g}$, we define the following function on $\mathfrak{g}^*$:

$$\begin{array}{rccl} H^X: & \mathfrak{g}^* & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & H^X(\xi) = \langle \xi, X\rangle \end{array}$$

Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit, and $\xi \in \mathcal{O}$. Identify $\mathcal{O}$ with $G/G_\xi$ via the orbit map

$$\begin{array}{rccc} \sigma_\xi : & G & \longrightarrow & \mathcal{O} \\ & g & \longmapsto & g \cdot \xi \end{array}$$

where $G_\xi = \{g \in G \mid g \cdot \xi = \xi\}$ is the stabiliser of $\xi$ in $G$. With this identification, the tangent space to $\mathcal{O}$ at $\xi$ is $\mathfrak{g}/\mathfrak{g}_\xi$, where $\mathfrak{g}_\xi$ is the Lie algebra of $G_\xi$.

For the coadjoint action of $G$ on $\mathfrak{g}^*$, the fundamental vector field $\underline{X}$ generated by $X \in \mathfrak{g}$ satisfies

$$\langle \underline{X}_\xi, Y \rangle = \langle \xi, [X, Y] \rangle, \qquad \xi \in \mathfrak{g}^*, Y \in \mathfrak{g}. \tag{2.3.1}$$

In particular, for any $X, Y \in \mathfrak{g}$, we have

$$\underline{X}(H_Y) = H^{[X,Y]} = \langle dH^Y, \underline{X} \rangle, \tag{2.3.2}$$

For each $\xi \in \mathfrak{g}^*$, there is defined on $\mathfrak{g}$ a skew-symmetric bilinear form $\omega_\xi$,

$$\omega_\xi(Y, X) := \langle \xi, [X, Y] \rangle, \qquad X, Y \in \mathfrak{g}. \tag{2.3.3}$$

The form $\omega_\xi$ descends to a nondegenerate skew-symmetric form on the quotient $\mathfrak{g}/\mathfrak{g}_\xi$: the kernel of $\omega_\xi$ is precisely $\mathfrak{g}_\xi$. Hence, we obtain a nondegenerate skew-symmetric bilinear form on the tangent space $T_\xi\mathcal{O}$, which we also denote $\omega_\xi$. In this way, we obtain a nondegenerate 2-form $\omega_{\mathcal{O}}$ on $\mathcal{O}$, known as the *Kostant-Kirillov-Souriau (KKS) form* [91, 120].

For $X, Y \in \mathfrak{g}$, (2.3.3) implies that $\omega_{\mathcal{O}}(\underline{X}, \underline{Y}) = H^{[Y,X]}$. Fixing $X \in \mathfrak{g}$, and using (2.3.1), we obtain, for all $Y \in \mathfrak{g}$,

$$\langle i_{\underline{X}}\omega_{\mathcal{O}}, \underline{Y} \rangle = \omega_{\mathcal{O}}(\underline{X}, \underline{Y}) = \langle dH^X, \underline{Y} \rangle.$$

As the vector fields $\underline{Y}$, for $Y \in \mathfrak{g}$, span the tangent spaces to $\mathcal{O}$ at each point, we have

$$i_{\underline{X}}\omega_{\mathcal{O}} = dH^X. \tag{2.3.4}$$

Applying the Lie derivative $\mathcal{L}_{\underline{X}}$ to $dH^Y$, for $X \in \mathfrak{g}$, this shows, together with (2.3.2),

$$\mathcal{L}_{\underline{X}} dH^Y = dH^{[X,Y]} = i_{\underline{[X,Y]}}\omega_{\mathcal{O}}. \tag{2.3.5}$$

Using the formula $[\mathcal{L}_{\underline{X}}, i_{\underline{Y}}] = i_{[\underline{X},\underline{Y}]}$, and the fact that $[\underline{X}, \underline{Y}] = \underline{[X, Y]}$ (Remark 2.1.2),

$$\mathcal{L}_{\underline{X}} i_{\underline{Y}}\omega_{\mathcal{O}} = i_{\underline{[X,Y]}}\omega_{\mathcal{O}} + i_{\underline{Y}}\mathcal{L}_{\underline{X}}\omega_{\mathcal{O}}$$

By (2.3.4) and (2.3.5), we obtain $i_{\underline{Y}}\mathcal{L}_{\underline{X}}\omega_{\mathcal{O}} = 0$, for all $X, Y \in \mathfrak{g}$. Hence, $\mathcal{L}_{\underline{X}}\omega_{\mathcal{O}} = 0$, for all $X \in \mathfrak{g}$ and $\omega_{\mathcal{O}}$ is $G$-invariant.

Using Cartan's formula, and (2.3.4), we find that, for every $X \in \mathfrak{g}$,

$$0 = \mathcal{L}_{\underline{X}}\omega_{\mathcal{O}} = di_{\underline{X}}\omega_{\mathcal{O}} + i_{\underline{X}}d\omega_{\mathcal{O}} = i_{\underline{X}}d\omega_{\mathcal{O}}$$

Since the fundamental vector fields $\underline{X}$, $X \in \mathfrak{g}$, span the tangent spaces to $\mathcal{O}$ at every point, $\omega_{\mathcal{O}}$ is closed.

In summary,

**Proposition 2.3.1** ([91, 120]). *Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit. Then, there exists a $G$-invariant symplectic form $\omega_{\mathcal{O}}$ on $\mathcal{O}$, the* Kirillov-Kostant-Siourau form. *The coadjoint action is Hamiltonian with moment map being the canonical inclusion*

$$\mu_{\mathcal{O}} : \mathcal{O} \hookrightarrow \mathfrak{g}^*$$

*so that the fundamental vector fields $\underline{X}$, for $X \in \mathfrak{g}$, admit Hamiltonian functions $H^X$.*

Restricting the action of $G$ to $T$, a coadjoint orbit (equipped with its KKS symplectic structure) is a Hamiltonian $T$-space. A moment map for the action is the composition

$$\mathcal{O} \xhookrightarrow{\mu_{\mathcal{O}}} \mathfrak{g}^* \longrightarrow \mathfrak{t}^*, \tag{2.3.6}$$

where the second map is the canonical projection, which we will also denote $\mu_{\mathcal{O}} : \mathcal{O} \to \mathfrak{t}^*$.

As a compact Hamiltonian $T$-space with moment map $\mu$, the image

$$\Delta_{\mathcal{O}} \coloneqq \mu_{\mathcal{O}}(\mathcal{O}) \subseteq \mathfrak{t}^*$$

of the moment map is a convex polytope (Theorem 2.1.9) with additional internal chamber/wall structure, which we now describe.

Choosing a $G$-invariant positive-definite inner product (for example, the Killing form) on $\mathfrak{g}$ induces a $G$-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, setting up a correspondence between adjoint and coadjoint orbits. Using this isomorphism, we consider $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$ (by identifying $\mathfrak{t}^* \subseteq \mathfrak{g}^*$ with $\mathfrak{t} \subseteq \mathfrak{g}$). An adjoint orbit admits the structure of a Hamiltonian $T$-space by pulling back the symplectic form on the corresponding coadjoint orbit. The moment map $\mu_{\mathcal{O}}$ for the $T$-action becomes orthogonal projection on to the subspace $\mathfrak{t}$.

**Remark 2.3.2.** In the proceeding discussion, we will fix such an identification and will refer to elements of $\mathfrak{g}^*$ as elements of $\mathfrak{g}$, without reference to the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$.

Suppose that $\mathcal{O} = \mathcal{O}_X \subseteq \mathfrak{g}$ is the adjoint orbit through $X \in \mathfrak{t}$. Then, $\mathcal{O} \cap \mathfrak{t} = W \cdot X$ (see [34, Ch.3]), and each point in the intersection is a fixed point of the $T$-action. Conversely, if $Y \in \mathcal{O}$ is a fixed point of the $T$-action then, for any $Z \in \mathfrak{t}$, we have $[Z, Y] = 0$, and $Z$ is an element of the centraliser of $\mathfrak{t}$ in $\mathfrak{g}$. But $T$ is a maximal torus so that its centraliser is itself. Hence, $Z \in \mathcal{O} \cap \mathfrak{t}$ and $Z = w \cdot X$, for some $w \in W$.

Combining the previous discussion with Theorem 2.1.9 proves the following:

**Theorem 2.3.3** (Kostant [90]). *Let $\xi \in \mathfrak{g}^*$, and $\mathcal{O} = G \cdot \xi$ be the coadjoint orbit through $\xi$, considered as a Hamiltonian $T$-space. Then, the moment polytope is realised as*

$$\Delta_{\mathcal{O}} = \mathrm{Conv}(W \cdot \xi). \tag{2.3.7}$$

*Moreover, each $w \cdot \xi \in \Delta_{\mathcal{O}}$ is a vertex.*

**Remark 2.3.4.** Theorem 2.3.3 appeared first as the Schur-Horn theorem: let $A$ be an $n \times n$ matrix with diagonal entries $a_1, \ldots, a_n$ and spectrum $\lambda_1 \geq \ldots \geq \lambda_n$. Then, $(a_1, \ldots, a_n)$ lies in the convex hull of $w \cdot (\lambda_1, \ldots, \lambda_n)$. This result was later generalised by Kostant to Theorem 2.3.3.

Let $\lambda_1, \ldots, \lambda_N \in \mathfrak{t}^*$ denote the fundamental weights and their $W$-conjugates, and let $H_i = H_{\lambda_i}$ be the stabiliser for the (co)adjoint action of $G$. Let $\mathfrak{h}_i$ be the Lie algebra of $H_i$. Then, $T$ is a maximal torus in $H_i$, for each $i$, so we can consider $W_i$, the Weyl group of the pair $(H_i, T)$. The Weyl group $W_i$ is a parabolic subgroup of $W$ (the Weyl group of $(G, T)$) and is generated by the reflections corresponding to those roots that are orthogonal to $\lambda_i$. The weight $\lambda_i$ generates a 1-dimensional torus $S_i \subseteq H_i$. In particular, by Proposition 2.1.7, any point in $\mathcal{O} = \mathcal{O}_X$ that is fixed by $S_i$ is a critical point of the moment map $\mu_{\mathcal{O}}$.

**Lemma 2.3.5.** *1) The fixed point set of $S_i$ is $\mathcal{O} \cap \mathfrak{h}_i$.*

*2) $\mu_{\mathcal{O}}(\mathcal{O} \cap \mathfrak{h}_i) = \bigcup_{w \in W} \operatorname{Conv}(W_i \cdot wX)$.*

*Proof.* (a) $Z \in \mathcal{O}$ is fixed by $S_i$ if and only if $[\lambda_i, Z] = 0$ if and only if $Z \in \mathfrak{h}_i$.

(b) Suppose that $\mathcal{O} = \mathcal{O}_X$, for $X \in \mathfrak{t}$. The intersection $\mathcal{O} \cap \mathfrak{h}_i \supseteq \mathcal{O} \cap \mathfrak{t} = W \cdot X$ is $H_i$-invariant, so it consists of the $H_i$-orbits passing through $W \cdot X$. Each orbit $H_i w \cdot X$, for $w \in W$, is a symplectic submanifold and becomes a Hamiltonian $T$-space with moment map being the restriction of $\mu_{\mathcal{O}}$ to $H_i w \cdot X$. Hence, by Theorem 2.3.3

$$\mu_{\mathcal{O}}(H_i w \cdot X) = \operatorname{Conv}(W_i w \cdot X) \tag{2.3.8}$$

and the result follows.

□

**Theorem 2.3.6** ([65, 58]). *Let $\mathcal{O} = \mathcal{O}_\xi$ be a coadjoint orbit, $\xi \in \mathfrak{g}^*$, consider as a Hamiltonian $T$-space with moment map $\mu_{\mathcal{O}} : \mathcal{O} \to \mathfrak{t}^*$. Let $\lambda_1, \ldots, \lambda_N \in \mathfrak{t}^*$ be collections of fundamental weights and all their $W$-conjugates, and denote the stabiliser of $\lambda_i$ in $G$ by $H_i$. Write $W_i$ for the Weyl group of the pair $(H_i, T)$. Then, the critical points of the moment map are the symplectic submanifolds*

$$H_i w \cdot \xi, \qquad w \in W, i = 1, \ldots, N. \tag{2.3.9}$$

*The codimension-1 walls in the moment polytope $\Delta_{\mathcal{O}}$ are the convex polytopes*

$$\operatorname{Conv}(W_i w \cdot \xi), \qquad w \in W, i = 1, \ldots, N. \tag{2.3.10}$$

*Proof.* Identify $\mathcal{O}$ with an adjoint orbit, so that $\mathcal{O} = \mathcal{O}_X$, for $X \in \mathfrak{t}$. A critical point $Y \in \mathcal{O}$ must be fixed by some positive dimensional subtorus $T' \subseteq T$ (Proposition 2.1.7). Hence, for any $Z \in \mathfrak{t}'$, where $\mathfrak{t}'$ is the Lie algebra of $T'$, we have $[Y, Z] = 0$. Thus, $Y$ lies in the centraliser of $\mathfrak{t}'$. The wall structure on $\Delta_{\mathcal{O}}$ implies that the centraliser of any point in $\mathcal{O}$ must be a subalgebra of one of the maximal centralisers $\mathfrak{h}_i$ ([58, Ch. 5]). Hence, $Y \in \mathfrak{h}_i$ □

**Example 2.3.7.** Let $G = \mathrm{SU}(2)$, $T \subseteq G$ the diagonal matrices. We identify $\mathfrak{g}^*$ with the set of traceless $2 \times 2$ Hermitian matrices $\mathcal{H}_2$

$$\begin{aligned} \mathrm{tr}: \quad \mathcal{H}_2 &\longrightarrow \mathfrak{g}^* \\ A &\longmapsto (X \mapsto i\,\mathrm{tr}(AX)) \end{aligned} \tag{2.3.11}$$

This map is $G$-equivariant and the moment map for the resulting Hamiltonian $T$-space $\mathcal{H}_2$ is projection onto the diagonal.

Any $G$-orbit is uniquely determined by a non-negative real number $\lambda \in \mathbb{R}_{\geq 0}$. Let $\mathcal{O} = \mathcal{O}_\lambda$ be the corresponding orbit. Thus, $A \in \mathcal{O}$ if and only if its eigenvalues are $\pm\lambda$. Theorem 2.3.3 implies that the top-left diagonal entry of $A$ must lie in the interval $[-\lambda, \lambda]$. By Theorem 2.3.6, the walls of the interval are $\{\pm\lambda\}$, and the chamber (equal to the set of regular values of the moment map) is the open interval $(-\lambda, \lambda)$.

We check this directly: consider a traceless Hermitian matrix

$$A = \begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix}, \qquad a, b, c \in \mathbb{R}.$$

such that $A$ has eigenvalues $\pm\lambda$. Then, we must have

$$\lambda^2 = -\det A = a^2 + b^2 + c^2 \geq a^2 \implies a \in [-\lambda, \lambda].$$

Moreover, if $a \in [-\lambda, \lambda]$ then $a^2 \leq \lambda^2$ and we can choose $z \in \mathbb{C}$ such that $|z|^2 = \lambda^2 - a^2$. Then, the matrix

$$A = \begin{bmatrix} a & z \\ \overline{z} & -a \end{bmatrix}$$

lies in $\mathcal{O}$. That the set of regular values for the moment map is $(-\lambda, \lambda)$ follows immediately.

**Example 2.3.8.** Let $G = \mathrm{SU}(3)$. Then, the moment polytope with chamber stucture for a generic (six dimensional) orbit is given in Figure 2.1.

## 2.4 Weight varieties

Let $G$ be a compact, semisimple Lie group, $T \subseteq G$ a maximal torus in $G$. Let $\mathcal{O} = \mathcal{O}_\xi \subseteq \mathfrak{g}^*$ be the coadjoint orbit through $\xi \in \mathfrak{g}^*$, considered as a Hamiltonian $T$-space with moment map $\mu : \mathcal{O} \to \mathfrak{t}^*$. Let $\Delta = \mu(\mathcal{O}) \subseteq \mathfrak{t}^*$ be the (convex) moment polytope, $\Delta^\circ$ the union of chambers in $\Delta$ (Definition 2.1.14). Recall that $\Delta^\circ$ is precisely the set of regular values of $\mu$.

The coadjoint action of $T$ on $\mathfrak{t}^*$ is trivial so that, for any $\xi \in \mathfrak{t}^*$, the stabiliser of $\xi$ in $T$ is $T$ itself. The equivariance of the moment map implies that the level set $\mu^{-1}(\xi)$ carries a (proper) $T$-action.

**Definition 2.4.1** (Knutson [86])**.** Let $\nu \in \mathfrak{t}^*$. The *$\nu$-weight variety of $\mathcal{O}_\xi$*, denoted $\mathcal{O}_\xi(\nu)$, is the symplectic reduction $\mathcal{O}_\xi /\!\!/ T(\nu)$ of $\mathcal{O}$ by $T$ at $\nu$.

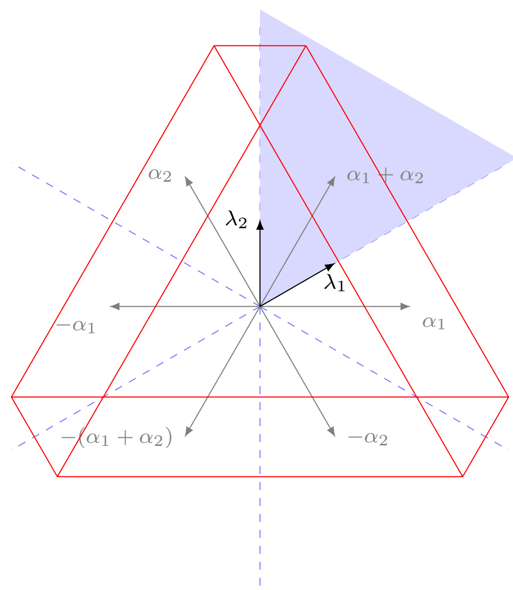


Figure 2.1: Generic hexagonal $SU(3)$ moment polytope. The chambers are the connected regions bounded by the interior lines.

**Remark 2.4.2.** Theorem 2.1.9 and Theorem 2.2.1 imply that weight varieties are defined (nonempty) whenever $\nu \in \Delta^\circ$.

**Remark 2.4.3.** By Theorem 2.2.1 we know that weight varieties are orbifolds. However, in type $A$ it is a fact (see [86, Ch. 1]) that weight varieties are always *manifolds*.

For the remainder of this section we describe some specific weight varieties in type $A$.

**Example 2.4.4.** Let $G = \mathrm{SU}(2)$ with maximal torus $T \cong S^1$ consisting of the diagonal matrices in $G$. Identify $\mathfrak{g}^*$ with the traceless $2 \times 2$ Hermitian matrices $\mathcal{H}_2$. A 2-dimensional coadjoint orbit $\mathcal{O}$ consists of those $A \in \mathcal{H}_2$ with distinct nonzero eigenvalues $\pm\lambda$. In Example 2.3.7 we saw that the a level set of the moment map, at a regular value $a \in (-\lambda, \lambda)$, could be identified with the circle $\mu^{-1}(a) = \{z \in \mathbb{C} \mid |z|^2 = \lambda^2 - a^2\}$. The $T$-action on the level set is is $t \cdot z = t^2 z$, $z \in \mu^{-1}(a)$, $t \in T$. In particular, the quotient is a point (which is to be expected).

**Example 2.4.5.** Let $G = \mathrm{SU}(3)$. Then, the coadjoint orbits have (real) dimension 2 or 6. A generic coadjoint orbit has dimension 6 and is diffeomorphic to the variety of complete flags in $\mathbb{C}^3$. Any weight variety $\mathcal{O}(\nu)$ of a generic coadjoint orbit $\mathcal{O}$ must be a compact, symplectic manifold having dimension 2. Using the Kirwan surjectivity theorem [84], there is a surjection from $H^*(\mu^{-1}(0))$ on to $H^*(\mathcal{O}(\nu))$. The level set $\mu^{-1}(0)$ is identified with a complex submanifold of the variety of complete flags in $\mathbb{C}^3$. In particular, it has cohomology in even degrees only. Thus, The Euler characteristic of $\mathcal{O}(\nu)$ is 2, so that $\mathcal{O}(\nu)$ is diffeomorphic to $S^2$.

**Example 2.4.6.** Let $G = U(1)^n \times U(2)$. Then, $G$ acts on $\mathrm{Mat}_{n\times 2}(\mathbb{C})$ by conjugation. By Proposition 2.2.3 and Example 2.2.7, we see that the symplectic reduction of $\mathrm{Mat}_{n\times 2}(\mathbb{C})$ by $G$ is the symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2,n)$ by $U(1)^n$. Hence, this symplectic reduction is a weight variety for a degenerate coadjoint orbit $\mathcal{O}$ of $U(n)$ diffeomorphic to $\mathrm{Gr}_{\mathbb{C}}(2,n)$. The Hamiltonian action of the torus $T = U(1)^n$ on $\mathrm{Gr}_{\mathbb{C}}(2,n)$ has associated moment map

$$\begin{array}{rccc} \mu_T: & \mathrm{Gr}_{\mathbb{C}}(2,n) & \longrightarrow & \mathfrak{t}^* \cong \mathbb{R}^n \\ & \mathrm{span}\{u,v\} & \longmapsto & \frac{1}{2}(|u_1|^2+|v_1|^2,\dots,|u_n|^2+|v_n|^2) \end{array}$$

Here $(u,v)$ is a unitary 2-frame in $\mathbb{C}^n$. Hence, the image of the moment map is

$$\Xi := \left\{ (r_1,\dots,r_n) \in \mathbb{R}^n \mid \sum_{i=1}^n r_i = 1,\ 0 \le r_i \le 1/2 \right\}$$

The critical values of moment polytope consists of those $r \in \Xi$ such that either

(a) $r_i = 0$, for some $i$, or

(b) $r_i = 1/2$, for some $i$, or

(c) there exists a subset $I \subseteq \{1,\dots,n\}$ with $|I|$ and $|I^c|$ at least two, and $\sum_{i\in I} r_i = \sum_{i\notin I} r_i$.

Points in the moment polytope satisfying one of the first two conditions are points of external walls: the interior walls are described by those points in the moment polytope satisfying the last condition. Therefore, the walls can be described as the subsets

$$\Xi_I := \left\{ (r_1,\dots,r_n) \in \Xi \mid \sum_{i\in I} r_i = 1/2 \right\}, \quad I \subseteq \{1,\dots,n\}.$$

Observe that $\Xi_I = \Xi_{I^c}$, for every $I \subseteq \{1,\dots,n\}$. For each $i \in I$, we write $\Xi_i$ instead of $\Xi_{\{i\}}$.

The set of regular values of $\mu_T$, $\Xi^\circ$, is the set of points such that $\sum_{i\in I} r_i \neq 0$, for any subset $I \subseteq \{1,\dots,n\}$.

**Remark 2.4.7.** If we scale the symplectic form on $\mathrm{Gr}_{\mathbb{C}}(2,n)$ by $\lambda > 0$ then the moment polytope $\Xi$ in Example 2.4.6 gets 'inflated' to

$$\Xi_\lambda = \left\{ (r_1,\dots,r_n) \in \mathbb{R}^n \mid \sum_{i=1}^n r_i = \lambda,\ 0 \le r_i \le \lambda/2 \right\}$$

As such, we should consider the following cone over $\Xi$

$$C(\Xi) := \left\{ (r_1,\dots,r_n) \in \mathbb{R}^n_{>0} \mid \sum_i r_i \neq 0,\ \frac{(r_1,\dots,r_n)}{\sum_i r_i} \in \Xi \right\}.$$

By abuse of notation we will write $r \in \Xi$ when we really mean $r \in C(\xi)$.

## 2.5 Algebraic viewpoint

In this section we briefly outline the relation between symplectic reductions and GIT quotients. First we recall the notion of the GIT quotient and then we apply the Kempf-Ness theorem (Theorem 2.5.5) in the setting of coadjoint orbits. For more details see [110].

Let $G$ be a complex connected reductive group with associated root datum $(X, R, X^\vee, R^\vee)$ and fix a choice of Borel subgroup $B \subseteq G$ and maximal torus $T \subseteq B$. Let $S = \{\alpha_i\}_{i\in I} \subseteq X$ be simple roots corresponding to this choice of $B$ and $P = P_J \supseteq B$ a standard parabolic subgroup corresponding to a subset $J \subseteq I$. The choice of a dominant weight $\lambda \in X^*(T)$ satisfying $\langle \lambda, \alpha^\vee\rangle = 0$, whenever $\alpha \in I$, and $\langle \lambda, \alpha^\vee\rangle > 0$, whenever $\alpha \in S \setminus I$, determines an embedding of the (partial) flag variety $i_\lambda : G/P \to \mathbb{P}(V_\lambda)$, where $V_\lambda$ is the finite dimensional irreducible representation of $G$ with highest weight $\lambda$ (see [70]). Let $\mathcal{L}_\lambda := i_\lambda^* \mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$ be the (ample) invertible sheaf of hyperplane sections associated to this embedding, and let $L_\lambda$ be the total space of $\mathcal{L}_\lambda$.

**Notation 2.5.1.** We write $G/_\lambda P$ to denote that we are considering the (partial) flag variety $G/P$ together with the projective embedding associated to $\mathcal{L}_\lambda$.

The Borel-Weil-Bott theorem [70] identifies the space of global sections $H^0(G/P, \mathcal{L}_\lambda) \cong V_{\lambda^*}$ as the irreducible representation of $G$ with highest weight $\lambda^* = -w_0\lambda$, where $w_0 \in W$ is the longest element. The homogeneous coordinate ring $R_\lambda$ associated to this embedding is

$$R_\lambda = \bigoplus_{n\geq 0} H^0(G/P, \mathcal{L}_\lambda^{\otimes n}) \cong \bigoplus_{n\geq 0} V_{n\lambda^*}. \tag{2.5.1}$$

**Remark 2.5.2.** For any dominant weight $\xi \in X_{\geq 0}$, there exists, up to a non-zero scalar, a unique $G$-invariant ring structure on the (graded) $G$-module $\bigoplus_{n\geq 0} V_{n\xi}$, known as the *Cartan product*, defined as follows: the irreducible representation $V_{(m+n)\xi}$ appears with multiplicity one in the tensor product $V_{n\xi} \otimes V_{m\xi}$. Hence, up to a non-zero scalar, there is a unique $G$-invariant surjection

$$V_{n\xi} \otimes V_{m\xi} \twoheadrightarrow V_{(n+m)\xi}.$$

This is how multiplication is defined in the ring $\bigoplus_{n\geq 0} V_{n\xi}$.

The line bundle $L_\lambda$ admits a $G$-linearisation. By restriction, $L_\lambda$ can also be considered as a $T$-linearised line bundle.

Now, let $\Delta(\lambda) := \mathrm{Conv}(W\lambda^*) \subseteq \mathfrak{t}^*$ be the convex hull of the $W$-orbit through $\lambda^*$, and choose $\mu \in X^*(T) \cap \Delta(\lambda)$. We can twist $L_\lambda$ by $\mu$ to obtain a $T$-linearised line bundle $L_\lambda(-\mu)$: as a line bundle, $L_\lambda(-\mu) = L_\lambda$, and we twist the action of $T$ in the fibres of $L_\lambda(-\mu)$ by $-\mu$. We let $\mathcal{L}_\lambda(-\mu)$ denote the sheaf of sections of $L_\lambda(-\mu)$.

It is straightforward to see the following:

**Lemma 2.5.3.** *The space of $T$-invariants $H^0\left(G/P, \mathcal{L}_\lambda(-\mu)^{\otimes n}\right)^T$ is the $\mu$-weight space in the (irreducible) representation $H^0(G/P, \mathcal{L}_\lambda^{\otimes n}) \cong V_{n\lambda^*}$.*

**Definition 2.5.4** ([86])**.** The *$\nu$-weight variety* of $G/\!\!/_{\lambda}P$ is the G.I.T. quotient defined by the $T$-linearised line bundle $L_\lambda(-\mu)$,

$$T_\mu \backslash\!\!\backslash G/_{\lambda}P := \operatorname{Proj} \bigoplus_{n \geq 0} H^0\left(G/P, \mathcal{L}(-\mu)^n\right)^T = \operatorname{Proj} \bigoplus_{n \geq 0} (V_{n\lambda^*})^{n\mu}$$

Let $K \subseteq G$ be a maximal compact subgroup, which we will assume is acting unitarily on $V_\lambda$. Let $H \subseteq T$ be a maximal compact torus in $T$ with $H \subseteq K$. Then, the quotient $X = G/_{\lambda}P$ is a Hamiltonian $H$-space with moment map $\mu : X \to \mathfrak{h}^*$.

Applying the Kempf-Ness theorem [110] we have the following result.

**Theorem 2.5.5.** *There is an inclusion $\mu^{-1}(0) \subseteq X^{ss}$ inducing a homeomorphism between the symplectic reduction and the GIT quotient*

$$\mu^{-1}(0)/H \cong T_\mu \backslash\!\!\backslash G/_{\lambda}P$$

*In particular, the symplectic reduction is a projective variety.*

We end this section with an elaboration of Example 2.2.4. We will come back to this example in Section 3.4.

Recall the construction of the moduli space of spatial $n$-gons $\mathcal{P}_{r,n}$, $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, from Example 2.2.4. Suppose that $r \in \mathbb{Z}^n_{>0}$. An application of the Kempf-Ness Theorem [110] implies that there is an identification

$$\mathcal{P}_{r,n} \cong (\mathbb{P}^1(\mathbb{C}))^n /\!\!/ \mathrm{PGL}_2(\mathbb{C})$$

The Gelfand-Macpherson correspondence [43] provides the following isomorphism of G.I.T. quotients

$$T_\mu \backslash\!\!\backslash \mathrm{SL}_n(\mathbb{C})/_{\lambda}P \cong (\mathbb{P}^1(\mathbb{C}))^n /\!\!/ \mathrm{PGL}_2(\mathbb{C})$$

where $P \subseteq \mathrm{SL}_n(\mathbb{C})$ is a maximal parabolic such that $G/_{\lambda}P \cong \mathrm{Gr}(2,n)$, $T \subseteq \mathrm{SL}_n(\mathbb{C})$ is a maximal torus. The linearisation defined by $\mu$ corresponds to the action of $T$ on $\mathbb{C}^n$ given by

$$\mathrm{diag}(t_1, \ldots, t_n) \longmapsto \mathrm{diag}(t^r t_1, \ldots, t^r t_n)$$

where $t^r = t_1^{r_1} \cdots t_n^{r_n}$ is the character defined by $r$.

Hence, we have the following result (recall Example 2.4.6).

**Theorem 2.5.6** (Hausmann-Knutson, [63])**.** *Let $r \in \mathbb{Z}^n_{>0}$. Then, the polygon space $\mathcal{P}_{r,n}$ admits the structure of a projective variety and can be identified with a weight variety. Specifially, the polygon space $\mathcal{P}_{r,n}$ is a symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2,n)$, the Grassmannian of 2-planes in $\mathbb{C}^n$, by the compact torus $H \subseteq T$ at $r \in \Xi^\circ$.*

**Remark 2.5.7.** To ensure compatibility of symplectic forms coming from our constructions of $\mathcal{P}_{r,n}$ (Example 2.2.4) and the symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2,n)$, we scale the symplectic form on $\mathrm{Gr}_{\mathbb{C}}(2,n)$ by $|r| > 0$ (cf. Remark 2.4.7). In particular, given $r \in \mathbb{R}^n_{>0}$, the moment polytope of the torus action on $\mathrm{Gr}_{\mathbb{C}}(2,n)$ used to define $\mathcal{P}_{r,n}$ as a symplectic quotient is

$$\Xi_{|r|} = \{s \in C(\Xi) \mid |s| = |r|\}.$$

# Chapter 3

# Mirror constructions

In this chapter we develop a new approach to computing quantum cohomology of weight varieties motivated by a conjecture of Teleman. Let $G$ be a semisimple complex algebraic group, $P \subseteq G$ a parabolic subgroup containing a maximal torus $T$. Using the mirror Landau-Ginzburg model $(M_P, f_P)$ of a partial flag variety $X = G/P$ introduced by Rietsch [115], the quantum cohomology of a symplectic reduction of $X$ by a compact torus $H \subseteq T$ at $\nu \in \mathfrak{h}^*$ is conjectured to be obtained by restricting the *superpotential* $f_P$ to a certain subvariety $Y_\nu$ of $M_P$ and computing the Jacobian ring. In fact, the mirror family $M_P$ is a subvariety of a Borel subgroup ${}^LB_-$ of the Langlands dual ${}^LG$ and the subvariety $Y_\nu$ is the fibre of the canonical homomorphism ${}^LB_- \to {}^LT$ over $\exp(2\pi i\nu)$. Here we canonically identify $\mathfrak{h}^*$ with the subalgebra ${}^L\mathfrak{h} \subseteq {}^L\mathfrak{t}$.

In Section 3.1 we recall the construction of the Rietsch mirror family Landau-Ginzburg $(M_P, f_P)$ and the definition of the ($T$-equivariant) superpotential. We briefly discuss the work of Rietsch relating the ($T$-equivariant) quantum cohomology to the Jacbobian ring of $f_P$ and the mirror conjectures that she proposed. In the short Section 3.2, we describe recent work of Teleman on topological actions of compact, connected Lie groups on Fukaya categories. Then, we introduce Teleman's conjecture on the construction of Fukaya categories of weight varieties and the consequences for quantum cohomoloy of weight varieties. Section 3.3 introduces several formulae for computing the superpotential $f_P$ and provides an explicit expression for $f_P$ with respect to a family of parameterisations of $M_P$. In Section 3.4 we provide a new conjectural description of the quantum cohomology of weight varieties, and introduce a conjectural presentation of these rings. We then focus on the problem in type $A$ and verify this presentation for weight varieties of low rank, paying particular attention to polygon spaces $\mathcal{P}_{r,n}$. Finally, in Section 3.5 we discuss further avenues of research and possible extensions of our work.

Let $G$ be a complex reductive algebraic group with root datum $(X, R, X^\vee, R^\vee)$. We will use the notation and conventions from Section 1.3. Let $T \subseteq B_\pm \subseteq G$ be a choice of maximal torus and opposite Borel subgroups, $N_\pm \subseteq B_\pm$ the unipotent radicals. Let $P = P_J \supseteq B_+$ be a standard parabolic subgroup. Let ${}^LG$ be the Langlands dual group, ${}^LT, {}^LB_\pm, {}^LN_\pm$ the corresponding subgroups (associated to the choice of simple coroots), and ${}^LP$ the dual standard parabolic subgroup (corresponding to the subset $J(P) \subseteq I$). At certain points in this chapter, we will restrict to the case when $G$ is semisimple.

## 3.1 Rietsch mirror construction

In this section we introduce a candidate for the ($B$-model) equivariant Landau-Ginzburg model $(M_P, f_P)$ of the ($A$-model) generalised flag variety $G/P$ proposed by Rietsch [115]. The mirror family $M_P$ is realised as a subvariety of the opposite Langlands dual subgroup ${}^LB_- \subseteq {}^LG$ and parameterised by $Z(L_{{}^LP})$, the centre of the unique Levi $L_{{}^LP} \subseteq {}^LP$ containing ${}^LT$. The superpotential $f_P$ should be considered as a family of holomorphic functions $f_P^t$, $t \in Z(L_P)$, on the fibres of the mirror family $M_P$.

To each $\mathbf{i} \in R(w_P^{-1})$ we will describe an explicit expression for the restriction of the *superpotential* $f_P$ to a dense open subset $U \subseteq M_P$, where $U \cong (\mathbb{C}^\times)^{\ell(w_P)}$. We will see that, with respect to this parameterisation, $f_P$ is a Laurent polynomial whose terms correspond to the divisor constituents of an anticanonical divisor in $G/P$. This is also observed in [94].

The verification that $M_P$ is the 'correct' mirror family takes the form of an identification of the equivariant quantum cohomology ring $qH^*(G/P)$ with the Jacobian ring of the superpotential $\mathrm{Jac}(f_P)$. This result is originally due to Rietsch.

Rietsch proposed a stronger statement of the mirror symmetry between $G/P$ and $M_P$ in the form of a *mirror conjecture*. We briefly discuss this conjecture and some partial verifications due to Lam, Lam-Templier [93], [94].

For ease of notation we swap the roles of $G$ and ${}^LG$. In particular, we are going to describe the mirror family $M_{{}^LP}$ to the flag variety ${}^LG/{}^LP$. By abuse of notation we will write $M_P$ instead of $M_{{}^LP}$.

Let $w_P^{-1} \in W^P$ be the minimal length coset representative having maximal length. We record the following elementary result (see [17]).

**Lemma 3.1.1.** *$w_P^{-1} = w_0 w_0^P$ and $\ell(w_P^{-1}) = \ell(w_P) = \ell(w_0) - \ell(w_0^P) = \dim G/P$.*

By Lemma 3.1.1 we have $w_P^{-1} w_0^P = w_0 = w_0^P w_P$ and

$$\overline{w_P^{-1}} \cdot \overline{w_0^P} = \overline{w}_0, \quad \overline{w_0^P} \cdot \overline{w}_P = \overline{w}_0.$$

Let $P^*$ be the standard parabolic subgroup containing $B_+$ and $\overline{w}_0 L_P \overline{w}_0^{-1}$. Thus, $J(P^*) = J(P)^*$, $w_0^P w_0 = w_0 w_0^{P^*}$ and

$$\overline{w_0^{P^*}} \cdot \overline{w_{P^*}^{-1}}, \quad \overline{w}_{P^*} \cdot \overline{w_0^{P^*}} = \overline{w}_0.$$

In particular, $w_P^{-1} = w_{P^*}$.

## The Landau-Ginzburg model $(M_P, f_P)$

Consider the incidence variety

$$Z_P := \{(t,b) \in Z(L_P) \times B_- \mid b \in N_+ Z(L_P)\overline{w}_P N_+\},$$

where $Z(L_P)$ is the centre of the Levi subgroup $L_P$. Projection onto the second factor $\mathrm{pr}_2 : Z_P \to B_-$ is an isomorphism

$$Z_P \cong M_P := B_- \cap N_+ Z(L_P)\overline{w}_P N_+ \subseteq B_-. \tag{3.1.1}$$

Let $\mathrm{pr}_1 : Z_P \to Z(L_P)$ be the projection onto the first factor. For $t \in Z(L_P)$, we can identify the fibre over $t$ as

$$\mathrm{pr}_1^{-1}(t) \cong M_P^t := B_- \cap N_+ t \overline{w}_P N_+ \tag{3.1.2}$$

We will require a unique decomposition of elements in $M_P$. Recall the Levi decomposition $P = L_P N_P$. Here $L_P$ is the reductive subgroup generated by $T$ and $\operatorname{im} x_j, \operatorname{im} y_j$, $j \in J(P)$, and $N_P = N_+(w_P)$, where for $w \in W$, we define

$$N_+(w) := \prod_{\substack{\alpha \in R^+ \\ w^{-1}(\alpha) \in R^-}} \operatorname{im} x_\alpha = N_+ \cap \overline{w} N_- \overline{w}^{-1}$$

We have $\overline{w} N_+(w) \overline{w}^{-1} \subseteq N_-$, for any $w \in W$.

The following result follows from the standard description of Bruhat cells in $G$ (see [122]).

**Lemma 3.1.2.** *Let $w \in W$.*

*(a) Any $x \in B_+ w B_+$ can be written uniquely as $x = zt\overline{w}u$, where $z \in N_+(w), t \in T, u \in N_+$.*

*(b) Any $x \in B_+ w B_+$ can be written uniquely as $x = v\overline{w}sy$, where $v \in N_+, s \in T, y \in N_+(w^{-1})$.*

Applying this result to $M_P = B_- \cap N_+ Z(L_P)\overline{w}_P N_+$, we obtain the unique decompositions

$$M_P = B_- \cap N_+(w_P) Z(L_P) \overline{w}_P N_+$$

and

$$M_P = B_- \cap N_+ \overline{w}_P Z(L_{P^*}) N_+(w_{P^*}).$$

**Definition 3.1.3.** (i) Define the *quantum structure map* to be the projection

$$\begin{array}{rccc} q: & M_P & \longrightarrow & Z(L_P) \\ & x = zt\overline{w}_P u & \longmapsto & t \end{array}$$

This map is well-defined. The fibres of $q$ are the subvarieties

$$M_P^t = B_- \cap N_+ t \overline{w}_P N_+$$

from (3.1.2).

(ii) Define the *equivariant structure map* to be the projection

$$\begin{array}{rrcl} e: & M_P \subseteq B_- = N_-T & \longrightarrow & T \\ & x = vs & \longmapsto & s \end{array}$$

As $B_-/N_- \cong T$, the map $e$ can be identified with the canonical quotient homomorphism.

**Definition 3.1.4.** Let ${}^LG/{}^LP$ be a generalised flag variety. Define the *mirror of* ${}^LG/{}^LP$ to be the subvariety

$$B_-^{w_P} := B_- \cap N_+\overline{w}_P N_+ \subseteq B_-. \tag{3.1.3}$$

We define the *mirror family* to be the subvariety $M_P$ defined in (3.1.1) considered as a (trivial) family over $Z(L_P)$ via the quantum structure map $q$. We will write $(M_P, q)$ to denote this family.

**Remark 3.1.5.** The mirror family $M_P$ (resp. mirror $B_-^{w_P}$) of ${}^LG/{}^LP$ is an example of a *double Bruhat cell* (resp. *reduced double Bruhat cell*). These are subvarieties of a reductive group $G$ of the form

$$B_-vB_- \cap B_+uB_+ \quad \text{(resp. } B_-vB_- \cap N_+\overline{u}N_+\text{)}$$

where $u, v \in W$ [36].

We record some straightforward properties of the fibres $M_P^t$.

**Proposition 3.1.6.**

*(a) Let $t \in Z(L_P)$. Then, $M_P^t$ is smooth variety of dimension* $\dim M_P^t = \dim \ell(w_P) = \dim {}^LG/{}^LP$*, isomorphic to $B_-^{w_P}$.*

*(b) Multiplication in $G$ induces an isomorphism of varieties*

$$\begin{array}{rrcl} m: & Z(L_P) \times B_-^{w_P} & \xrightarrow{\sim} & M_P \\ & (t, z\overline{w}_P u) & \longmapsto & tz\overline{w}_P u \end{array} \tag{3.1.4}$$

*Proof.* (a) That $M_P^t$ is isomorphic to $B_-^{w_P}$ is immediate. The map

$$\begin{array}{rcl} M_P^t & \longrightarrow & G/B_- \\ x & \longmapsto & b\overline{w}_0 B_- \end{array} \tag{3.1.5}$$

identifies $B_-^{w_P}$ with the *(open) Richardson variety*

$$\mathcal{R}^-_{w_0^P, w_0} := \left(B_+\overline{w_0^P}B_- \cap B_-\overline{w}_0 B_-\right)/B_- \subseteq G/B_-.$$

The varieties $\mathcal{R}_{w_0^P, w_0}$ are known to be smooth of dimension $\ell(w_0) - \ell(w_0^P) = \ell(w_P)$, see [21]. We also have

$$\begin{aligned}\ell(w_0) - \ell(w_0^P) &= (\dim {}^L G - \dim {}^L B_+) - (\dim {}^L P - \dim {}^L B_+) \\ &= \dim {}^L G - \dim {}^L P \\ &= \dim {}^L G / {}^L P.\end{aligned}$$

(b) This is obvious.

□

**Remark 3.1.7.** For any $t \in Z(L_P)$, we can embed $M_P^t$ in $G/B_+$ using the map

$$\begin{array}{ccc} M_P^t & \longrightarrow & G/B_+ \\ x & \longmapsto & x^{-1}\overline{w_0^P} B_+ \end{array}$$

and then use the canonical projection $p : G/B_+ \to G/P$ to embed $M_P^t$ in $G/P$. In this way, $G/P$ can be considered to be a compactification of the fibres $M_P^t$, $t \in Z(L_P)$, of the mirror family $(M_P, q)$. The image is a (projected) open Richardson variety [87] and $M_P^t$ can be considered as an open subvariety of $G/P$.

As an open (projected) Richardson variety, the image of the embedding $M_P^t \to G/P$ is the complement of an anticanonical divisor $\partial_{G/P}$ in $G/P$ [87, Lemma 5.4], where $\partial_{G/P}$ is the multiplicity-free union of the divisors $D^i$, $i \in I$, and $D_i$, $i \notin J(P)$,

$$D^i := \overline{p(\mathcal{R}_{w_0^P}^{w_0 s_i})}, \quad \text{and} \quad D_i := \overline{p(\mathcal{R}_{s_i w_0^P}^{w_0})}.$$

Here $p : G/B_+ \to G/P$ is the canonical projection and the Richardson variety $\mathcal{R}_u^v \subseteq G/B_+$ is the intersection of the Schubert cell $B_-\overline{u}B_+/B_+$ with the opposite Schubert cell $B_+\overline{v}B_+/B_+$.

We now proceed to define the (equivariant) superpotential associated with $M_P$.

**Definition 3.1.8.** For $i \in I$, define the *elementary characters* $\chi_i : N_+ \to \mathbb{A}^1$ uniquely determined by $\chi_i(x_j(a)) = \delta_{ij} \cdot a$. The *standard regular character* is the character

$$\chi := \sum_{i \in I} \chi_i. \tag{3.1.6}$$

**Definition 3.1.9.** Define the *superpotential function* $f_P$ to be the holomorphic function

$$\begin{array}{rccc} f_P : & M_P = B_- \cap N_+ Z(L_P) \overline{w}_P N_+ & \longrightarrow & \mathbb{C} \\ & z t \overline{w}_P u & \longmapsto & \chi(z) + \chi(u) \end{array} \tag{3.1.7}$$

For $t \in Z(L_P)$, define $f_P^t : M_P^t \to \mathbb{C}$ to be the restriction of $f_P$ to a fibre of $q$.

**Remark 3.1.10.** Writing $x = zt\overline{w}_P u \in B_- \cap N_+(w_P)t\overline{w}_P N_+$, with $z \in N_+(w_P)$, $t \in Z(L_P)$, $u \in N_+$ uniquely determined by $x$, we have

$$f_P(x) = \chi(z) + \chi(u) = \sum_{i \notin J(P)} \chi_i(z) + \chi(u).$$

For the remainder of this section we assume that $G$ is semisimple. We will also require the ${}^L T$*-equivariant superpotential*. This is a holomorphic function defined on $M_P \times {}^L\mathfrak{t}$ which is essentially the map

$$(x, h) \longmapsto f_P(x) + \exp(\langle h, \log \pi^0(x) \rangle).$$

Here we have made the canonical identification ${}^L\mathfrak{t} \cong \mathfrak{t}^*$.

Define the variety $\tilde{M}_P$ by the fibre diagram

$$\begin{array}{ccc} \tilde{M}_P & \longrightarrow & \mathfrak{t} \\ \downarrow & & \downarrow{\scriptstyle \exp} \\ M_P & \xrightarrow[e]{} & T \end{array}$$

Hence, $\tilde{M}_P$ may be identified with $\{(b; y) \in M_P \times \mathfrak{t} \mid b\exp(-y) \in N_-\}$. For $t \in Z(L_P)$, consider the correspondence

$$\tilde{M}_P^t := \{(b, y) \in M_P^t \times \mathfrak{t} \mid b\exp(-y) \in N_-\}.$$

The projection

$$\begin{array}{rccc} c_P : & \tilde{M}_P & \longrightarrow & M_P \\ & (b; y) & \longmapsto & b \end{array}$$

is a covering map and there is a commutative diagram

$$\begin{array}{ccc} \tilde{M}_P & \xrightarrow{c_P} & M_P \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ Z(L_P) & = & Z(L_P) \end{array} \tag{3.1.8}$$

induces a covering on fibres

$$\begin{array}{ccc} \tilde{M}_P^t & \longrightarrow & M_P \\ (b, y) & \longmapsto & b \end{array}$$

In (3.1.8), both projections $\mathrm{pr}_1$ corrspond to projections on to the first factor. Define the holomorphic function

$$\begin{array}{rccc} \tilde{\phi} : & \tilde{M}_P \times {}^L\mathfrak{t} & \longrightarrow & \mathbb{C} \\ & (b, y; h) & \mapsto & \exp(\langle h, y \rangle) \end{array} \tag{3.1.9}$$

Here we identify ${}^L\mathfrak{t}$ with $\mathfrak{t}^*$.

**Definition 3.1.11.** Define the ${}^LT$*-equivariant superpotential function* to be the (multi-valued) holomorphic function

$$f_{P,{}^LT} := f_P + \ln \tilde{\phi} : \quad M_P \times {}^L\mathfrak{t} \longrightarrow \mathbb{C}$$

**Remark 3.1.12.** It is immediate from the definition of $\tilde{\phi}$ that the logarithmic derivative in the direction of $\tilde{M}_P$ is independent of $y$; that is, the logarithmic deriviative of $\tilde{\phi}$ depends only on the $M_P$ directions. In particular, fixing $h \in {}^L\mathfrak{t}$, we can talk about the (logarithmic) critical points of $\tilde{\phi}(\ ;h)$ in a fibre $M_P^t$ in the original mirror family $M_P$. As such, we can define

$$M_{P,{}^LT}^{\text{crit}} := \{(b;h) \in M_P \times {}^L\mathfrak{t} \mid b \text{ is a critical point of } (f_P + \ln \tilde{\phi}(\ ;h))_{|M_P^t},\ t \in Z(L_P)\}$$

to be the set of (logarithmic) critical points of $\tilde{\phi}$ in a fibre $M_P^t$ of $M_P$.

**Remark 3.1.13.** For our purposes, we will be interested in the critical points of the ${}^LT$-equivariant superpotential function $f_{P,{}^LT}$ restricted to $M_P^t \times \{h\}$, for fixed $t \in Z(L_P)$ and $h \in {}^L\mathfrak{t}$. In particular, the equivariant part will not come into consideration when determining the critical points of $f_{P,{}^LT}^t$.

## Quantum cohomology and mirror conjectures

We will briefly indicate why the mirror family $M_P$, together with equivariant superpotential $f_{P,{}^LT}$, are the 'correct' Landau-Ginzburg $B$-model to be considered as the (equivariant) mirror to ${}^LG/{}^LP$. We will describe Rietsch's construction of (a localisation of) the ${}^LT$-equivariant (small) quantum cohomology $qH_T^*({}^LG/{}^LP)$ and recent results of Lam and Lam-Temperlier on a *mirror conjecture* formulated by Rietsch in [115, Conjecture 8.2]. In this section we assume that $G$ is semisimple.

The *(small) quantum cohomology ring of* ${}^LG/{}^LP$, $qH^*({}^LG/{}^LP)$, is a deformation of the usual cohomology ring $H^*({}^LG/{}^LP) \equiv H^*({}^LG/{}^LP, \mathbb{C})$ with $k = \dim H^2({}^LG/{}^LP)$ parameters, admitting the structure of a $\mathbb{C}[q_1, \dots, q_k]$-module. As a $\mathbb{C}[q_1, \dots, q_k]$-module we have

$$qH^*({}^LG/{}^LP) \cong H^*({}^LG/{}^LP) \otimes \mathbb{C}[q_1, \dots, q_k].$$

The ring structure is defined by deforming the usual cup product, with new (deformed) structure constants defined in terms of genus 0, 3-point Gromov-Witten invariants. The ${}^LT$-*equivariant quantum cohomology*, $qH_{{}^LT}^*({}^LG/{}^LP)$, is defined in terms of equivariant genus 0, 3-point Gromov-Witten invariants, and is a module over $\mathbb{C}[q_1, \dots, q_k]$ and $H^*(B{}^LT) \cong \mathbb{C}[{}^L\mathfrak{t}]$. It can be considered as a deformation of the usual ${}^LT$-equivariant cohomology $H_{{}^LT}^*({}^LG/{}^LP)$. See [30], [42] and [11] for further details about (equivariant) Gromov-Witten invariants in general, and [82] for the (equivariant) Gromov-Witten invariants of (partial) flag varieties.

The small quantum cohomology of full flag varieties ${}^LG/{}^LB_+$ has seen significant progress over the past two decades. Presentations of $qH^*({}^LG/{}^LB_+)$ were given by Givental-Kim [50], Ciocan-Fontanine [29] and Kim [83] and identified with the regular functions of the nilpotent leaf of the Toda lattice of the Langlands dual $G$. In [49], Givental proved a mirror conjecture relating oscillatory integrals on the mirror manifold with solutions to his quantum $D$-module.

Building on the (unpublished) work of D. Peterson, Rietsch obtained the following result for all partial flag varieties.

**Theorem 3.1.14** (Rietsch, [115, Theorem 4.1]). *There exists an isomorphism*

$$qH^*_{{}^LT}({}^LG/{}^LP)[q_1^{-1},\dots,q_k^{-1}] \cong \mathbb{C}[M^{\mathrm{crit}}_{P,{}^LT}] \tag{3.1.10}$$

*between (a localisation of) the ${}^LT$-equivariant quantum cohomology of ${}^LG/{}^LP$ and the coordinate ring of the (possibly non-reduced) variety $M^{\mathrm{crit}}_{P,{}^LT}$ (i.e. the* Jacobian ring of *$f_{P,{}^LT}$). The quantum parameters on the right hand side of the isomorphism arise from the quantum structure map $q : M^{\mathrm{crit}}_{P,{}^LT} \to Z(L_P)$, and the equivariant structure is given by projection $M^{\mathrm{crit}}_{P,{}^LT} \to {}^L\mathfrak{t}$ onto the second factor.*

Specialising the equivariant parameters to 0 gives the following identification of the non-equivariant quantum cohomology with the Jacobian ring of the superpotential

**Corollary 3.1.15.** *There is an isomorphism*

$$qH^*({}^LG/{}^LP) \cong \mathbb{C}[M^{\mathrm{crit}}_P] \tag{3.1.11}$$

*where the right hand side is the (possibly non-reduced) variety*

$$M^{\mathrm{crit}}_P := \{b \in M_P \mid b \textit{ is a critical point of } (f_P)_{|M^t_P},\ t \in Z(L_P)\} \tag{3.1.12}$$

In [115] Rietsch proposed the following (${}^LT$-equivariant) mirror conjecture:

**Conjecture 3.1.16** (Rietsch, [115, Conjecture 8.2]). A full set of solutions to the ${}^LT$-equivariant quantum differential equations of ${}^LG/{}^LP$ (defined in [47], [30], for example) is given by the period integrals

$$S_\Gamma(t,h) = \int_{\Gamma_t} \exp(f_P/\hbar)\tilde{\phi}(\ ,h)\omega_t \tag{3.1.13}$$

where $\Gamma = \{\Gamma_t\}_{t\in Z(L_P)}$ is a continuous family of cycles in the fibres $M^t_P$, and $\omega_t$ is a family of non-vanishing to forms on the fibres.

Conjecture 3.1.16 can be considered as a strengthening of Theorem 3.1.14. Conjecture 3.1.16 has been shown to hold in several cases.

**Theorem 3.1.17** (Lam, [93]). *Let ${}^LP = {}^LB_+$, so that ${}^LG/{}^LB_+$ is a full flag variety. Then, Conjecture 3.1.16 holds.*

Recall that a Dynkin node $i \in I$ is *miniscule* if the set of weights of $V(\varpi_i)$, where $\varpi_i$ is a fundamental weight, are extremal. A parabolic subgroup $P$ is *miniscule* if $J(P) = I \setminus \{i\}$, where $i$ is miniscule.

If ${}^LP \subseteq {}^LG$ is miniscule then the partial flag varieties ${}^LG/{}^LP$ includes Grassmannians, orthogonal Grassmannians and even dimensional quadrics as examples.

**Theorem 3.1.18** (Lam-Templier, [94]). *Let ${}^LP \subseteq {}^LG$ be a miniscule parabolic subgroup. Then, Conjecture 3.1.16 holds.*

## 3.2 A conjectural mirror construction for weight varieties

In his 2014 ICM address, Teleman [125] described a conjectural mirror construction for symplectic reductions $M /\!\!/ G$, with $G$ a compact, connected Lie group and $M$ a compact Hamiltonian $G$-space. This construction is a consequence of a proposed general framework focusing on topological actions of $G$ on Fukaya categories arising from Hamiltonian $G$-spaces and gauging topological quantum field theories (TQFTs). We will briefly describe this conjecture when $M$ is a flag variety of $G$, omitting the majority of the (conjectural) details and definitions. For the general story we refer to [125], and the references therein.

Let $G$ be a compact, connected Lie group and $T \subseteq G$ be a maximal torus. Suppose that $M = G/L \cong \mathcal{O}_q \subseteq \mathfrak{g}^*$ is a coadjoint orbit for $G$ with its Kirillov-Kostant-Souriau symplectic structure given by $q$. Then, $M$ is a Hamiltonian $G$-space and, upon restriction to $T$, can be considered as a Hamiltonian $T$-space.

**Definition 3.2.1** ([16]). The *Bezrukavnikov-Mirkovic-Finkelberg space*, $\mathrm{BFM}(G)$, is the holomorphic symplectic reduction of $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ by conjugation under $G_{\mathbb{C}}$ (the complexification of $G$), where $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ denotes the (open) submanifold of elements that are regular in the cotangent fibre.

**Remark 3.2.2** ([16], [125, Theorem 5.1]).

1) If $G = T$ then $\mathrm{BFM}(T) = T^*T_{\mathbb{C}}$.

2) The zero fibre of the moment map for the Hamiltonian $G_{\mathbb{C}}$-space $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ is the *universal centraliser*

$$Z_{\mathrm{reg}} = \{(g, \nu) \in G_{\mathbb{C}} \times (\mathfrak{g}^*_{\mathbb{C}})_{\mathrm{reg}} \mid g \cdot \nu = \nu,\ \nu \text{ regular}\} \tag{3.2.1}$$

The space $Z_{\mathrm{reg}}$ is smooth with stabilisers of constant dimension. As such, the symplectic manifold structure is evident.

In [125], Teleman considers the (conjectural) 2-category $\sqrt{\mathfrak{Coh}}(\mathrm{BFM}(G^\vee))$, the *Kapustin-Rozansky-Saulina (KRS) 2-category* of $\mathrm{BFM}(G^\vee)$ ([73], [74]), which (conjecturally) contains, for example, ($G$-equivariant) Fukaya categories of coadjoint orbits $\mathfrak{F}(G/L, q)$ and the dg-category $\mathfrak{Coh}(L)$ of coherent sheaves on smooth holomorphic Lagrangians $L \subseteq \mathrm{BFM}(G^\vee)$ as objects. To be more precise, $\sqrt{\mathfrak{Coh}}(\mathrm{BFM}(G^\vee)$ is the sheaf of global sections of a sheaf of $\mathcal{O}_{\mathrm{BFM}(G^\vee)}$-linear 2-categories, over the space $\mathrm{BFM}(G^\vee)$.

**Remark 3.2.3.** Given holomorphic Lagrangians $L, L' \subseteq \mathrm{BFM}(G^\vee)$, which we consider as objects in $\sqrt{\mathfrak{Coh}}(\mathrm{BFM}(G^\vee))$, their Hom-category $\mathrm{Hom}(L, L')$ will be a sheaf of categories supported on the intersection $L \cap L'$ and equivalent to the *matrix factorization category* $\mathrm{MF}(L, \Psi)$, for some holomorphic function $\Psi : L \to \mathbb{C}$. For more details see [125, Sec. 3].

**'Theorem' 3.2.4** ([125])**.** *The space* $\mathrm{BFM}(G^\vee)$ *admits a smooth Lagrangian foliation, parameterised by pairs* $(L, q)$*, where* $T \subseteq L \subseteq G$ *is a Levi subgroup and* $q \in Z(L^\vee_{\mathbb{C}})$*. Moreover, the leaves of this foliation (conjecturally) arise as the support of the* $G$*-equivariant Fukaya categories* $\mathfrak{F}(G/L, q)$*.*

**Remark 3.2.5.**

1) The quote marks appearing in 'Theorem' 3.2.4 should be interpreted as follows: the existence of a smooth foliation of $\mathrm{BFM}(G^\vee)$ of the type indicated is proved in [125, Theorem 6.8]. However, the statement concerning Fukaya categories relies on (yet unproven) equivalences of categories predicted by *homological mirror symmetry*, and on the (conjectural) construction of the KRS 2-category.

2) The story here is formally analogous to the Borel-Weil construction of irreducible representions of $G$. The appearance of the Fukaya category $\mathfrak{F}(G/L, q)$ arises from *symplectic induction* of the category of vector spaces admitting actions of $L$ (passing through $q \in Z(L^\vee_{\mathbb{C}})$).

The flag variety $(G/L, q)$ admits the structure of a Hamiltonian $T$-space. Therefore, the $T$-equivariant Fukaya category $\mathfrak{F}(G/L, q)$ is an object in $\mathrm{BFM}(T^\vee) = T^*T^\vee_{\mathbb{C}}$. We denote its holomorphic Lagrangian support $\Lambda(q) \subseteq \mathrm{BFM}(T^\vee)$.

**Conjecture 3.2.6** (Teleman, [125])**.** Let $\nu$ be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian $T$-action. Let $t \in Z({}^L L_{\mathbb{C}})$ denote the symplectic structure on $G/L$. Then, the Fukaya category of the symplectic reduction $(G/L) /\!\!/ T(\nu)$ can be computed as the category $\mathrm{Hom}(S_\nu, \Lambda(t))$, where $S_\nu$ is the cotangent fibre over $\exp(\nu) \in T^\vee$.

At the level of quantum cohomology, we can reformulate the conjecture as follows:

**Conjecture 3.2.7.** Let $\nu$ be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian $T$-action. Let $t \in Z({}^L L_{\mathbb{C}})$ denote the symplectic structure on $G/L$. Then, the quantum cohomology of the symplectic reduction $(G/L) /\!\!/ T(\nu)$ can be computed as the Jacobian ring of the restriction of the $T$-equivariant superpotential to the fibre of the equivariant structure map $e : M_P \to {}^L T$ lying over $\exp(2\pi i \nu)$. Here we canonically identify $\mathfrak{t}^* \cong {}^L\,\mathfrak{t}$. The quantum structure comes from the variation of $t \in Z(L)$.

Moreover, if $G$ has nontrivial (finite) centre $Z$, then the number of critical points appears with multiplicity $|Z|$.

## 3.3 Formulae for the superpotential

In this section we describe formulae that will allow us to compute $f_P$. This will be essential in our approach to computing the quantum cohomology of weight varieties.

First, we have the following formula for $f_P$ in terms of $x \in M_P$.

**Lemma 3.3.1.** *For any $x \in N_+ Z(L_P)\overline{w}_P N_+$ we have*

$$f_P(x) = \chi\left(\pi^+(\overline{w}_P^{-1} x)\right) + \sum_{i \notin J(P)} \chi_i\left(\pi^+(\overline{w_P^{-1}}^{-1} x^\iota)\right)$$

*Here $g \mapsto g^\iota$ is the positive inverse defined in Section 1.3.*

*Proof.* Let $x = zt\overline{w}_P u$, with $z \in N_+(\overline{w}_P), u \in N_+$. Then, $\pi^+(\overline{w}_P^{-1} x) = u$. Let $P^*$ be the standard parabolic containing $B_+$ and $\overline{w}_0 L_P \overline{w}_0$. Write $u = u_L v$, where $u_L \in N_+ \cap L_{P^*}$, and $v \in N_+(w_P^{-1})$. Define $v' := \overline{w}_P u_L \overline{w}_P^{-1} \in N_+ \cap L_P$; in particular, $tv' = v't$. Hence,

$$x = zt\overline{w}_P u = zt\overline{w}_P u_L v = zv' t\overline{w}_P v$$

and

$$\pi^+(\overline{w_P^{-1}}^{-1} x^\iota) = \pi^+(\overline{w_P^{-1}}^{-1} v^\iota \overline{w_P^{-1}} t^{-1} (zv')^\iota) = (zv')^\iota$$

Here we have used that $v \in N_+(w_p^{-1})$.

For any $i \notin J(P)$, $\chi_i((zv)^\iota) = \chi_i(z)$: the fact that $\chi_i(n^\iota) = \chi_i(n)$, for all $i \in I$, $n \in N_+$, follows from the definition of the map $n \mapsto n^\iota$. Hence, by Remark 3.1.10, for $x = zt\overline{w}_P u \in N_+(w_P)t\overline{w}_P N_+$,

$$f_P(x) = \sum_{i \notin J(P)} \chi_i(z) + \chi(u) = \sum_{i \notin J(P)} \chi_i(\pi^+(\overline{w_P^{-1}}^{-1} x^\iota)) + \chi(\pi^+(\overline{w}_P^{-1} x))$$

$\square$

In the case that $G$ is simply connected we can use the identity

$$\chi_i(\pi^+(g)) = \frac{\Delta_{\varpi_i, s_i \varpi_i}(g)}{\Delta_{\varpi_i, \varpi_i}(g)}$$

to obtain a formula for $f_P$ using generalised minors (recall Section 1.3).

**Lemma 3.3.2.** *Let $G$ be simply-connected. For any $g \in N_+ Z(L_P)\overline{w}_P N_+$,*

$$f_P(g) = \sum_{i \in I} \frac{\Delta_{w_P \omega_i, s_i \omega_i}(g)}{\Delta_{w_P \omega_i, \omega_i}(g)} + \sum_{i \notin J(P)} \frac{\Delta_{w_0 s_i \omega_i, w_i}(g)}{\Delta_{w_0 \omega_i, \omega_i}(g)}$$

**Remark 3.3.3.** Observe that the terms in the above descriptions of $f_P$ correspond to the divisors $D^i$, $i \in I$, $D_i$, $i \notin J(P)$, defined in Remark 3.1.7. This is analogous to the situation for mirror symmetry of Fano toric varieties. Further discussion can be found in [80, Chapter 2].

We will also make use of the following description of $f_P$ due to Lam-Templier [94]. In earlier work we obtained a similar expression but we follow the presentation in [94].

For any $w \in W$, define the variety

$$N_+^w \coloneqq B_-\overline{w}B_- \cap N_+ \subseteq N_+ \tag{3.3.1}$$

This variety is a reduced Bruhat cell (Remark 3.1.5).

**Lemma 3.3.4.** *(a) There is an isomorphism*

$$\begin{array}{rccc} \eta: & B_-^{w_P} & \longrightarrow & N_+^{w_P^{-1}} \\ & x & \longmapsto & \pi^+(\overline{w}_P^{-1}x) \end{array}$$

*(b) There is an injection*

$$\begin{array}{rccc} \tau: & N_+^{w_P^{-1}} & \longrightarrow & N_+(w_P) \\ & u & \longmapsto & \pi^+((\overline{w}_P u)^{-1})) \end{array}$$

*Proof.* (a) If $x = z\overline{w}_P u \in B_- \cap N_+(\overline{w}_P)\overline{w}_P N_+$ then $\pi^+(\overline{w}_P^{-1}x) = u$. Now, observe that

$$u^{-1} = x^{-1}z\overline{w}_P = x^{-1}\overline{w}_P(\overline{w}_P^{-1}z\overline{w}_P) \in B_-\overline{w}_P B_-$$

so that $u \in B_- w_P^{-1} B_- \cap N_+$. Hence, $\eta$ is well-defined. Conversely, if $u \in N_+^{w_P^{-1}}$ then $u^{-1} \in N_+^{w_P}$ and $u\overline{w}_P^{-1} \in B_- N_+(w_P)$. Then, the inverse to $\eta$ is seen to be

$$u \longmapsto \pi^-(u^{-1}\overline{w}_P^{-1})^{-1}.$$

(b) In the course of the proof above we saw that $\pi^+(u^{-1}\overline{w}_P^{-1}) \in N_+(w_P)$. In fact, we find

$$\tau(u) = \eta(u)(\overline{w}_P u)^{-1}$$

so that $\tau$ is injective.

∎

Lemma 3.3.4 implies that the restriction of the superpotential $f_P$ to a fibre of $q$ can be defined as a map on $N_+^{w_P^{-1}}$: we can trivialise the mirror family

$$\begin{array}{ccc} Z(L_P) \times N_+^{w_P^{-1}} & \xrightarrow{m\circ(\mathrm{id}\times\eta^{-1})} & M_P \\ (t,u) & \longmapsto & t\eta^{-1}(u) = t\tau(u)\overline{w}_P u \end{array} \tag{3.3.2}$$

Hence, if $x = zt\overline{w}_P u \in M_P^t$ then, as function of $(t,u) \in Z(L_P) \times N_+^{w_P^{-1}}$,

$$f_P^t(t,u) = \chi(t\tau(u)t^{-1}) + \chi(u). \tag{3.3.3}$$

**Lemma 3.3.5.** *Let $u \in N_+^{w_P^{-1}}$. Then,*

$$\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}}) = \overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0,$$

*where $P^*$ is the standard parabolic subgroup containing $B_+$ and $\overline{w}_0 L_P \overline{w}_0^{-1}$.*

*Proof.* Let $u \in N_+^{w_P^{-1}}$. Then, $x = \tau(u)\overline{w}_P u \in B_-^{w_P}$ and $u = \overline{w}_P^{-1}\tau(u)^{-1}x$. Hence,

$$\overline{w}_0^{-1}u^T = \overline{w}_0^{-1}x^T\tau(u)^{-T}\overline{w}_P = \overline{w}_0^{-1}x^T\overline{w}_0\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0\overline{w_0^{P^*}}^{-1} \tag{3.3.4}$$

Here we use that $w_P w_0^{P^*} = w_0$, with $\ell(w_P)+\ell(w_0^{P^*}) = \ell(w_0)$, so that $\overline{w}_P = \overline{w}_0\overline{w_0^{P^*}}^{-1}$. Finally, $x \in B_-$ so that $\overline{w}_0^{-1}x^T\xi\overline{w}_0 \in B_-$ and the result follows. □

Since, for any $i \in I$,

$$\overline{w}_0^{-1}y_i(a)\overline{w}_0 = x_{i^*}(-a),$$

we see that

$$\chi_i(\tau(u)) = \chi_{i^*}(\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0)$$

and

$$\sum_{i \notin J(P)} \chi_i(\tau(u)) = \sum_{i \notin J(P)} \chi_{i^*}(\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0) = \sum_{i \notin J(P^*)} \chi_i(\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}}))$$

**Lemma 3.3.6.** *For $u \in N_+^{w_P^{-1}}$, $i \in I$,*

$$\chi_i(\tau(u)) = \chi_{i^*}(\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}})).$$

*Hence, if $x = zt\overline{w}_P u \in B_- \cap N_+(w_P)Z(L_P)\overline{w}_P N_+$ then*

$$f_P(x) = \chi(u) + \sum_{i \notin J(P^*)} \alpha_{i^*}(t)\chi_i(\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}}))$$

*In particular, when $G$ is simply-connected we have*

$$f_P^t(zt\overline{w}_P u) = \chi(u) + \sum_{i \notin J(P^*)} \alpha_{i^*}(t)\frac{\Delta_{w_0^{P^*}s_i\varpi_i, w_0\varpi_i}(u)}{\Delta_{\varpi_i, w_0\varpi_i}(u)}$$

*Proof.* Noting that $\chi_i(tnt^{-1}) = \alpha_i(t)\chi_i(n)$, for any $n \in N_+$, the result follows from (3.3.3) and Lemma 3.3.5. For the last formula recall the definition of the generalised minors in Section 1.3 and note that, for any $i \notin J(P^*)$, $w_0^{P^*}\varpi_i = \varpi_i$. □

We will now describe a formula for the restriction of $f_P$ to a family of open subsets $U_{\mathbf{i}} \subseteq N_+^{w_P^{-1}}$, $\mathbf{i} \in R(w_P^{-1})$, each of which is isomorphic to a complex algebraic torus.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in R(w)$, $w \in W$. Define the map

$$\begin{array}{rccc} x_{\mathbf{i}}: & (\mathbb{C}^\times)^r & \longrightarrow & N_+^w \\ & (a_1, \ldots, a_r) & \longmapsto & x_{i_1}(a_1) \cdots x_{i_r}(a_r) \end{array}$$

An essential property of the maps $x_{\mathbf{i}}$ is the following result.

**Lemma 3.3.7** (Fomin-Zelevinsky, [36, Theorem 1.2])**.** *Let $\mathbf{i} \in R(w)$, $w \in W$. Then, $x_{\mathbf{i}}$ is an open embedding.*

**Definition 3.3.8.** Let $\mathbf{i} \in R(w_P^{-1})$. Define the open embedding

$$\begin{array}{rccc} j_{\mathbf{i}}: & Z(L_P) \times (\mathbb{C}^\times)^{\ell(w_P)} & \longrightarrow & M_P \\ & (t, a) & \longmapsto & t\tau(x_{\mathbf{i}}(a))\overline{w}_P x_{\mathbf{i}}(a) \end{array} \tag{3.3.5}$$

We call $j_{\mathbf{i}}$ the *FZ-parameterisation in the direction* $\mathbf{i}$.

**Proposition 3.3.9** ([94])**.** *Assume $G$ is simply-connected. Let $\mathbf{i} = (i_1, \ldots, i_r) \in R(w_P^{-1})$. Then,*

$$f_P(j_{\mathbf{i}}(t, a)) = a_1 + \ldots + a_r + \sum_{i \notin J(P^*)} \alpha_{i^*}(t) F_i(a)$$

*where $F_i(a) \in \mathbb{Z}_{\geq 0}[a_1^{\pm}, \ldots a_r^{\pm}]$ is a Laurent polynomial with nonnegative integer coefficients.*

*Proof.* We use Lemma 3.3.6. Let $\mathbf{i} = (i_1, \ldots, i_r) \in R(w_P^{-1})$. First, we observe that if $u = x_{\mathbf{i}}(a)$ then

$$\chi_i(u) = \sum_{i_j = i} \chi_{i_j}(x_{i_j}(a_j)) = \sum_{i_j = i} a_{i_j}.$$

Hence, $\chi(u) = a_1 + \ldots + a_r$. Finally, [14, Theorem 5.8] shows that $\Delta_{w_0^{P^*} s_i \varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))$ is a polynomial with nonegative integer coefficients, and [14, Corollary 9.4] shows that $\Delta_{w_0^{P^*} \varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))$ is a monomial. $\square$

## 3.4 Computing the quantum cohomology of polygon spaces

In this section we describe a new approach to computing the quantum cohomology of a class of weight varieties in type $A$: the *polygon spaces* $\mathcal{P}_{r,n}$ (see Examples 2.2.4, 2.2.6 and Section 2.5). First, we set up our notation specific to this setting.

Let $G = \mathrm{SL}_{n+1}(\mathbb{C})$, and write $I = \{1, \ldots, n\}$. Choose $T$ to be the maximal torus consisting of diagonal matrices and write $t = \mathrm{diag}(t_1, \ldots, t_{n+1})$ for elements of $T$. Let

$H \subseteq T$ be a maximal torus in $T$, identified with $(S^1)^n$. We set $B_+$ to be the subgroup of upper triangular matrices with unipotent radical $N_+$ being the subgroup of upper triangular unipotent matrices. The opposite Borel $B_-$ consists of lower triangular matrices and its unipotent radical is $N_-$, the subgroup of lower triangular unipotent matrices. If $P \supseteq B_+$ is a standard parabolic, $J(P) = \{j_1, \ldots, j_l\} \subseteq I$, so that $\{k_1, \ldots, k_m\} = I \setminus J(P)$, then $P$ consists of those upper block-triangular matrices having blocks of the form

$$\begin{bmatrix} A_1 & * & * & * & * \\ 0 & A_2 & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & * \\ 0 & 0 & \cdots & 0 & A_{m+1} \end{bmatrix} \in G$$

where $A_1$ is $k_1 \times k_1$, $A_i$ is $(k_i - k_{i-1}) \times (k_2 - k_1)$, for $i = 2, \ldots, m$, and $A_{m+1}$ is $(n + 1 - k_m) \times (n+1-k_m)$. The Levi subgroup is the subgroup of block-diagonal matrices in $P_J$. In particular, $Z(L_P)$ is identified with an algebraic torus of rank $m$.

The Langlands dual group is ${}^LG = \mathrm{PGL}_{n+1}(\mathbb{C})$ which we identify with $G/Z$, $Z \subseteq G$ is the (finite, cyclic) centre of $G$. The dual torus ${}^LT$ is identified with $T/Z$. The corresponding subgroups ${}^LB_\pm$ and ${}^LN^\vee_\pm$ are the images of corresponding subgroups of $G$. We identify ${}^LN_\pm$ with $N_\pm$ (there is a unique lift under the canonical quotient homomorphism). For standard parabolic $P \subseteq G$ we identify ${}^LP$ with the image of $P$ in ${}^LG$.

Let $(X, R, X^\vee, R^\vee)$ be the root datum of $G$. The weight lattice $X = \mathrm{Hom}(T, \mathbb{G}_m)$ admits a basis of fundamental weights $\varpi_1, \ldots, \varpi_n$,

$$\begin{array}{rccc} \varpi_i : & T & \longrightarrow & \mathbb{C}^\times \\ & \mathrm{diag}(t_1, \ldots, t_{n+1}) & \longmapsto & t_1 \cdots t_i \end{array}$$

The positive roots corresponding to $B_+$ are $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n+1\}$, where

$$\begin{array}{rccc} \alpha_{ij} : & T & \longrightarrow & \mathbb{C}^\times \\ & \mathrm{diag}(t_1, \ldots, t_{n+1}) & \longmapsto & t_i t_j^{-1}, \end{array}$$

and corresponding simple roots $\alpha_i \coloneqq \alpha_{i,i+1}$, $i = 1, \ldots, n$. In particular,

$$\alpha ij = \alpha_i + \ldots + \alpha_j, \quad i < j.$$

The simple coroots are $S^\vee = \{\alpha_i^\vee \mid i = 1, \ldots, n\}$, where

$$\begin{array}{rccl} \alpha_i^\vee : & \mathbb{C}^\times & \longrightarrow & T \\ & c & \longmapsto & \mathrm{diag}(1, \ldots, \underbrace{c, c^{-1}}_{i,i+1}, \ldots, 1) \end{array}$$

The Weyl group is $W = S_n$. Define $s_i$, $i = 1, \ldots, n$, to be the standard adjacent transpositions. The longest element is the permutation

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n \cdots s_1. \tag{3.4.1}$$

For a standard parabolic subgroup $P_J$, $J = \{j_1, \ldots, j_l\}$, the Weyl group of the pair $(L_P, T)$ is the subgroup generated by $s_{j_1}, \ldots, s_{j_l}$ and can be identified with a product of permutation groups. The longest element $w_0^P$ is the product of the longest elements for each of these permutation groups.

For each $i \in I$, we have the root subgroups

$$\begin{aligned} x_i : \ \mathbb{C} &\longrightarrow N_+ \\ c &\longmapsto \mathbb{I} + cE_{i,i+1} \end{aligned}$$

and

$$\begin{aligned} y_i : \ \mathbb{C} &\longrightarrow N_+ \\ c &\longmapsto \mathbb{I} + cE_{i+1,i} \end{aligned}$$

Here $E_{ij}$ is the matrix with 1 in the $ij$-entry and 0s elsewhere, $\mathbb{I}$ is the identity matrix.

The monomial matrix representative $\overline{s}_i =\in N_G(T)$, $i \in I$, is the the image of the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in the $\mathrm{SL}_2$-triple generated by $\operatorname{im} x_i$ and $\operatorname{im} y_i$.

## The quantum cohomology of a point

As a sanity check, we describe the simplest case of a (proper) parabolic subgroup $P \subseteq G$ of maximal dimension. This case will also provide highlights of the methods we use in the next section when we consider polygon spaces.

Let $J = \{2, \ldots, n\}$ and $P = P_J$ be the parabolic subgroup

$$P = \left\{ \begin{bmatrix} a & b^t \\ 0 & C \end{bmatrix} \in G \right\}.$$

Then, ${}^L G / {}^L P \cong \mathbb{P}^n_{\mathbb{C}}$. As $\mathbb{P}^n_{\mathbb{C}}$ is a toric variety under the (diagonal) action of ${}^L T$ any symplectic reduction will be a point. Hence, Conjecture 3.2.7 predicts a single critical point for the superpotential $f_P$ when restricted to a generic fibre of the equivariant structure map $e : M_P \to T$. Let's verify that this does indeed hold.

The Levi subgroup $L_P \subseteq P$ is

$$L_P = \left\{ \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix} \in G \right\},$$

and $Z(L_P)$ is the subgroup of diagonal matrices $\{\mathrm{diag}(a, a^{-1}, a^{-1}, \ldots, a^{-1}) \mid a \in \mathbb{C}^\times\}$.

The parabolic subgroup is $W_P = \langle s_2, \ldots, s_n \rangle \subseteq W$ and the longest element in $W_P$ is $w_0^P$. Using the reduced expressions

$$w_0^P = s_2 s_3 s_2 \cdots s_n \cdots s_2, \quad \text{and} \quad w_0 = s_n s_{n-1} s_n s_{n-2} s_{n-1} s_n \cdots s_n,$$

we find that

$$w_P^{-1} = w_0 w_0^P = s_n \cdots s_1.$$

As any two reduced expressions are related by a sequence of braid relations, this last expression implies that $R(w_P^{-1}) = \{(n, \ldots, 1)\}$. Let $\mathbf{i} = (n, \ldots, 1) \in R(w_P^{-1})$ be this unique reduced expression.

The parabolic $P^*$ containing $B_+$ and $\overline{w}_0 L_P \overline{w}_0^{-1}$ is such that $J(P^*) = \{1, \ldots, n-1\}$. Hence, $w_0^{P^*}$ is the longest element in the permutation group $\langle s_1, \ldots s_{n-1} \rangle \cong S_{n-1}$.

Using Lemma 3.3.6 we compute $f_P^t$, $t \in Z(L_P)$, with respect to the FZ-parameterisation in the direction $\mathbf{i}$. We have

$$\begin{aligned} f_P(j_{\mathbf{i}}(t, a)) &= a_1 + \ldots + a_n + \sum_{i \notin J(P^*)} \alpha_{i^*}(t) \frac{\Delta_{w_0^{P^*} s_i \varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))} \\ &= a_1 + \ldots + a_n + \alpha_1(t) \left( \frac{\Delta_{w_0^{P^*} s_n \varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a))} \right) \end{aligned}$$

Identifying generalised minors with matrix minors (see Section 1.3) we find

$$\Delta_{w_0^{P^*} s_n \varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a)) = \Delta_{12\cdots(n-1)(n+1), 23\cdots(n+1)}(x_{\mathbf{i}}(a))$$

and

$$\Delta_{\varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a)) = \Delta_{12\cdots n, 23\cdots(n+1)}(x_{\mathbf{i}}(a)).$$

Using induction on $n$, it's straightforward to see that

$$x_{\mathbf{i}}(a_1, \ldots, a_n) = \begin{bmatrix} 1 & a_n & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Hence, $\Delta_{12\cdots n, 23\cdots(n+1)}(x_{\mathbf{i}}(a)) = 1$ and, since $\Delta_{12\cdots n, 23\cdots(n+1)}$ is the determinant of the top right $n \times n$ matrix,

$$\Delta_{12\cdots n, 23\cdots(n+1)}(x_{\mathbf{i}}(a)) = a_1 \cdots a_n$$

Hence,

$$f_P(j_{\mathbf{i}}(t, a)) = a_1 + \ldots + a_n + \frac{\alpha_1(t)}{a_1 \cdots a_n}. \tag{3.4.2}$$

**Remark 3.4.1.** The superpotential $f_P$ computed in (3.4.2) is precisely the well-known superpotential associated to projective space [30].

By Conjecture 3.2.7, the quantum cohomology of $\mathbb{P}^n_{\mathbb{C}} /\!\!/ {}^L T$ (which is a point) is obtained by restricting (3.4.2) to a fibre of $e : M_P \to B_-/N_- = T$ and computing the Jacobian ring. We will now compute determine $e$ with respect to the FZ-parameterisation $j_{\mathbf{i}}$.

Let $u \in N_+^{w_P^{-1}}$, $t \in Z(L_P)$. Then, $j_{\mathbf{i}}(t,u) = t\eta^{-1}(u)$ and the image of $j_{\mathbf{i}}(t,u)$ under $e$ is $te(\eta^{-1}(u))$. Observe that $e(\eta^{-1}(u)) = \pi^0(\eta^{-1}(u))$.

Let $\eta(x) = u$ so that $x = \tau(u)\overline{w}_P u$. We saw in the proof of Lemma 3.3.5 that

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0.$$

As $\tau(u) \in N_+(w_P)$ we obtain $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1})$. Therefore,

$$\pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}) = \pi^0(\overline{w}_0^{-1} x^T \overline{w}_0) = \overline{w}_0^{-1}\pi^0(x)\overline{w}_0$$

In particular, we can compute

$$e(\eta^{-1}(u)) = \overline{w}_0 \pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}})\overline{w}_0^{-1}.$$

We will require to use the fact that we are in a special situation, namely that $P$ is very large.

**Proposition 3.4.2.** *Let $u \in N_+^{w_P^{-1}}$. Then, $e(\eta^{-1}(u))$ is uniquely determined by the diagonal entries of the matrix $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$.*

*Proof.* We recall the notation immediately preceding the statement of the Proposition. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0$$

and $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1}) = N_+(w_{P^*})$. The subgroup

$$N_+(w_{P^*}) = \prod_{\substack{\alpha \in R^+ \text{ s.t.} \\ w_{P^*}^{-1}(\alpha) \in R^-}} \operatorname{im} x_\alpha = \prod_{i=1}^{n} \operatorname{im} x_{\alpha_{in}}$$

consists of unipotent matrices $n \in N_+$ of the form

$$n = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & * \\ 0 & 1 & 0 & \cdots & \cdots & * \\ 0 & 0 & 1 & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & * \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{3.4.3}$$

Write $x = vs \in N_-T$, with $v \in N_-$, $s \in T$. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \left(\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0\right) \overline{w}_0^{-1} s \overline{w}_0 \left(\overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0\right)$$

As $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0$ is a matrix of the form (3.4.3) and $\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0 \in N_-$, the element $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$ is a matrix of the form

$$\begin{bmatrix} d_1(u) & 0 & 0 & \cdots & * \\ * & d_2(u) & 0 & \cdots & * \\ * & * & d_3(u) & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}$$

Projecting onto the first $n$ diagonal entries allows us to define a morphism of varieties

$$\begin{array}{rccl} g: & N_+^{w_P^{-1}} & \longrightarrow & T \\ & u & \longmapsto & g(u) \coloneqq \operatorname{diag}(d_1(u), \dots, d_n(u), (d_1(u) \cdots d_n(u))^{-1}) \end{array}$$

Then, by construction, we have

$$e(\eta^{-1}(u)) = \overline{w}_0 g(u) \overline{w}_0^{-1}.$$

$\square$

We illustrate the proof of the above Proposition with an example.

**Example 3.4.3.** Consider $G = \mathrm{SL}_4(\mathbb{C})$, $\mathbf{i} = (3, 2, 1)$. Then,

$$\overline{w}_0^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{w_0^{P^*}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then, for

$$u = x_{\mathbf{i}}(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_3 & 0 & 0 \\ 0 & 1 & a_2 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \begin{bmatrix} -a_1 & 0 & 0 & 1 \\ 1 & -a_2 & 0 & 0 \\ 0 & 1 & -a_3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute

$$\eta^{-1}(u) = \begin{bmatrix} a_1^{-1}a_2^{-1}a_3^{-1} & 0 & 0 & 0 \\ 1 & a_3 & 0 & 0 \\ 0 & 1 & a_2 & 0 \\ 0 & 0 & 1 & a_1 \end{bmatrix}$$

If we let $ty = \operatorname{diag}(-a_1, -a_2, -a_3, -(a_1a_2a_3)^{-1})$, then we have

$$e(\eta^{-1}(u)) = \overline{w}_0 y \overline{w}_0^{-1}.$$

A straightforward generalisation of Example 3.4.3 provides the following computation of $e$ restricted to the FZ-parameterisation.

**Proposition 3.4.4.** *Let* $\mathbf{i}$ *be the unique reduced expression for* $w_P^{-1}$*, where* $P$ *is the standard parabolic subgroup with* $J(P) = \{2, \dots, n\}$*. Let* $j_{\mathbf{i}}$ *be the FZ-parameterisation in the direction* $\mathbf{i}$*,* $e : M_P \to T$ *the equivariant structure map. Then,*

$$e(j_{\mathbf{i}}(t, a_1, \dots, a_n)) = t \operatorname{diag}((a_1 \cdots a_n)^{-1}, a_n, a_{n-1}, \dots, a_1) \in T$$

Hence, we are comforted to see that the intersection of the fibres of the quantum structure map $q$ and the equivariant structure map $e$ is a single point.

**Theorem 3.4.5.** *Conjecture 3.2.7 holds for symplectic reductions of* ${}^LG/{}^LP \cong \mathbb{P}^n_{\mathbb{C}}$.

**Remark 3.4.6.** If we swap the role of $G$ and ${}^LG$, so that ${}^LG = \mathrm{SL}_{n+1}(\mathbb{C})$, then Conjecture 3.2.7 states that we expect $|Z({}^LG)|$ critical points of $f_P$. Indeed, the computation can proceed as above, and $f_P$ (in the FZ-parameterisation) is equal to the expression (3.4.2).

In this situation, determining the equivariant structure map with respect to the FZ-parameterisation is similar to Proposition 3.4.2: for any $u \in N_+^{w_P^{-1}}$, $e(\eta^{-1}(u)) \in T \subseteq \mathrm{PGL}_{n+1}(\mathbb{C})$ is determined by the diagonal entries of $\overline{w_0}^{-1} u^T \overline{w_0^{P*}}$. We compute, for $t \in Z(L_P)$, $u \in N_+^{w_P^{-1}}$,

$$e(j_{\mathbf{i}}(t, u)) = t \operatorname{diag}((a_1^2 a_2 \cdots a_n)^{-1}, a_n a_1^{-1}, \dots, a_2 a_1^{-1}, 1) \in T$$

Here we are choosing the unique representative of elements in $T = {}^LT/Z({}^LG)$ whose last diagonal entry is 1.

Then, the intersection of the fibres of $q^{-1}(t)$ and $e^{-1}(s)$, where $t = \operatorname{diag}(c, 1, \dots, 1)Z \in Z(L_P)$, $s = \operatorname{diag}(c_1, \dots, c_n, 1)Z \in T$, can be identified with the set

$$\left\{ a \in \mathbb{C}^\times \mid a^{n+1} = \frac{c}{c_1 \cdots c_n} \right\}.$$

Hence, we have $n + 1 = |Z({}^LT)|$ critical points of the restriction of $f_P^t$ to a fibre of the equivariant structure map.

## The quantum cohomology of polygon spaces

In this section we will outline a new approach to computing the quantum cohomology of the class of weight varieties realised as symplectic reductions of the complex Grassmannian of 2-planes $\mathrm{Gr}_{\mathbb{C}}(2, n+1)$: these are the polygon spaces $\mathcal{P}_{r,n+1}$ (Example 2.2.4).

Let $P \subseteq \mathrm{SL}_{n+1}(\mathbb{C})$ be a standard parabolic subgroup such that $J(P) = \{1, 3, \ldots, n+1\}$ or $J(P) = \{1, 2, \ldots, n-2, n\}$. Recall that, in the former case ${}^L G/{}^L P \cong \mathrm{Gr}_{\mathbb{C}}(2, n+1)$ and, in the latter case ${}^L G/{}^L P$ is isomorphic to the Grassmannian of $n$-planes in $\mathbb{C}^{n+1}$ (by duality, this is the same as $\mathrm{Gr}_2((\mathbb{C}^{n+1})^*)$).

Let $P \subseteq G$ be the parabolic subgroup

$$P = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in G \mid A \text{ is } 2 \times 2 \right\}$$

with Levi subgroup

$$L_P = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in P \right\}$$

and $Z(L_P) \cong \mathbb{G}_m$. Therefore, we are considering the case $J(P) = \{1, 3, \ldots, n\}$. The parabolic subgroup $P^*$ is the standard parabolic subgroup such that $J(P^*) = \{1, 2, \ldots, n-2, n\}$. The parabolic subgroup $W_P = \langle s_1, s_3, \ldots, s_n \rangle \subseteq W$ and is isomorphic to a product of permutation groups $S_2 \times S_{n-1}$. The longest element in $W_P$ is $w_0^P$.

By Theorem 2.5.6, any symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2, n+1) \cong {}^L G/{}^L P$ is a polygon space $\mathcal{P}_{r,n+1}$. Here $r \in \mathbb{R}^{n+1}_{>0}$ is such that $|r|$ corresponds to the Kahler form on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ defined via its realisation as a symplectic reduction of complex affine space (Example 2.2.7). In this setting, we are considering $\mathbb{R}^{n+1} \cong \mathfrak{h}'$, where $H \subseteq H' \subseteq U(n+1)$ is the maximal diagonal torus. Since $G = \mathrm{SL}_{n+1}(\mathbb{C})$, we project $\mathfrak{u}(n+1)$ along $\mathbb{R}(1, \ldots, 1)$ onto $\mathfrak{su}(n+1)$ and write $\hat{r} \in \mathfrak{h}$ for the image of $r$ under this projection. In this way, we can associate to $r$ the element $t(r) \in T$ where, if $\hat{r} = (\hat{r}_1, \ldots, \hat{r}_{n+1})$, we define

$$t(r) \coloneqq \mathrm{diag}(\exp(2\pi i \hat{r}_1), \ldots, \exp(2\pi i \hat{r}_1)) \in T \subseteq \mathrm{SL}_{n+1}(\mathbb{C}).$$

We now proceed to describe our main conjecture: an explicit description of the quantum cohomology of $\mathcal{P}_{r,n}$. First, we require the following technical result.

**Lemma 3.4.7.** $w_{P^*}^{-1} = w_0 w_0^{P^*} = s_2 \cdots s_n s_1 \cdots s_{n-1}$.

*Proof.* Consider the following reduced expressions

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n \cdots s_1, \quad \text{and} \quad w_0^{P^*} = s_1 s_2 s_1 \cdots s_{n-2} \cdots s_1 s_n$$

For $i = 1, \ldots, n$, write $w_i = s_i s_{i-1} \cdots s_2 s_1$, so that $w_0 = w_1 w_2 \cdots w_n$. First we note that, for any $n$,

$$w_n w_1 w_2 \cdots w_{n-2} = w_1 \cdots w_{n-3} s_n s_{n-1}. \tag{3.4.4}$$

Indeed, proceeding by induction we find

$$\begin{aligned} s_n(w_{n-1}w_1\cdots w_{n-3})w_{n-2} &= s_n(w_1\cdots w_{n-4})s_{n-1}s_{n-2}w_{n-2} \\ &= s_n w_1\cdots w_{n-4}s_{n-1}w_{n-3} \\ & w_1\cdots w_{n-3}s_n s_{n-1} \end{aligned}$$

The last equality follows because $s_j w_i = w_i s_j$, whenever $j > i+1$.

Thus, assuming the result holds for $n-1$ we have,

$$\begin{aligned} w_0 w_0^{P^*} &= w_1 w_2\cdots w_n w_1 w_2\cdots w_{n-2}s_n \\ &= w_1\cdots w_{n-1}w_1\cdots w_{n-3}s_n s_{n-1}s_n, \quad \text{by (3.4.4)}, \\ &= w_1\cdots w_{n-1}w_1\cdots w_{n-3}s_{n-1}s_n s_{n-1} \\ &= s_2\cdots s_{n-1}s_1\cdots s_{n-2}s_n s_{n-1}, \quad \text{by induction}, \\ &= s_2\cdots s_n s_1\cdots s_{n-1}. \end{aligned}$$

□

Hence, $w_P^{-1} = s_{n-1}\cdots s_1 s_n\cdots s_2$.

Let $\mathbf{i} = (n-1, n-2, \ldots, 1, n, n-1, \ldots, 2) \in R(w_P^{-1})$. Using Lemma 3.3.6 we compute $f_P^t$, $t \in Z(L_P)$, with respect to the FZ-parameterisation in the direction $\mathbf{i}$. By Lemma 3.3.6, we have

$$\begin{aligned} f_P(j_{\mathbf{i}}(t,a)) &= a_1 + \ldots + a_n + \sum_{i\notin J(P^*)} \alpha_{i^*}(t)\frac{\Delta_{w_0^{P^*}s_i\varpi_i, w_0\varpi_i}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_i, w_0\varpi_i}(x_{\mathbf{i}}(a))} \\ &= a_1 + \ldots + a_n + \alpha_2(t)\left(\frac{\Delta_{w_0^{P^*}s_{n-1}\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a))}\right) \end{aligned}$$

Identifying generalised minors with matrix minors (see Section 1.3), and using

$$w_0^{P^*} = s_1 s_2 s_1 s_3\cdots s_{n-2}\cdots s_1 s_n,$$

we find

$$\Delta_{w_0^{P^*}s_{n-1}\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a)) = \Delta_{2\cdots(n-1)(n+1), 3\cdots(n+1)}(x_{\mathbf{i}}(a))$$

and

$$\Delta_{\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a)) = \Delta_{1\cdots(n-1), 3\cdots n(n+1)}(x_{\mathbf{i}}(a)).$$

Now,

$$x_{\mathbf{i}}(a_1,\ldots,a_{n-1},b_1,\ldots,b_{n-1}) = x_{n-1}(a_1)\cdots x_1(a_{n-1})x_n(b_1)\cdots x_2(b_{n-1})$$

Using

$$x_{n-1}(a_1)\cdots x_1(a_{n-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$x_n(b_1)\cdots x_2(b_{n-1}) = \begin{bmatrix} 1 & b_{n-1} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & b_{n-2} & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

we find

$$x_{n-1}(a_1)\cdots x_1(a_{n-1})x_n(b_1)\cdots x_2(b_{n-1}) = \begin{bmatrix} 1 & a_{n-1} & a_{n-1}b_{n-1} & 0 & \cdots & 0 \\ 0 & 1 & a_{n-2}+b_{n-1} & a_{n-2}b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1b_1 \\ 0 & 0 & 0 & 0 & \cdots & b_1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

In particular,

$$\Delta_{1\cdots(n-1),3\cdots(n+1)}(x_{\mathbf{i}}(a,b)) = a_1\cdots a_{n-1}b_1\cdots b_{n-1}.$$

An induction argument gives

$$\Delta_{2\cdots(n-1)(n+1),3\cdots(n+1)}(x_{\mathbf{i}}(a,b)) = a_1\cdots a_{n-2} + a_1\cdots a_{n-3}b_{n-1} + \cdots + a_1b_3\cdots b_{n-1} + b_2\cdots b_{n-1}$$

Hence,

$$\begin{aligned} f_P(j_{\mathbf{i}}(t,a,b)) &= a_1 + \ldots + a_{n-1} + b_1 + \ldots + b_{n-1} \\ &\quad + \alpha_2(t)\frac{a_1\cdots a_{n-2} + a_1\cdots a_{n-3}b_{n-1} + \cdots + b_2\cdots b_{n-1}}{a_1\cdots a_{n-1}b_1\cdots b_{n-1}} \end{aligned} \tag{3.4.5}$$

This expression for the superpotential is related to previous mirror constructions of [49] and [10] in the following way.

First we introduce the definition of a Gelfand-Tsetlin quiver.

**Definition 3.4.8.** Let $P \subseteq G$ be a standard parabolic subgroup. A *Gelfand-Tsetlin quiver of shape* $P$ is a quiver $\mathrm{GT}_P$ with underlying set of vertices $\mathcal{V}_P = \{\alpha \in R^+ \mid w_P^{-1}(\alpha) \in -R^+\}$, and arrows defined by

$$\alpha \longrightarrow \beta \qquad \text{if and only if} \qquad \beta = \alpha + \alpha_i, \quad \text{for some } i \in I.$$

We include two additional vertices $v_h, v_t$ and two additional arrows

$$\alpha_{13} \longrightarrow v_h \qquad \text{and} \qquad v_h \longrightarrow \alpha_{2,n+1}.$$

For an arrow $a \in \mathrm{GT}_P$, we define $h(a)$ to be the head of $a$, $t(a)$ to be the tail of $a$.

For $P$ such that $J(P) = \{1, 3 \dots, n\}$, the Gelfand-Tsetlin quivers take the form

$$\begin{tikzcd}
v_h & \alpha_{13} \arrow[l] & \alpha_{14} \arrow[l] & \cdots \arrow[l] & \alpha_{1,n+1} \arrow[l] \\
 & \alpha_{23} \arrow[u] & \alpha_{24} \arrow[l] \arrow[u] & \cdots \arrow[l] & \alpha_{2,n+1} \arrow[l] \arrow[u] \\
 & & & & v_t \arrow[u]
\end{tikzcd}$$

Given $\mathrm{GT}_P$, a Gelfand-Tsetlin quiver of shape $P$, where $J(P) = \{1, 3, \dots, n\}$, we define a family of monomial transformations of $(\mathbb{C}^\times)^{2(n-1)}$. Let $t \in Z(L_P)$ and $q = \alpha_2(t) \in \mathbb{C}^\times$. Let $(a_1, \dots, a_{n-1}, b_1, b_{n-1})$ denote the standard coordinates on $(\mathbb{C}^\times)^{2(n-1)}$. Associate the variables $z_{1i}, z_{2i}$, $i = 1, \dots, n-1$, to the vertices of $\Gamma$ as follows

$$\begin{tikzcd}
q & z_{1,n-1} \arrow[l] & z_{1,n-2} \arrow[l] & \cdots \arrow[l] & z_{11} \arrow[l] \\
 & z_{2,n-1} \arrow[u] & z_{2,n-2} \arrow[l] \arrow[u] & \cdots \arrow[l] & z_{21} \arrow[l] \arrow[u] \\
 & & & & 1 \arrow[u]
\end{tikzcd}$$

and define the following monomial transformation of $(\mathbb{C}^\times)^{2(n-1)}$:

$$\begin{aligned}
z_{1i} &:= \frac{q}{a_{n+2-i} \cdots a_{n-1}}, && i = 3, \dots, n-1, \\
z_{2i} &:= b_1 \cdots b_i, && i = 3, \dots .n-2, \\
z_{2,n-1} &:= \frac{q}{a_{n-1} b_{n-1}}.
\end{aligned} \tag{3.4.6}$$

The following result is a generalisation of Givental's mirror construction for the complete flag variety [49], and is similar to an observation of Marsh-Rietsch [105]. An analogous construction of the superpotential given a Gelfand-Tsetlin quiver of type $P$ was given in [10] (see also [35] for a physical derivation).

**Proposition 3.4.9.** *Let $P \subseteq \mathrm{SL}_{n+1}(\mathbb{C})$ be a standard parabolic subgroup such that $J(P) = \{1, 3, \dots, n\}$ with Levi subgroup $P$ containing the maximal torus $T$ of diagonal matrices. Let $\mathbf{i} = (n-1, \dots, 1, n, \dots, 2) \in R(w_P^{-1})$. Composing the FZ-parameterisation in the direction $\mathbf{i}$ with the inverse of the monomial transformation (3.4.6), the superpotential $f_P$ takes the form*

$$f_p = \sum_{a \in \mathrm{GT}_P} \frac{z_{h(a)}}{z_{t(a)}}$$

**Remark 3.4.10.** Given a Gelfand-Tsetlin quiver of an arbitrary standard parabolic subgroup $P$, we conjecture that there exists a monomial transformation on $(\mathbb{C}^\times)^{\ell(w_P)}$ giving rise to a similar description of the superpotential.

We now compute the equivariant structure map $e : M_P \to T$ with respect to the FZ-parameterisation in the direction $\mathbf{i}$.

Let $u \in N_+^{w_P^{-1}}$, $t \in Z(L_P)$. Then, $j_{\mathbf{i}}(t, u) = t\eta^{-1}(u)$ and the image of $j_{\mathbf{i}}(t, u)$ under $e$ is $te(\eta^{-1}(u))$. Observe that $e(\eta^{-1}(u)) = \pi^0(\eta^{-1}(u))$.

Let $\eta(x) = u$ so that $x = \tau(u)\overline{w}_P u$. We saw in the proof of Lemma 3.3.5 that

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0.$$

As $\tau(u) \in N_+(w_P)$ we obtain $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1})$. Therefore,

$$\pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}) = \pi^0(\overline{w}_0^{-1} x^T \overline{w}_0) = \overline{w}_0^{-1} \pi^0(x) \overline{w}_0$$

In particular, we can compute

$$e(\eta^{-1}(u)) = \overline{w}_0 \pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}) \overline{w}_0^{-1}.$$

**Proposition 3.4.11.** *Let $u \in N_+^{w_P^{-1}}$. Then, $e(\eta^{-1}(u))$ is uniquely determined by the diagonal entries of the matrix $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$ and $u^{-1}\overline{w}_P^{-1}$.*

*Proof.* We recall the notation immediately preceding the statement of the Proposition. The proof is similar to the proof of Proposition 3.4.2. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0$$

and $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1}) = N_+(w_{P^*})$. The subgroup

$$N_+(w_{P^*}) = \prod_{\substack{\alpha \in R^+ \text{ s.t.} \\ w_{P^*}^{-1}(\alpha) \in R^-}} \operatorname{im} x_\alpha = \prod_{i=1}^{n-1} \operatorname{im} x_{\alpha_{i,n}} \times \prod_{i=1}^{n-1} \operatorname{im} x_{\alpha_{i,n-1}}$$

consists of unipotent matrices $n \in N_+$ of the form

$$n = \begin{bmatrix} 1 & 0 & 0 & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3.4.7}$$

Write $x = vs \in N_-T$, with $v \in N_-$, $s \in T$. In particular, $s = e(\eta^{-1}(u))$. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \left(\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0\right) \overline{w}_0^{-1} s \overline{w}_0 \left(\overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0\right)$$

As $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0$ is a matrix of the form (3.4.7) and $\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0 \in N_-$, the element $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$ is a matrix of the form

$$\begin{bmatrix} d_1(u) & 0 & 0 & \cdots & 0 & * & * \\ * & d_2(u) & 0 & \cdots & 0 & * & * \\ * & * & d_3(u) & \cdots & 0 & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \end{bmatrix}$$

Consider the map that projects onto the first $n-1$ diagonal entries of $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$

$$\begin{aligned} h: \quad N_+^{w_P^{-1}} &\longrightarrow (\mathbb{C}^\times)^{n-1} \\ u &\longmapsto h(u) := (d_1(u), \ldots, d_{n-1}(u)) \end{aligned}$$

Then, by construction, we have

$$\overline{w}_0 \operatorname{diag}(h(u), 1, 1) \overline{w}_0^{-1} = (1, 1, s_3, \ldots, s_{n+1}),$$

where $s = \operatorname{diag}(s_1, \ldots, s_{n+1}) \in T$.

The diagonal entries $s_1, s_2$ can be determined by the following argument. For $u \in N_+^{w_P^{-1}}$, and $x = vs$ such that $\eta(x) = u$, we have

$$u^{-1} \overline{w}_P^{-1} = \eta(u)^{-1} \tau(u) = (s^{-1} v^{-1} s) s^{-1} \tau(u) \in N_- T N_+(w_P).$$

By a similar analysis as above, we have $u^{-1}\overline{w}_P^{-1}$ is a matrix whose top left $2 \times 2$ block is of the form

$$\begin{bmatrix} s_1^{-1} & 0 \\ * & s_2^{-1} \end{bmatrix}$$

Hence, the map that projects onto the first two diagonal entries of $u^{-1}\overline{w}_P^{-1}$

$$g = (g_1, g_2) : \quad N_+^{w_P^{-1}} \longrightarrow (\mathbb{C}^\times)^2$$

determines the remaining diagonal entries of $s = e(\eta^{-1}(u))$.

Finally, define

$$\begin{aligned} f : \quad N_+^{w_P^{-1}} &\longrightarrow T \\ u &\longmapsto \operatorname{diag}(g_1(u)^{-1}, g_2(u)^{-1}, 1 \ldots, 1)\overline{w}_0 \operatorname{diag}(h(u), 1, 1)\overline{w}_0^{-1} \end{aligned}$$

By construction, $f(u) = e(\eta^{-1}(u))$. □

We highlight the proof of Proposition 3.4.11 with an example.

**Example 3.4.12.** Let $G = \mathrm{SL}_5(\mathbb{C})$ and $\mathbf{i} = (3, 2, 1, 4, 3, 2) \in R(w_P^{-1})$. Then,

$$\overline{w}_0^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \overline{w_0^{P*}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \overline{w}_P^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$u = x_{\mathbf{i}}(a_1, a_2, a_3, b_1, b_2, b_3) = \begin{bmatrix} 1 & a_3 & a_3 b_3 & 0 & 0 \\ 0 & 1 & a_2 + b_3 & a_2 b_2 & 0 \\ 0 & 0 & 1 & a_1 + b_2 & a_1 b_1 \\ 0 & 0 & 0 & 1 & b_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\overline{w}_0^{-1} u^T \overline{w_0^{P*}} = \begin{bmatrix} a_1 b_1 & 0 & 0 & 1 & -b_1 \\ -(a_1 + b_2) & a_2 b_2 & 0 & 0 & 1 \\ 1 & -(a_2 + b_3) & a_3 b_3 & 0 & 0 \\ 0 & 1 & -a_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Projecting onto the first three diagonal entries gives $h(u) = (a_1 b_1, a_2 b_2, a_3 b_3)$.

Also,

$$u^{-1}\overline{w}_P^{-1} = \begin{bmatrix} a_1 a_2 a_3 & 0 & 1 & -a_3 & a_2 a_3 \\ -(a_1 a_2 + a_1 b_3 + b_2 b_3) & b_1 b_2 b_3 & 0 & 1 & -(a_2 + b_3) \\ a_1 + b_2 & -b_1 b_2 & 0 & 0 & 1 \\ -1 & b_1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Projecting onto the first two diagonal entries gives $g(u) = (a_1 a_2 a_3, b_1 b_2 b_3)$.

Then, the Proposition shows that $e(\eta^{-1}(u))$ is the matrix

$$\begin{aligned}&\mathrm{diag}((a_1a_2a_3)^{-1},(b_1b_2b_3)^{-1},1,1,1)\cdot\overline{w}_0\,\mathrm{diag}(a_1b_1,a_2b_2,a_3b_3,1,1)\overline{w}_0^{-1}\\ =&\,\mathrm{diag}((a_1a_2a_3)^{-1},(b_1b_2b_3)^{-1},a_3b_3,a_2b_2,a_1b_1)\in T,\end{aligned}$$

in agreement with

$$\eta^{-1}(u)=\begin{bmatrix}(a_1a_2a_3)^{-1} & 0 & 0 & 0 & 0\\ \frac{a_1a_2+a_1b_3+b_2b_3}{a_1a_2a_3b_1b_2b_3} & (b_1b_2b_3)^{-1} & 0 & 0 & 0\\ 1 & a_3 & a_3b_3 & 0 & 0\\ 0 & 1 & a_2+b_3 & a_2b_2 & 0\\ 0 & 0 & 1 & a_1+b_2 & a_1b_1\end{bmatrix}$$

Proposition 3.4.11 allows us to determine an explicit formula for $e$ in the $FZ$-parameterisation.

**Theorem 3.4.13.** *Let $\mathbf{i}=(n-1,n-2,\dots,1,n,n-1,\dots,2)\in R(w_P^{-1})$, where $P\subseteq G$ is the standard parabolic with $J(P)=\{1,3\dots,n\}$. Let $j_{\mathbf{i}}$ be the FZ-parameterisation in the direction $\mathbf{i}$, $e:M_P\to T$ the equivariant structure map. Then,*

$$\begin{aligned}&e(j_{\mathbf{i}}(t,a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}))\\ =&\,t\cdot\prod_{j=3}^{n+1}\left(\alpha_{1j}^\vee(a_{n+2-j})\alpha_{2j}^\vee(b_{n+2-j})\right)\in T.\end{aligned}$$

*Proof.* This follows from a calculation similar to Example 3.4.12. □

**Corollary 3.4.14.** *Fix $t\in Z(L_P)$.*

*(a) For any $j=1,\dots,n$,*

$$\varpi_j\left(e(j_{\mathbf{i}}(t,a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}))\right)=\begin{cases}\varpi_1(t)(a_1\cdots a_{n-1})^{-1}, & \text{if } j=1,\\ \varpi_2(t)(a_1\cdots a_{n-1}b_1\cdots b_{n-1})^{-1}, & \text{if } j=2,\\ \varpi_j(t)a_{n+2-j}b_{n+2-j}, & \text{if } j=3,\dots,n.\end{cases}$$

*(b) Let $c_1,\dots,c_n\in\mathbb{C}^\times$ be generic. Then,*

$$\frac{\mathbb{C}[a_1^\pm,\dots,a_{n-1}^\pm,b_1^\pm,\dots,b_{n-1}^\pm]}{\langle c_i-\varpi_i(e(j_{\mathbf{i}}(t,\underline{a},\underline{b})))\rangle_{i\in I}}\cong\frac{\mathbb{C}[b_1^\pm,\dots,b_{n-1}^\pm]}{\langle\varpi_1(t)c_2b_1\cdots b_{n-1}-\varpi_2(t)c_1\rangle}$$

## Quantum cohomology of polygon spaces in low rank

In this section we will verify Conjecture 3.2.7 for polygon spaces $\mathcal{P}_{r,n}$ with $n=4,5$.

### Quantum cohomology of the moduli space of 4-gons $\mathcal{P}_{r,4}$

Let $n+1=4$. Let $P \subseteq \mathrm{SL}_4(\mathbb{C})$ be the standard parabolic subgroup of upper block-triangular matrices such that $J(P)=\{1,3\}$; then $P=P^*$. Therefore, $\mathbb{Z}(L_P)=\{\mathrm{diag}(a,a,a^{-1},a^{-1}) \mid a \in \mathbb{C}^\times\}$. Let $\mathbf{i}=(2,1,3,2) \in R(w_P^{-1})$. With respect to the FZ-parameterisation in the direction $\mathbf{i}$, (3.4.5) implies that the superpotential takes the form

$$f_P(t,a_1,a_2,b_1,b_2)=a_1+a_2+b_1+b_2+\alpha_2(t)\frac{a_1+b_2}{a_1a_2b_1b_2}, \quad t \in Z(L_P), a_1,a_2,b_1,b_2 \in \mathbb{C}^\times.$$

Theorem 3.4.13 shows that, with respect to the FZ-parameterisation, the equivariant structure map $e$ takes the form

$$e(t,a_1,a_2,b_1,b_2)=t\,\mathrm{diag}((a_1a_2)^{-1},(b_1b_2)^{-1},a_2b_2,a_1b_1).$$

We trivialise $T$ as follows

$$\begin{array}{ccc} T & \longrightarrow & (\mathbb{C}^\times)^3 \\ t & \longmapsto & (\varpi_1(t),\varpi_2(t),\varpi_3(t)) \end{array}$$

Then, for $c=(c_1,c_2,c_3) \in (\mathbb{C}^\times)^3$, and $t=\mathrm{diag}(q,q,q^{-1},q^{-1}) \in Z(L_P)$, the intersection of the fibres $e^{-1}(c) \cap q^{-1}(t)$ is described by the equations

$$\frac{q}{a_1a_2}=c_1, \quad \frac{q^2}{a_1a_2b_1b_2}=c_2, \quad \frac{q}{a_1b_1}=c_3$$

Hence, we can eliminate, say, $a_1, a_2$ and $b_2$ and the restriction of the superpotential $f_P^t$ to the fibre $e^{-1}(c)$ is

$$f_P^t(b_1)=\frac{q}{c_3b_1}+\frac{c_3b_1}{c_1}+b_1+\frac{c_1q}{c_2b_1}+c_2\left(\frac{q}{c_3b_1}+\frac{c_1q}{c_2b_1}\right)$$

Here we have used $\alpha_2(t)=q^2$. Hence, for generic $c$, we obtain

$$\mathrm{Jac}(f_P^t) \cong \frac{\mathbb{C}[b]}{(b^2-q)}.$$

Recall from Example 2.2.4 that a weight variety constructed as the symplectic reduction of $\mathrm{Gr}(2,4)$ is diffeomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Hence, our construction recovers the well-known quantum cohomology ring of $\mathbb{P}^1_{\mathbb{C}}$.

**Theorem 3.4.15.** *Let $X=\mathrm{Gr}_{\mathbb{C}}(2,4)=\mathrm{SL}_4(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, $(M_P,F_P)$ the Rietsch mirror family. Let $e: M_P \to {}^LT$ be the equivariant structure map. Let $\mathcal{P}_{r,4}$, $r \in \mathbb{Z}^4_{>0}$, be the space of 4-gons realised as the symplectic reduction of $X$. Then, the quantum cohomology of $\mathcal{P}_{r,4}$ can be computed as the Jacobian ring of the restriction of $f_P$ to a generic fibre of $e$. In particular, Conjecture 3.2.7 is verified.*

**Quantum cohomology of the moduli space of 5-gons $\mathcal{P}_{r,5}$**

Let $n+1=5$. Let $P \subseteq \mathrm{SL}_5(\mathbb{C})$ be the standard parabolic subgroup of upper block-triangular matrices such that $J(P)=\{1,3,4\}$; then $P^*$ is the parabolic subgroup such that $J(P^*)=\{1,2,4\}$. Therefore, $\mathbb{Z}(L_P)=\{\mathrm{diag}(a,a,a^{-1},a^{-1},a^{-1}) \mid a \in \mathbb{C}^\times\}$. Let $\mathbf{i}=(3,2,1,4,3,2) \in R(w_P^{-1})$. With respect to the FZ-parameterisation in the direction $\mathbf{i}$, (3.4.5) implies that the superpotential takes the form

$$f_P(t,a_1,a_2,a_3,b_1,b_2,b_3)=a_1+a_2+a_3+b_1+b_2+b_3+\alpha_2(t)\frac{a_1a_2+a_1b_3+b_2b_3}{a_1a_2a_3b_1b_2b_3}$$

for $t \in Z(L_P), a_1,a_2,a_3,b_1,b_2,b_3 \in \mathbb{C}^\times$.

As above, we trivialise $T$ using the fundamental weights

$$\begin{array}{ccc} T & \longrightarrow & (\mathbb{C}^\times)^4 \\ t & \longmapsto & (\varpi_1(t),\varpi_2(t),\varpi_3(t),\varpi_4(t)) \end{array}$$

**Theorem 3.4.16.** *Let $X=\mathrm{Gr}_{\mathbb{C}}(2,5)=\mathrm{SL}_4(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, $(M_P,F_P)$ the Rietsch mirror family. Let $e: M_P \to {}^LT$ be the equivariant structure map. Let $\mathcal{P}_{r,5}$, $r \in \mathbb{Z}^4_{>0}$, be the space of 4-gons realised as the symplectic reduction of $X$. Let $r \in \{(1,1,1,1,2),(1,2,2,3)\}$. Then, for the quantum cohomology of $\mathcal{P}_{r,5}$ can be computed as the Jacobian ring of the restriction of $f_P$ to a generic fibre of $e$.*

## 3.5 Future directions

It would be interesting to extend the methods developed in this chapter to further classes of weight varieties in type $A$. One particular class where Conjecture 3.2.7 could be verified is for certain symplectic reductions of complete flag varieties ${}^LG/{}^LB_+$.

Let $\lambda \in \mathfrak{t}^*_+$ be generic and $\mathcal{O}_\lambda \subseteq \mathfrak{sl}_n$ the corresponding coadjoint orbit. By mapping a complete flag to its 1-dimensional constituent, we obtain a bundle

$$\mathcal{O}_\lambda \longrightarrow \mathbb{P}^{n-1}_{\mathbb{C}}$$

This bundle is a $\mathrm{SL}_n(\mathbb{C})$-equivariant symplectic fibration (see [58]) with fibre being a complete flag variety for $\mathrm{SL}_{n+1}(\mathbb{C})$. The Minimal Coupling Theorem (see [58, Chapter 4]) states that if the fibres $F$ of an equivariant symplectic fibration $X \to B$ are small enough then the symplectic reduction of $X$ at $\mu$, for an open subset of $\mu$'s, is a bundle over the symplectic reduction of $B$ with fibre $F$. In this situation, 'small enough' means that $\lambda$ is in a certain open neighbourhood of the line through the first fundamental weight. Hence, as the reduction of $\mathbb{P}^{n-1}_{\mathbb{C}}$ is a point, this implies that the reduction of $\mathcal{O}_\lambda$ is a complete flag variety for $\mathrm{SL}_{n-1}(\mathbb{C})$.

Therefore, for $\lambda$ close enough to $\varpi_1$ we expect that the approach developed in this thesis to compute the quantum cohomology of weight varieties of $\mathrm{SL}_n(\mathbb{C})$ will recover the quantum cohomology of a complete flag variety for $\mathrm{SL}_{n-1}(\mathbb{C})$. The quantum cohomology rings of complete flag varieties are known by [50], allowing us to verify our computation.

# Chapter 4

# Crystals

This Chapter investigates the appearance of combinatorial structures from representation theory in the mirror symmetry for flag varieties. Let $G$ be a reductive complex algebraic group with Lie algebra $\mathfrak{g}$. Associated to $\mathfrak{g}$ is the Drinfeld-Jimbo quantised universal enveloping algbra $U_q(\mathfrak{g})$, a Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$. The representation theory of $U_q(\mathfrak{g})$ is similar to the representation theory of $\mathfrak{g}$, and admits sufficient extra structure to be able to better understand the combinatorial representation theory of $\mathfrak{g}$. The key is Lusztig's *canonical basis* $\mathcal{B}$ of the positive part of $U_q(\mathfrak{g})$ [99]. $\mathcal{B}$ is canonical in the sense that it gives rise to canonical bases of all finite dimensional simple $U_q(\mathfrak{g})$-modules via the standard (co)Verma module construction.

Understanding the basis $\mathcal{B}$ itself is complicated but there exist several useful parameterisations of $\mathcal{B}$. For $\mathbf{i}$, a reduced expression for the longest element $w_0$ in the Weyl group of $\mathfrak{g}$, the *string parameterisation* $c_{\mathbf{i}}$ of Littelmann [95] provides a parameterisation of $\mathcal{B}$ by the lattice points in a rational polyhedral cone $C_{\mathbf{i}} \subseteq \mathbb{R}^{\ell(w_0)}$, the *string cone (in the direction* $\mathbf{i}$*)*. The *extended string cone* $\underline{C}_{\mathbf{i}} \subseteq \mathbb{R}^{\mathrm{rk}(\mathfrak{g})+\ell(w_0)}$ is a modification of $C_{\mathbf{i}}$ that 'remembers' the way in which $\mathcal{B}$ interacts with the finite dimensional simple $U_q(\mathfrak{g})$-modules.

The canonical basis $\mathcal{B}$ gives rise to a rich combinatorial structure known as a Kashiwara crystal [77]. These combinatorial objects model the representation theory of $\mathfrak{g}$ and provide effective approaches to studying tensor product multiplicities. Birational analogues of Kashiwara crystals, known as *geometric crystals*, were introduced by Berenstein-Kazhdan [12]. From a geometric crystal, one can construct a Kashiwara crystal via *tropicalisation*. For us, tropicalisation is a functor Trop from the class $\mathcal{V}$ of *positive varieties* (birational to algebraic tori) to Set such that, if $f : X \to \mathbb{A}^1$ is a rational function, $X \in \mathcal{V}$ birational to the algebraic torus $S$, then $\mathrm{Trop}(f) : X^\vee(S) \to \mathbb{Z}$ is a piecewise linear function.

A remarkable fact is that the Rietsch mirror family $(M_P, f_P)$ introduced in Section 3.1 is part of a geometric crystal. This has been observed in [93], [94], and used to prove mirror conjectures of Rietsch [115]. Our main result, Theorem 4.4.5, uses a family of non-standard parameterisation $j_{\mathbf{i}}$ of $M_P$, $\mathbf{i}$ a reduced expression of $w_0$, to explicitly show that the tropical locus $\{\mathrm{Trop}(f_P) \geq 0\}$ can be identified with the lattice points in the extended string cone $\underline{C}_{\mathbf{i}}(\mathbb{Z})$. Specifically, with respect to the parameterisation $j_{\mathbf{i}}$, we recover precisely the inequalities defining $\underline{C}_{\mathbf{i}}$.

In Section 4.1 we recall background from the theory of quantised universal enveloping algebras. Section 4.2 introduces Lustig's canonical basis $\mathcal{B}$, its consequences for representation theory and a brief account of the role of $\mathcal{B}$ in determining combinatorial tensor product multiplicity formulae. We define several parameterisations of $\mathcal{B}$ including the the family of string parameterisations due to Littelmann. We conclude this section by introducing the extended string cone $\underline{C}_{\mathbf{i}}$ and the $\lambda$-inequalities that define it. In Section 4.3, we give a brief account of Kashiwara's theory of crystals and their geometric counterparts developed by Berenstein-Kazhdan. In this section we develop the tool of tropicalisation, realised as a functor from a certain class of varieties to Set. Section 4.4 introduces a non-standard parameterisation of the Rietsch mirror $(M_B, f_B)$, and we state and prove our main result Theorem 4.4.5. We conclude with a discussion illuminating intriguing similarities between the hierarchy of a family of toric degenerations on the $A$-model side (introduced in [113]) and the crystal structure obtained in Theorem 4.4.5.

## 4.1 Some quantum algebra

In this section we recall some of the structure theory of quantised universal enveloping algebras associated to the Lie algebra $\mathfrak{g}$ of a reductive complex algebraic group $G$ and their representation theory.

**Convention 4.1.1.** Throughout this section $G$ will be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$ and Lie algebra $\mathfrak{g}$. We adopt the conventions and notation from Section 1.3. We will assume that $X = \Pi$ is the lattice of integral weights.

The Cartan matrix $[c_{ij}]_{i,j\in I}$, where $c_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$, $i, j \in I$, is symmetrisable. Let $d_i \in \mathbb{Z}_{>0}$, $i \in I$, be such that $d_i c_{ij} = d_j c_{ji}$. We assume that the integers $\{d_i\}_{i\in I}$ are pairwise relatively prime.

Let $\mathbb{C}(q)$ be the field of rational functions. Given $n \in \mathbb{Z}$ we define

$$[n]_q \coloneqq \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-3)} + q^{-(n-1)} \in \mathbb{C}(q).$$

Set $[0]_q! \coloneqq 1$, $[n]_q! \coloneqq [n]_q[n-1]_q \cdots [1]_q$, for $n > 0$, and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q \coloneqq \frac{[m]_q!}{[n]_q![m-n]_q!}, \quad 0 \leq n \leq m.$$

### Quantised universal enveloping algebras

**Definition 4.1.2.** The *(Drinfeld-Jimbo) quantised universal enveloping algebra associated to* $\mathfrak{g}$, $U_q(\mathfrak{g})$, or simply $U$ when there is no risk of confusion, is the associative $\mathbb{C}(q)$-algebra with unit generated by the elements $E_i, F_i$, $i \in I$, and $K_h$, $h \in X^\vee$, such that the function

$$\begin{array}{ccc} \mathbb{C}(q)[X^\vee] & \longrightarrow & U_q(\Psi(G)) \\ e^h & \longmapsto & K_h \end{array}$$

is a homomorphism of (commutative, unital) $\mathbb{C}(q)$-algebras (here $\mathbb{C}(q)[X^\vee]$ is the group algebra of $X^\vee$ over $\mathbb{C}(q)$), and such that the following relations hold:

(1) $K_h E_i K_{-h} = q^{\langle \alpha_i, h\rangle} E_i$, for $i \in I, h \in X^\vee$,

(2) $K_h F_i K_{-h} = q^{-\langle \alpha_i, h\rangle} F_i$, for $i \in I, h \in X^\vee$,

(3) $E_i F_j - F_j E_i = \delta_{ij} \frac{K_{d_i \alpha_i^\vee} - K_{-d_i \alpha_i^\vee}}{q^{d_i} - q^{-d_i}}$, for $i, j \in I$,

(4) for every $i \neq j$,

$$\sum_{r+s=1-c_{ij}} (-1)^r E_i^{(r)} E_j E_i^{(s)} = \sum_{r+s=1-c_{ij}} (-1)^r F_i^{(r)} F_j F_i^{(s)} = 0.$$

Here $E_i^{(r)} = E_i/[r]_{q^{d_i}}!$ and $F_i^{(r)} = F_i/[r]_{q^{d_i}}!$ are the *q-divided powers.*

We write $U^0(\mathfrak{g})$, or simply $U^0$ when there is no risk of confusion, for the image of the homomorphism $\mathbb{C}(q)[X^\vee] \to U_q(\Psi(G))$ described above. A standard argument shows that $U^0 \cong \mathbb{C}(q)[X^\vee]$.

Denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the $\mathbb{C}(q)$-subalgebra generated by $E_i$, $i \in I$, (resp. $F_i$, $i \in I$). When there is no risk of confusion we write $U^+$ (resp. $U^-$).

We write $U_q(\mathfrak{g})^{\geq 0}$ (resp. $U_q(\mathfrak{g})^{\leq 0}$) to be the subalgebra generated by $E_i$, $i \in I$, and $K_h$, $h \in X^\vee$, (resp. $F_i$, $i \in I$, and $K_h$, $h \in X^\vee$). When there is no risk of confusion we write $U^{\geq 0}$ (resp. $U^{\leq 0}$).

For any $i \in I$, we define $U_q(\mathfrak{g})_i$ to be the $\mathbb{C}(q^{d_i})$-subalgebra generated by $E_i, F_i, K_{\pm d_i \alpha_i^\vee}$. $U_q(\Psi(G))_i$ is a subalgebra isomorphic to $U_q(\mathfrak{sl}_2)$ (this follows from Proposition 4.1.5 below). When there is no risk of confusion we write $U_i$.

Define, for every $i \in I$, $h \in X^\vee$,

$$\deg(K_h) = 0, \text{ and } \deg(E_i) = -\deg(F_i) = \alpha_i.$$

The defining relations of $U_q(\mathfrak{g})$ are homogeneous and $U_q(\mathfrak{g})$ is a $Q$-graded algebra. Furthermore, there is the root space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where

$$U_q(\mathfrak{g})_\alpha = \{u \in U_q(\mathfrak{g}) \mid K_h u K_{-h} = q^{\langle h, \alpha\rangle} u, \text{ for all } h \in X^\vee\}.$$

When there is no risk of confusion we write $U_\alpha$ in place $U_q(\mathfrak{g})_\alpha$.

We will also make use of the following involutions:

(i) the *bar involution*, $u \mapsto \overline{u}$, is the $\mathbb{C}$-algebra automorphism of $U_q(\Psi(G))$ such that

$$\overline{q} = q^{-1}, \quad \overline{E}_i = E_i, \quad \overline{F}_i = F_i, \quad \overline{K}_h = K_{-h}. \tag{4.1.1}$$

An element $u \in U_q(\mathfrak{g})$ is *bar-invariant* if $u = \overline{u}$.

(ii) the $\mathbb{C}(q)$-algebra automorphism $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, uniquely determined by

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_h) = K_{-h}, \quad (i \in I, h \in X^\vee). \tag{4.1.2}$$

(iii) The $\mathbb{C}(q)$-algebra antiautomorphism $\iota : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, uniquely determined by

$$\iota(E_i) = E_i, \quad \iota(F_i) = F_i, \quad \iota(K_h) = K_{-h}, \quad (i \in I, h \in X^\vee). \tag{4.1.3}$$

Observe that $\iota$ is $Q$-graded.

**Remark 4.1.3.** The quantised universal enveloping algebra $U = U_q(\mathfrak{g})$ can be given the structure of a non-commutative, non-cocommutative Hopf algebra $(U, \Delta, S, \varepsilon)$, where

$$\Delta(E_i) = E_i \otimes 1 + K_{d_i\alpha_i^\vee} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-d_i\alpha_i^\vee} + 1 \otimes F_i, \quad \Delta(K_h) = K_h \otimes K_h, \tag{4.1.4}$$

$$S(E_i) = -K_{-d_i\alpha_i^\vee} E_i, \quad S(F_i) = -F_i K_{d_i\alpha_i^\vee}, \quad S(K_h) = K_{-h}, \tag{4.1.5}$$

$$\varepsilon(K_h) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \tag{4.1.6}$$

for $i \in I$, $h \in X^\vee$. With this structure of a Hopf algebra $U_q(\mathfrak{g})$ is a *Hopf algebra deformation* of $U(\mathfrak{g})$, the universal enveloping algebra associated to $\mathfrak{g}$ (equipped with its usual (noncommutative, cocommutative) Hopf algebra structure).

As a Hopf algebra $U_q(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$ as follows: in the specialisation $q \to 1$, the Hopf algebra structure given to $U_q(\mathfrak{g}$ specialises to the Hopf algebra $U(\mathfrak{g})$. Details can be found in [66, Chapter 3], [32, Chapter 6], or [69, Chapter 4].

**Remark 4.1.4.** The antipode map $S$ given in Remark 4.1.3 is an antiautomorphism of $\mathbb{C}(q)$-algebras.

A standard application of the Hopf algebra structure on $U_q(\Psi(G))$ is the existence of a *triangular decomposition*, originally due to Rosso:

**Proposition 4.1.5** (Rosso, [118])**.** *Let $U = U_q(\Psi(G))$. Then, $U = U^- \otimes U^0 \otimes U^+$.*

**Corollary 4.1.6.** *The subalgebras $U^\pm$ are completely determined by the relations in Definition 4.1.2 (4). In particular, the automorphism $\omega$ induces isomorphisms of $\mathbb{C}(q)$-algebras $U^\pm \cong \mathrm{U}^\mp$.*

*Also, $U^{\geq 0} \cong U^0 \otimes U^+$, $U^{\leq 0} \cong U^- \otimes U^0$.*

**Remark 4.1.7.** Using the involution $\omega$ and Corollary 4.1.6, we also have $U = U^+ \otimes U^0 \otimes U^-$.

Proposition 4.1.5 and Corollary 4.1.6 imply that we can obtain a basis for $U_q(\mathfrak{g})$ once we've specified a basis for $U_q^+(\mathfrak{g})$ (equivalently $U_q^-(\mathfrak{g})$). In Section 4.2 we will construct several bases for $U^{\pm}$: the *PBW-type bases* and Lusztig's *canonical basis*.

**Remark 4.1.8.** Given any symmetrisable Kac-Moody algebra $\mathfrak{g}$ with associated Cartan datum $(X, S, X^\vee, S^\vee)$ one can define an associated quantised universal enveloping algebra $U_q(\mathfrak{g})$. For further details see [66].

## Representation theory

In light of Remark 4.1.3, aspects of the representation theory of $U_q(\mathfrak{g})$ should closely resemble the representation theory of $G$. Some precise statements on how the representation theory of $U_q(\mathfrak{g})$ is a 'deformation' of the representation theory of $G$ are given in [27, Chapter 6].

We record the basic definitions and results from the representation theory of $U_q(\mathfrak{g})$ that we need. Let $U = U_q(\mathfrak{g})$, and denote the category of $U$-modules by $U$-mod. A $U$-module $V$ is a *weight module* if there is a *weight space decomposition*

$$V = \bigoplus_{\lambda \in X} V_\lambda, \quad \text{where } V_\lambda = \{v \in V \mid K_h v = q^{\langle \lambda, h \rangle} v, \;\; \text{for all } h \in X^\vee\}.$$

A weight $\lambda \in X$ is called a *weight of* $V$ if $V_\lambda \neq 0$, in which case $V_\lambda$ is called a *weight space*. Denote by $\mathrm{wt}(V) \subseteq X$ the set of weights of $V$. If $\lambda \in \mathrm{wt}(V)$ then any nonzero $x \in V_\lambda$ is called a *weight vector*. Observe that, for a $U$-module $V$, the generators of $U$ permute weight spaces:

$$E_i V_\lambda \subseteq V_{\lambda+\alpha_i}, \quad \text{and} \quad F_i V_\lambda \subseteq V_{\lambda-\alpha_i}. \tag{4.1.7}$$

A vector $v \in V$ is called a *highest weight vector of weight* $\lambda$ (resp. *lowest weight vector of weight* $\lambda$) if there exists $\lambda \in \mathrm{wt}(V)$ such that $v \in V_\lambda$ and

$$U^+ v = 0, \quad \text{and} \quad V = Uv, \quad (\text{resp.} \quad U^- v = 0, \quad \text{and} \quad V = Uv).$$

By Proposition 4.1.5, weight modules $V$ admitting a highest weight vector (resp. lowest weight vectors) are cyclic $U^-$-modules (resp. cyclic $U^+$-modules). A weight module $V$ is a *highest weight module with highest weight* $\lambda$ (resp. *lowest weight module with lowest weight* $\lambda$) if there exists a highest weight vector $v \in V_\lambda$ (resp. if there exists a lowest weight vector $v \in V_\lambda$). If $V$ is a highest/lowest weight module of highest/lowest $\lambda$ then $\dim V_\lambda = 1$.

Let $V$ be a weight module such that $\dim V_\lambda < \infty$, for all $\lambda \in X$, and such that there exists $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_l \in X$ so that

$$\mathrm{wt}(V) \subseteq \{\leq \lambda_1\} \cup \cdots \cup \{\leq \lambda_k\} \cup \{\geq \mu_1\} \cup \cdots \cup \{\geq \mu_l\}.$$

Here $\{\leq \lambda\} := \{\nu \in X \mid \nu \leq \lambda\}$ and $\{\geq \mu\} := \{\nu \in X \mid \nu \geq \mu\}$. The *character of* $V$ is

$$\mathrm{ch}\, V := \sum_{\lambda \in X} \dim V_\lambda e^\lambda \in \mathbb{Z}[[X]],$$

where $\mathbb{Z}[[X]]$ is the formal group ring of $X$.

Let $\mathcal{O}^q \subseteq U$-mod be the full subcategory of finitely-generated weight modules $V$ with finite dimensional weight spaces. Define $\mathcal{O}^q_+ \subseteq \mathcal{O}^q$ to be the full subcategory of modules for which $E_i$, $i \in I$, acts locally nilpotently: for any $v \in V$, and any $i \in I$, there exists $r$ such that $E_i^r v = 0$. Analogously, define $\mathcal{O}^q_- \subseteq \mathcal{O}^q$ to be the full subcategory of modules for which $F_i$, $i \in I$, acts locally nilpotently. Define $\mathcal{O}^q_{int} = \mathcal{O}^q_+ \cap \mathcal{O}^q_-$ to be the full subcategory consisting of those $U$-modules $V \in \mathcal{O}^q$ for which $E_i, F_i$, $i \in I$, act locally nilpotently. Objects in $\mathcal{O}^q_{int}$ are called *integrable $U$-modules.*

**Remark 4.1.9.** The category of weight modules considered above is often referred to as the category of Type $\mathbf{1}$ $U$-modules. See [69, Chapter 5] for further details.

Now we introduce some endofunctors on $\mathcal{O}^q$. They will restrict to give endofunctors on $\mathcal{O}^q_{int}$.

The automorphism $\omega$ defined in (4.1.2) induces an autoequivalence on $\mathcal{O}^q$, $V \mapsto {}^{\omega}V$: as a vector space ${}^{\omega}V = V$ but we define the *twisted* $U$-action $*$ on $V$

$$u * v := \omega(u)v, \quad u \in U, v \in V. \tag{4.1.8}$$

Then, $({}^{\omega}V)_\lambda = V_{-\lambda}$ and twisting induces an equivalence $\mathcal{O}^q_\pm \xrightarrow{\sim} \mathcal{O}^q_\mp$. In particular, we obtain an autoequivalence of $\mathcal{O}^q_{int}$.

For any $V \in \mathcal{O}^q$, with $V = \bigoplus_{\lambda \in X} V_\lambda$ and $\dim V_\lambda < \infty$, we define the *graded dual of* $V$ to be

$$V_* := \bigoplus_\lambda V_\lambda^*, \quad \text{where } V_\lambda^* = \mathrm{Hom}_{\mathbb{C}(q)}(V_\lambda, \mathbb{C}(q)). \tag{4.1.9}$$

We provide $V_*$ with the structure of a $U$-module as follows: for $x \in U$, $f \in V_*$, define $xf \in V_*$ by

$$(xf)(u) = f(S(x)u), \quad u \in U. \tag{4.1.10}$$

Observe that, if $V \in \mathcal{O}^q$, $\lambda, \mu \in \mathrm{wt}(V)$, and $f \in V_\lambda^*$, $v \in V_\mu$, then

$$(K_h f)(v) = q^{-\langle \mu, h\rangle} f(v), \quad h \in X^\vee.$$

Hence, $(V_*)_\lambda = V^*_{-\lambda}$. Therefore, we have an endofunctor on $\mathcal{O}^q$, $V \mapsto V_*$. If $V \in \mathcal{O}^q_+$ (resp. $V \in \mathcal{O}^q_-$) then $V_* \in \mathcal{O}^q_-$ (resp. $V_* \in \mathcal{O}^q_+$): indeed, for $i \in I$, we have

$$F_i^r (V_*)_\lambda \subseteq (V_*)_{\lambda - r\alpha_i} = V^*_{-\lambda + r\alpha_i}$$

so that, if $V \in \mathcal{O}^q_+$ then $V_{-\lambda+r\alpha_i} \equiv 0$, whenever $-\lambda \in \mathrm{wt}(V)$ and $r$ is sufficiently large. Hence, $V \mapsto V_*$ restricts to give an endofunctor on $\mathcal{O}^q_{int}$.

**Proposition 4.1.10.** *(a) The functor $V \mapsto V_*$ is exact, and,*

*(b) $(\mathcal{O}^q_\pm)_* \subseteq \mathcal{O}^q_\mp$. In particular, $(\mathcal{O}^q_{int})_* \subseteq \mathcal{O}^q_{int}$.*

We introduce some examples of $U$-modules that we use throughout this thesis. They are the quantum analogues of (co)Verma modules from the representation theory of $U(\mathfrak{g})$.

**Example 4.1.11.** (a) Let $\lambda \in X$. Define

$$\Delta(\lambda) \coloneqq U/\left(\sum_{i\in I} UE_i + \sum_{h\in X^\vee} U(K_h - q^{\langle\lambda,h\rangle}1)\right) \in \mathcal{O}^q_- \tag{4.1.11}$$

and

$$\nabla(\lambda) \coloneqq U/\left(\sum_{i\in I} UF_i + \sum_{h\in X^\vee} U(K_h - q^{\langle\lambda,h\rangle}1)\right) \in \mathcal{O}^q_+. \tag{4.1.12}$$

Denote the left ideal in (4.1.11) (resp. (4.1.12)) by $J^-_\lambda$ (resp. $J^+_\lambda$). The triangular decomposition (Proposition 4.1.5) implies that $\Delta(\lambda)$ is a highest weight module having highest weight $\lambda$, with highest weight vector $1 + J^-_\lambda$, and that $\nabla(\lambda)$ is a lowest weight module having lowest weight $\lambda$, with lowest weight vector $1 + J^+_\lambda$.

As a cyclic $U^-$-module, $\Delta(\lambda) \cong U^-$; as a cyclic $U^+$-module, $\nabla(\lambda) \cong U^+$. Moreover, ${}^\omega\Delta(\lambda) = \nabla(-\lambda)$ and ${}^\omega\nabla(\lambda) = \Delta(-\lambda)$.

(b) Let $a = (a_i)_{i\in I}, b = (b_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$. Define the left ideal

$$J_{a,b,\lambda} \coloneqq \sum_{i\in I} UE_i^{a_i+1} + \sum_{i\in I} UF_i^{b_i+1} + \sum_{h\in X^\vee} U(K_h - q^{\langle\lambda,h\rangle}1) \subseteq U. \tag{4.1.13}$$

A straightforward but lengthy calculation shows that the quotient $U/J_{a,b,\lambda} \in \mathcal{O}^q_{int}$ (see [69, Lemma 5.7]).

**Definition 4.1.12.** (a) Let $\lambda \in X_+$ be a dominant weight and $b = (b_i) \in \mathbb{Z}^I_{\geq 0}$ be defined by $b_i = \langle\lambda, \alpha_i^\vee\rangle$. Define $V^q(\lambda) \in \mathcal{O}^q_{int}$ to be the $U$-module

$$V^q(\lambda) \coloneqq U/J_{0,b,\lambda}.$$

(b) Let $\lambda \in X_-$ be an antidominant weight and $a = (a_i) \in \mathbb{Z}^I_{\geq 0}$ be defined by $a_i = -\langle\lambda, \alpha_i^\vee\rangle$. Define $V_q(\lambda) \in \mathcal{O}^q_{int}$ to be the $U$-module

$$V_q(\lambda) \coloneqq U/J_{a,0,\lambda}.$$

The following result shows that the combinatorics of the representation theory of $U$ is 'the same as' the corresponding representation theory of $G$. For details see [66, Chapter 3], [69, Chapter 5].

**Proposition 4.1.13.** *(a) The category $\mathcal{O}^q_{int}$ is semisimple and closed under taking tensor products and graded duals. The objects of $\mathcal{O}^q_{int}$ are precisely the finite-dimensional $U$-modules.*

*(b) (i) Let $\lambda \in X_+$ be dominant. Then, $V^q(\lambda)$ is a finite-dimensional irreducible highest weight module with highest weight $\lambda$.*

*(ii) Let $\lambda \in X_-$ be antidominant. Then, $V_q(\lambda)$ is a finite-dimensional irreducible lowest weight module with lowest weight $\lambda$.*

*(c) If $V \in \mathcal{O}^q_{int}$ is irreducible then $V$ is a highest weight module.*

*(d) If $V \in \mathcal{O}^q_{int}$ is a highest weight module with highest weight $\lambda \in X$ then $\lambda \in X_+$ and $V \cong V^q(\lambda)$. Similarly, if $V \in \mathcal{O}^q_{int}$ is a lowest weight module having lowest weight $\lambda \in X$ then $\lambda \in X_-$ and $V \cong V_q(\lambda)$.*

*(e) Let $\lambda \in X_+$ be dominant. let $V(\lambda)$ be the finite-dimensional irreducible $G$-module with highest weight $\lambda$. Then, $\operatorname{ch} V(\lambda) = \operatorname{ch} V^q(\lambda)$.*

**Remark 4.1.14.** Using the twisted action (4.1.8), Proposition 4.1.13 implies that the irreducible highest weight module ${}^{\omega}V^q(\lambda)$ is a lowest weight module having lowest weight $-\lambda$, so that ${}^{\omega}V^q(\lambda) \cong V_q(-\lambda)$. Similarly, we have $V^q(\lambda)_* \cong V_q(-\lambda)$.

In fact, as in the classical setting, we have, for $\lambda \in X_+$,

$$V_q(-\lambda) \cong V^q(-w_0(\lambda)).$$

**Remark 4.1.15.** The subalgebras $U^{\geq 0}$ and $U^{\leq 0}$ are Hopf subalgebras and most of the above constructions can be defined by restricting the Hopf algebra structure from $U$.

The subalgebras $U^{\pm}$ are not Hopf subalgebras (they are not closed under $\Delta$, for example); however, we can define a (twisted) Hopf algebra structure. We describe this Hopf algebra structure for $U^+$, the structure on $U^-$ is obtained via $\omega$. Define a (twisted) multiplication (on homogeneous elements)

$$\begin{array}{ccc} U^+ \otimes U^+ \times U^+ \otimes U^+ & \longrightarrow & U^+ \otimes U^+ \\ (a \otimes b, c \otimes d) & \longmapsto & q^{\langle \deg b, (\deg c)^\vee \rangle} ac \otimes bd. \end{array}$$

Define

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes 1, \quad S(E_i) = -E_i, \quad \varepsilon(E_i) = 0, \quad (i \in I) \tag{4.1.14}$$

Then, $(U^+, \Delta, S, \varepsilon)$ is a (twisted) Hopf algebra. For further discussion see [99, Chapter 1].

## 4.2 Bases and parameterisations

In this section we give several constructions of bases for the quantised universal enveloping algebra $U = U_q(\mathfrak{g})$. Using Proposition 4.1.5, it suffices to obtain a basis for either $U^+$ or $U^-$.

## PBW-type bases

For $\mathfrak{g}$ of simply-laced, finite type, Lusztig introduces in [100] an action on $U$ of the braid group covering $W$. In subsequent joint work with M. Dyer [101, Appendix], and again still for simply-laced, finite type $\mathfrak{g}$, Lusztig uses the braid group action to determine a collection of bases $\mathcal{B}_{\mathbf{i}} \subseteq U^+$ depending on a reduced expression $\mathbf{i}$ of the longest element $w_0 \in W$. This construction is a quantum analogue of the existence of the well-known PBW basis for $U(\mathfrak{g})$. As such, the bases $\mathcal{B}_{\mathbf{i}} \subseteq U^+$ are known as PBW-type bases. Saito later extended this construction to $\mathfrak{g}$ of arbitrary finite type and obtained $PBW$-type bases of $U^+$ (see [119]).

We will briefly outline this construction. Following [101], define an algebra automorphism $T_i$ of $U$, $i \in I$, defined by

$$T_i(E_i) = -F_i K_{d_i\alpha_i^\vee}, \quad T_i(F_i) = -K_{-d_i\alpha_i^\vee} E_i, \tag{4.2.1}$$

$$T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r q^{-d_i s} E_i^{(r)} E_j E_i^{(s)}, \quad T_i(F_j) = \sum_{r+s=-c_{ij}} (-1)^r q^{d_i s} F_i^{(s)} F_j F_i^{(r)} \quad j \in I, j \neq i,$$

$$T_i(K_h) = K_{s_i(h)}, \quad h \in X^\vee.$$

It can be seen that $T_i^{-1} = \iota \circ T_i \circ \iota$.

The following result is fundamental (see [69, Chapter 8]).

**Proposition 4.2.1** (Lusztig [100], Saito [119])**.** *The collection $\{T_i \mid i \in I\}$ satisfy the braid relations associated to the Weyl group of $G$.*

Hence, there is an action of the braid group covering $W$ on $U$. If $w = s_{i_1} \cdots s_{i_k} \in W$, with $\ell(w) = k$, then the automorphism $T_w \coloneqq T_{i_1} \cdots T_{i_k}$ is well-defined. Observe that, for each $i \in I$,

$$T_i(U_\alpha) \subseteq U_{s_i(\alpha)}, \quad \alpha \in Q.$$

The following Lemma can be found in [69, Proposition 8.20]).

**Lemma 4.2.2.** *Let $w = s_{i_1} \cdots s_{i_k} \in W$, $\ell(w) = k$, such that*

$$s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_j \in S, \quad \text{for some } j \in I.$$

*Then, $T_{s_{i_1} \cdots s_{i_{k-1}}}(E_{i_k}) = E_j \in U^+$.*

Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ be a reduced expression for the longest element $w_0 \in W$, and $t = (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0}$. Define

$$p_{\mathbf{i}}(t) \coloneqq E_{i_1}^{(t_1)} T_{i_1}\left(E_{i_2}^{(t_2)}\right) \cdots (T_{i_1} \cdots T_{i_{m-1}})\left(E_{i_m}^{(t_m)}\right). \tag{4.2.2}$$

Then,

$$\deg p_{\mathbf{i}}(t) = \sum_{j=1}^m t_j \beta_j, \tag{4.2.3}$$

where $\beta_j \coloneqq s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Define

$$\mathcal{B}_{\mathbf{i}} \coloneqq \{p_{\mathbf{i}}(t) \mid t \in \mathbb{Z}^m_{\geq 0}\}. \tag{4.2.4}$$

**Proposition 4.2.3** (Lusztig, [101, Appendix]). *For every reduced expression $\mathbf{i} \in R(w_0)$ the set $\mathcal{B}_{\mathbf{i}}$ is a (homogeneous) $\mathbb{C}(q)$-basis of $U^+$, called a* PBW-type basis of $U^+$.

**Remark 4.2.4.** Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. Then, $\mathbf{i}$ induces a total ordering on the set of positive roots determined by the simple roots $S \subseteq R$ as follows: define $\beta_1 = \alpha_{i_1}$ and, for each $j > 1$, define

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}).$$

Then, the sequence $\beta_1, \ldots, \beta_m \in R_+$ consists of distinct positive roots and provides an enumeration of $R_+$. We call the sequence $\beta_1, \ldots, \beta_m$ the *root sequence* associated to $\mathbf{i}$.

Recall the PBW theorem (see [68, Chapter 17]) for the universal enveloping algebra $U(\mathfrak{n}_+)$: for any ordered basis $x_1, \ldots, x_m$ of $\mathfrak{n}_+$, the set of monomials

$$\{x_1^{a_1} \cdots x_m^{a_m} \mid (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0}\} \subseteq U(\mathfrak{n}_+)$$

is a $\mathbb{C}$-basis of $U(\mathfrak{n}_+)$. In particular, if $\mathbf{i} \in R(w_0)$ then the corresponding root sequence $\beta_1, \ldots, \beta_m$ and root vectors $x_i := e_{\beta_i} \in \mathfrak{n}_+(\beta_i)$, $i = 1, \ldots, m$, induces a $\mathbb{C}$-basis $B_{\mathbf{i}}$ of $U(\mathfrak{n}_+)$. Here $e_\alpha \in \mathfrak{n}_+$ is a root vector of weight $\alpha$.

The bases $\mathcal{B}_{\mathbf{i}}$ are $q$-analogues of the PBW-bases defined for $U(\mathfrak{n}_+)$. In fact, in the $q \to 1$ limit, $\mathcal{B}_{\mathbf{i}}$ specialises to $B_{\mathbf{i}}$ (see [101], Appendix).

**Remark 4.2.5.** Recall from (4.1.2) the isomorphism of algebras $\omega : U^+ \xrightarrow{\sim} U^-$. Using this isomorphism, a PBW-type basis $\mathcal{B}_{\mathbf{i}}$ gives rise to a basis of $U^-$. Using (4.2.1) we have

$$T_i(\omega(E_j)) = (-q^{-d_i})^{\langle \alpha_j, \alpha_i^\vee \rangle} \omega(T_i(E_j)), \quad i, j \in I.$$

Hence, if $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ the set

$$\left\{ F_{i_1}^{(t_1)} T_{i_1} \left( F_{i_2}^{(t_2)} \right) \cdots (T_{i_1} \cdots T_{i_{m-1}}) \left( F_{i_m}^{(t_m)} \right) \mid (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0} \right\} \subseteq U^- \tag{4.2.5}$$

is a basis of $U^-$, called a *PBW-type basis of* $U^-$.

Recall the antiautomorphism $\iota$ of $U$ given in (4.1.3) and the involution $i \mapsto i^*$ from Section 1.3. We will see that $\iota$ induces a permutation on the set $\{\mathcal{B}_{\mathbf{i}} \mid \mathbf{i} \in R(w_0)\}$ of PBW-type bases. First, we require some notation.

**Definition 4.2.6.** Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a sequence. Define

$$\mathbf{i}^* := (i_1^*, \ldots, i_r^*), \quad \mathbf{i}^{op} = (i_r, \ldots, i_1). \tag{4.2.6}$$

The operations $\mathbf{i} \mapsto \mathbf{i}^*$ and $\mathbf{i} \mapsto \mathbf{i}^{op}$ commute with each other,

$$(\mathbf{i}^{op})^* = (\mathbf{i}^*)^{op}, \quad \text{for } \mathbf{i} \in I^r.$$

If $\mathbf{i} \in R(w_0)$ then $\mathbf{i} \mapsto \mathbf{i}^*$, $\mathbf{i} \mapsto \mathbf{i}^{op}$, are (commuting) permutations of $R(w_0)$.

**Proposition 4.2.7.** *Let* $\mathbf{i} \in R(w_0)$ *and let* $\mathbf{i}' = (\mathbf{i}^{op})^*$*. Then,* $\iota(p_{\mathbf{i}}(t)) = p_{\mathbf{i}'}(t^{op})$*, for any* $t = (t_1, \dots, t_m) \in \mathbb{Z}^m_{\geq 0}$*, and where* $t^{op} = (t_m, \dots, t_1)$*.*

*Proof.* Let $\mathbf{i} = (i_1, \dots, i_m) \in R(w_0)$ and denote $\mathbf{i}' = (\mathbf{i}^{op})^* = (i'_1, \dots, i'_m) \in R(w_0)$. For any $k \in \{1, \dots, m\}$, we have

$$s_{i_{k-1}} \cdots s_{i_1} s_{i'_1} \cdots s_{i'_{m-k+1}} = w_0,$$

and

$$s_{i_{k-1}} \cdots s_{i_1} s_{i'_1} \cdots s_{i'_{m-k}} (\alpha_{i'_{k-m+1}}) = \alpha_{i_k}.$$

By Lemma 4.2.2, we obtain

$$T_{w_0 s_{i'_{k-m+1}}} (E_{i_{m-k+1}}) = E_{i_k}.$$

Hence,

$$T_{i'_1} \cdots T_{i'_{m-k}} (E_{i'_{m-k+1}}) = T^{-1}_{i_1} \cdots T^{-1}_{i_{k-1}} (E_{i_k}) = \iota(T_{i_1} \cdots T_{i_{k-1}} (E_k))$$

Applying the antiautomorphism $\iota$ to $p_{\mathbf{i}}(t) \in \mathcal{B}_i$, $t = (t_1, \dots, t_m) \in \mathbb{Z}^m_{\geq 0}$, we obtain

$$\begin{aligned}
\iota(p_{\mathbf{i}}(t)) &= \iota\left(E^{(t_1)}_{i_1} T_{i_1}\left(E^{(t_2)}_{i_2}\right) \cdots (T_{i_1} \cdots T_{i_{m-1}})\left(E^{(t_m)}_{i_m}\right)\right) \\
&= \iota\left(T_{i_1} \cdots T_{i_{m-1}}\left(E^{(t_m)}_{i_m}\right)\right) \cdots \iota\left(T_{i_1}\left(E^{(t_2)}_{i_2}\right)\right) \iota(E^{(t_1)}_{i_1}) \\
&= E^{(t_m)}_{i'_1} T_{i'_1}\left(E^{(t_{m-1})}_{i'_2}\right) \cdots (T_{i'_1} \cdots T_{i'_{m-1}})\left(E^{(t_1)}_{i'_m}\right) \\
&= p_{\mathbf{i}'}(t^{op}) \in \mathcal{B}_{\mathbf{i}'}.
\end{aligned}$$

□

An immediate consequence is the following:

**Corollary 4.2.8.** *Let* $\mathbf{i} \in R(w_0)$ *and let* $\mathbf{i}' = (\mathbf{i}^{op})^*$*. Then,* $\iota(\mathcal{B}_{\mathbf{i}}) = \mathcal{B}_{\mathbf{i}'}$*.*

**Remark 4.2.9.** There is an analogous result for the PBW-type bases of $U^-$ (see Remark 4.2.5): the antiautomorphism $\iota$ is an involutive permutation on the set of PBW-type bases of $U^-$.

## Canonical bases and Lusztig parameterisation

In [97] Lusztig provided a simultaneous modification of the PBW-type bases, called the *canonical basis*: each of the PBW-type bases is related to the canonical basis by a unitriangular change of basis (with respect to some order). Originally the construction of the canonical basis was restricted to simply-laced, finite type $\mathfrak{g}$ as it relied on results of Ringel relating $ADE$ quantised enveloping algebras $U^+$ with the Hall algebra of the type $ADE$ quiver. Moreover, Lusztig's construction made essential use of deep results in algebraic geometry and topology coming from the theory of perverse sheaves and intersection cohomology. A favourable feature of Lusztig's canonical basis is that it provides a construction of a *canonical basis* for each finite dimensional irreducible $U$-module admitting remarkable consequences (for example, Theorem 4.2.15).

Independently and simultaneously, Kashiwara provided an elementary algebraic construction of a *global basis* of $U^-$ admitting similar consequences for the representation theory of $U$. Kashiwara's construction had the advantage of working for an arbitrary symmetrisable Kac-Moody algebra and the only 'geometry' required was a basic result on the triviality of vector bundles on $\mathbb{P}^1$ (which is essentially an algebraic problem). We will discuss Kashiwara's result further in Section 4.3.

Lusztig later [102] extended his construction of the canonical basis of $U^+$ to the (positive part of) quantised universal enveloping algebras associated arbitrary symmetric Kac-Moody algebras, and outlined a construction for the non-symmetric setting. Again, his construction relied on deep results from the theory of perverse sheaves and intersection cohomology. Complete details of Lusztig's topological construction of the canonical basis can be found in [99, Part II].

In this section we recall the essential features of Lusztig's results and indicate some of the consequences for the determination of tensor product multiplicities in representation theory.

**Theorem 4.2.10** (Lusztig, [97])**.** *Let $U = U_q(\mathfrak{g})$.*

*(a) The $\mathbb{Z}[q^{-1}]$-submodule $\mathcal{L} = \operatorname{span}_{\mathbb{Z}[q^{-1}]} \mathcal{B}_{\mathbf{i}} \subseteq U^+$ is independent of $\mathbf{i}$.*

*(b) The $\mathbb{Z}$-basis $B = \mathcal{B}_{\mathbf{i}} + \mathcal{L} \subseteq \mathcal{L}/q^{-1}\mathcal{L}$ is independent of $\mathbf{i}$.*

*(c) The projection $\mathcal{L} \to \mathcal{L}/q^{-1}\mathcal{L}$ induces a $Q$-graded isomorphism $f : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/q^{-1}\mathcal{L}$ of $\mathbb{Z}$-modules. The $\mathbb{Z}$-basis $\mathcal{B} \coloneqq f^{-1}(B)$ is a $\mathbb{Z}[q^{-1}]$-basis of $\mathcal{L}$ and consists of bar-invariant, homogeneous elements.*

**Corollary 4.2.11.** *$\iota(\mathcal{B}) = \mathcal{B}$.*

*Proof.* By Corollary 4.2.8 and Theorem 4.2.10(a), the antiautomorphism $\iota$ induces an action on $\mathcal{L}/q^{-1}\mathcal{L}$ and a permutation of $B \subseteq \mathcal{L}/q^{-1}\mathcal{L}$. Moreover, $\iota$ commutes with $^- : U \to U$ and therefore preserves $\mathcal{L} \cap \overline{\mathcal{L}}$. Also, $\iota$ commutes with the natural projection $\pi : \mathcal{L} \to \mathcal{L}/q^{-1}\mathcal{L}$, so that $f \circ \iota = \iota \circ f$. The result follows. ☐

The $\mathbb{C}(q)$-basis $\mathcal{B} \subseteq U^+$ is Lusztig's *canonical basis*. By Theorem 4.2.10, $\mathcal{B}$ is the unique homogeneous basis of $U^+$ such that

(i) for every $b \in \mathcal{B}$, $b = \overline{b}$,

(ii) for every $\mathbf{i} \in R(w_0)$, $t \in \mathbb{Z}^m_{\geq 0}$, there is a unique $b = b_{\mathbf{i}}(t) \in \mathcal{B}$ such that $b - p_{\mathbf{i}}(t)$ is a linear combination of elements in $\mathcal{B}_{\mathbf{i}}$ with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$.

Define $\mathcal{B}_\alpha \coloneqq \mathcal{B} \cap U_\alpha$, $\alpha \in Q$. Observe that $\mathcal{B}_0 = \{1\}$.

**Definition 4.2.12.** Let $\mathbf{i} \in R(w_0)$. The map

$$\begin{array}{rccc} b_{\mathbf{i}}: & \mathbb{Z}^m_{\geq 0} & \longrightarrow & \mathcal{B} \\ & t & \longmapsto & b_{\mathbf{i}}(t) \end{array}$$

is called the *Lusztig parameterisation of $\mathcal{B}$ (in the direction $\mathbf{i}$)*. If $b = b_{\mathbf{i}}(t)$ then we call $t$ the *Lusztig data of $b$ (in the direction $\mathbf{i}$)*.

**Remark 4.2.13.** Using the isomorphism $\omega : U^+ \xrightarrow{\sim} U^-$, the image of $\mathcal{B}$ in $U^-$ can be shown to be a basis possessing analogous properties as those described in Theorem 4.2.10.

The canonical basis $\mathcal{B} \subseteq U_q^+(\mathfrak{g})$ admits remarkable consequences for the representation theory of $U_q(\mathfrak{g})$ and, by Proposition 4.1.13, for the representation theory of $G$ itself. The following technical result is essential.

**Lemma 4.2.14.** *Let $i \in I$, $r \geq 0$.*

*(a) $\mathcal{B} \cap U^+E_i^r$ spans $U^+E_i^r$.*

*(b) $\mathcal{B} \cap E_i^rU^+$ spans $E_i^rU^+$.*

*Proof.* (a) Let $b \in \mathcal{B}_\nu$, where $\nu = \sum_{i\in I} \nu_i\omega_i$, and fix $i \in I$. Define $s_i(b) \in \mathbb{Z}_{\geq 0}$ to be the largest integer $r$ such that $0 \leq r \leq \nu_i$ and satisfying the condition:

$(A)$ there exists $z' \in U^+$ such that $b$ appears with nonzero coefficient in $z'E_i^r$

Observe that $b \mapsto s_i(b)$ is well-defined: $(A)$ is always satisfied when $r = 0$. Using [102, Section 11.6], we have $b \in U^+E_i^{s_i(b)}$.

Let $z \in U^+E_i^r$. By Theorem 4.2.10, we can write $z = \sum_{b\in\mathcal{B}} a_b b$, $a_b \in \mathbb{C}(q)$. Suppose $a_b \neq 0$. Then, degree considerations give $r \leq \nu_i$ and $r \leq s_i(b)$ by definition of $s_i(b)$. Hence, $b \in U^+E_i^{s_i(b)} \subseteq U^+E_i^r$.

(b) Applying the antiautomorphism $\iota$ to $U^+$, the result follows from (a) and Corollary 4.2.11.

□

**Theorem 4.2.15.** *Let $\lambda \in X_-$ be an antidominant weight, $V_q(\lambda)$ the corresponding irreducible lowest weight module (see Definition 4.1.12). Then, if $V_q(\lambda) \cong U^+/J_\lambda$ as a cyclic $U^+$-module then $\mathcal{B} \cap J_\lambda$ spans $J_\lambda$. Equivalently, $\{b + J_\lambda \mid b \notin J_\lambda\}$ spans $V_q(\lambda)$.*

*Proof.* Let $\lambda = -\sum_{i\in I} c_i\omega_i$, with $c_i \in \mathbb{Z}_{\geq 0}$. As a $U^+$-module we have $V_q(\lambda) \cong U^+/\sum_{i\in I} U^+E_i^{c_i+1}$ and it suffices to show that $\mathcal{B} \cap U^+E_i^n$ spans $U^+E_i^n$, for every $i \in I$ and every $n \geq 0$. This follws from Lemma 4.2.14(a).

□

**Definition 4.2.16.** Let $\lambda \in X_+$, $v_\lambda \in V_q(w_0(\lambda))$ be a lowest weight vector. Define

$$\mathcal{B}(\lambda) \coloneqq \{b \in \mathcal{B} \mid bv_\lambda \neq 0\}.$$

Therefore,

$$\mathcal{B}(\lambda) = \{b \in \mathcal{B} \mid b \notin U^+ E_i^{-\langle w_0(\lambda), \alpha_i^\vee \rangle + 1},\ i \in I\}.$$

**Proposition 4.2.17.** *Let* $\mathbf{i} \in R(w_0)$*. Then, for any* $\lambda \in X_+$*,* $\mathcal{B}(\lambda)$ *is parameterised by a subset of Lusztig data via the Lusztig parameterisation of* $\mathcal{B}$ *in the direction* $\mathbf{i}$*.*

In the remainder of this section we illustrate an application of the canonical basis to determining combinatorial tensor product multiplicity formulae. First we recall the basic problem.

For $\lambda \in X_+$, we write $V(\lambda)$ for the irreducible $\mathfrak{g}$-module, and let $\mathrm{wt}(\lambda) \coloneqq \mathrm{wt}\, V(\lambda)$ be the set of weights of $V(\lambda)$.

Let $\lambda, \mu, \nu \in X_+$. Then, the tensor product $V(\lambda) \otimes V(\mu)$ decomposes as a direct sum of irreducibles

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_\nu V(\nu)^{c^\nu_{\lambda,\mu}}$$

We would like to determine manifestly positive combinatorial models that compute the nonnegative integers $c^\nu_{\lambda,\mu}$. Such combinatorial models are called *Littlewood-Richardson rules* in reference to the type $A$ model involving skew-tableaux.

Gelfand-Zelevinsky proposed in [44], [45], an approach to determining the tensor product multiplicities $c^\nu_{\lambda,\mu}$ by counting lattice points in polytopes (see also [5]). Their argument relied on the notion of a *good basis* in $V(\lambda)$ that we will now describe.

**Definition 4.2.18.** Let $\lambda, \gamma \in X_+$, $\beta \in \mathrm{wt}(\lambda)$. Define the *$\gamma$-primitive $\beta$-weight vectors in* $V(\lambda)$ to be the nonzero elements in the following subspace

$$V(\lambda; \beta, \gamma) \coloneqq \{v \in V(\lambda)_\beta \mid e_i^{\langle \gamma, \alpha_i^\vee \rangle + 1}(v) = 0,\ i \in I\}.$$

The relation between $V(\lambda; \beta, \gamma)$ and $c^\nu_{\lambda,\mu}$ is given by the following result due to Kostant [89, Lemma 4.1].

**Proposition 4.2.19.** *Let* $\lambda, \mu, \nu \in X_+$*, Then,*

$$c^\nu_{\lambda,\mu} = \dim V(\lambda; \nu - \mu, \mu).$$

*Proof.* Let $\lambda, \mu, \nu \in X_+$. Then,

$$\begin{aligned} c^\nu_{\lambda,\mu} &= \dim \mathrm{Hom}_{\mathfrak{g}}(V(\nu), V(\lambda) \otimes V(\mu)) \\ &= \dim \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu), V(\lambda) \otimes V(\mu)) \\ &= \dim \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu) \otimes V(\mu)^*, V(\lambda)) \end{aligned}$$

Any $f \in \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu) \otimes V(\mu)^*, V(\lambda))$ is determined by $f(1_\nu \otimes v_{-\mu})$, where $v_{-\mu} \in V(\mu)^*$ is a lowest weight vector. Then, we must have

$$f(1_\nu \otimes v_{-\mu}) \in V(\lambda)_{\nu-\mu}.$$

Moreover, we must have $e_i^{\langle \mu, \alpha_i^\vee \rangle + 1}(v_{-\mu}) = 0$, for each $i \in I$. Therefore, there is an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu) \otimes V(\mu)^*, V(\lambda)) & \longrightarrow & V(\lambda; \nu - \mu, \mu) \\ f & \longmapsto & f(1_\nu \otimes v_{-\mu}). \end{array}$$

The result follows. □

A *good basis for* $V(\lambda)$, $\lambda \in X_+$, is a weight basis $B \subseteq V(\lambda)$ such that $B \cap V(\lambda; \beta, \gamma)$ spans, for all $\beta \in \mathrm{wt}(\lambda)$, $\gamma \in X_+$. In particular, by Proposition 4.2.19, a subset of a good basis of $V(\lambda)$ computes $c^\nu_{\lambda,\mu}$.

Mathieu showed proved the existence of good bases in [106] using Frobenius splitting methods (in particular, his proof is restricted to finite type). Lusztig provided a proof using the canonical basis; his approach extends to arbitrary symmetrisable type (upon equating Lusztig's canonical basis with Kashiwara's global basis).

**Proposition 4.2.20** (Mathieu [106], Lusztig [98, Section 4])**.** *Let $\lambda \in X_+$. Then, there exists a good basis of $V(\lambda)$.*

*Proof.* Let $\lambda \in X_+$ and consider the $U_q(\mathfrak{g})$-module $V^q(\lambda)$. There is an analogous definition of the space of $\nu$-primitive $\mu$-weight vectors in $V^q(\lambda)$. For $\beta, \gamma \in X_+$, define

$$I_\beta := \sum_{i \in I} U^+ E_i^{\langle \beta, \alpha_i^\vee \rangle + 1}, \quad \text{and} \quad J_\gamma := \sum_{i \in I} E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1} U_+.$$

By Lemma 4.2.14, $\mathcal{B} \cap I_\beta$ spans $I_\beta$, $\mathcal{B} \cap J_\gamma$ spans $J_\gamma$, and $\mathcal{B} \cap I_\beta \cap J_\gamma$ spans $I_\beta \cap J_\gamma$, for all $\beta, \gamma \in X_+$.

Recall from Remark 4.1.14 that $V^q(\lambda) = V_q(w_0(\lambda)) \cong U^+/I_{-w_0(\lambda)}$. Hence, for any $\gamma \in X_+$ there is an isomorphism (of vector spaces)

$$J_\gamma/(I_{-w_0(\lambda)} \cap J_\gamma) \cong J_\gamma V^q(\lambda)$$

and $(\mathcal{B} \cap J_\gamma) \setminus (\mathcal{B} \cap I_{-w_0(\lambda)} \cap J_\gamma)$ maps to a basis of $J_\gamma V^q(\lambda) = \sum_{i \in I} E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1} V^q(\lambda)$. We have just shown that $\mathcal{B}(\lambda) \cap J_\gamma$ maps to a weight basis $B_{\lambda,\gamma}$ of $\sum_{i \in I} E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1} V^q(\lambda)$, for any $\gamma \in X_+$.

If we consider the dual space $V^q(\lambda)^*$ as a $U$-module then the annihilator of $J_\gamma V^q(\lambda)$ in $V^q(\lambda)^*$ is seen to be the subspace

$$(J_\gamma V^q(\lambda))^\perp = \{\xi \in V^q(\lambda)^* \mid E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1}(\xi) = 0, \ i \in I\}.$$

Hence, $(J_\gamma V^q(\lambda))^\perp$ is spanned by part of the dual basis $B^*_{\lambda,\gamma}$. Hence, at the specialisation $q = 1$ we obtain a good basis for $V(\lambda)^* \cong V(-w_0(\lambda))$ and the result follows. □

Let $\mathbf{i} \in R(w_0)$, with corresponding root sequence $\beta_1, \ldots, \beta_{\ell(w_0)}$, and $b_{\mathbf{i}}$ the corresponding Lusztig parameterisation in direction $\mathbf{i}$. Let $b \in \mathcal{B}$ be such that $b = b_{\mathbf{i}}(t)$. Then, (recall (4.2.3))

$$\deg b = \deg b_{\mathbf{i}}(t) = \sum_{j=1}^{\ell(w_0)} t_j \beta_j.$$

As $V(\lambda; \nu - \mu, \mu) \subseteq V(\lambda)_{\nu-\mu}$, Proposition 4.2.19 implies that the tensor product multiplicity $c^{\nu}_{\lambda,\mu}$ is equal to the cardinality of a subset of lattice points in the following polytope sitting inside the *Kostant partition space* $\mathbb{R}^{R^+}$:

$$\{t \in \mathbb{R}^{R^+}_{\geq 0} \mid \lambda - \nu + \mu = \sum_{j=1}^{\ell(w_0)} t_j \beta_j\} \subseteq \mathbb{R}^{R^+}. \tag{4.2.7}$$

**Remark 4.2.21.** Berenstein-Zelevinsky show that the subset whose lattice points count $c^{\nu}_{\lambda,\mu}$ is a polytope embedded in the space (4.2.7) and explicitly determine a set of defining inequalities [14, Theorem 2.3]. A remarkable feature of their description is that the inequalities they obtain are determined by the representation theory of the Langlands dual group ${}^{L}G$.

## Canonical bases and string parameterisations

There is another, quite different, parameterisation of the canonical basis $\mathcal{B}$ called the *string parameterisation*. The string parameterisation was introduced by Kashiwara in his work on Littelmann's generalised Demazure character formulae [77]. Similar parameterisations were considered by Littelmann [95], and Berenstein-Zelevinsky [14]. In this section we introduce a string parameterisation that is different, but equivalent to, the string parameterisation defined in [14, Section 3]. We relate our string parameterisation to Berenstein-Zelevinsky's in Remark 4.2.25. We will describe Kashiwara's original parameterisation in Section 4.3.

Let $V$ be a $U^-$-module satisfying the following property: for any nonzero $v \in V$, $i \in I$, $F_i^r v = 0$, for sufficiently large $r$. We will call a $U^-$-module with this property *locally nilpotent*. For locally nilpotent $U^-$-module $V$, the function

$$\begin{array}{rccl} c_i: & V \setminus \{0\} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ & v & \longmapsto & \max\{r \in \mathbb{Z}_{\geq 0} \mid F_i^r v \neq 0\}. \end{array} \tag{4.2.8}$$

is well-defined.

**Definition 4.2.22.** Let $V$ be a locally nilpotent $U^-$-module, $v \in V$ nonzero. Given any sequence $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ define the *string of $v$ in the direction* $\mathbf{i}$ to be

$$c_{\mathbf{i}}(v) = (t_1, \ldots, t_r) \in \mathbb{Z}^r_{\geq 0},$$

where we define recursively

$$t_1 = c_{i_1}(v),\ t_2 = c_{i_2}\left(F_{i_1}^{t_1}(v)\right), \ldots,\ t_r = c_{i_r}\left(F_{i_{r-1}}^{t_{r-1}} \cdots F_{i_1}^{t_1}(v)\right).$$

The *string map of $V$ in the direction* $\mathbf{i}$ is the function

$$\begin{aligned} c_{\mathbf{i}}: \quad V\setminus\{0\} &\longrightarrow \mathbb{Z}^r_{\geq 0} \\ v &\longmapsto c_{\mathbf{i}}(v). \end{aligned}$$

We define string maps of $U^+$ in the direction $\mathbf{i} \in R(w_0)$ that give rise to a family of parameterisations of the canonical basis $\mathcal{B}$. These *string parameterisations* are different to the Lusztig parameterisations in Definition 4.2.12. First, we need to specify a locally nilpotent action of $U^-$ on $U^+$.

In [69, Chapter 6], there is described a graded perfect pairing

$$(\,,\,): \quad U^- \times U^+ \longrightarrow \mathbb{C}(q)$$

satisfying

$$(\omega(x), \omega(y)) = (y, x) = (\iota(y), \iota(x)), \quad y \in U^-, x \in U^+,$$

and such that, if $x \in U^+(\alpha)$, $y \in U^-(\beta)$, with $\alpha + \beta \neq 0$, then $(y, x) = 0$. Composing with $\omega$ we obtain a nondegenerate symmetric bilinear form on $U^+$

$$\begin{aligned} (\,,\,)': \quad U^+ \times U^+ &\longrightarrow \mathbb{C}(q) \\ (u, v) &\longmapsto (\omega(u), v). \end{aligned} \tag{4.2.9}$$

For $i \in I$, let $L_i$ be the linear operator on $U^+$ adjoint to left multiplication by $E_i$: $L_i$ is uniquely specified by the condition that

$$(L_i(y), x)' = (y, E_i x)', \quad y, x \in U^+. \tag{4.2.10}$$

Then, for all $\mu \in Q_+$,

$$L_i(U^+_\mu) \subseteq U^+_{\mu - \alpha_i}. \tag{4.2.11}$$

Since $U^-$ and $U^+$ are isomorphic algebras we obtain an action of $U^{-,op}$, the opposite algebra of $U^-$, on $U^+$, uniquely determined by

$$F_i \bullet y := L_i(y), \quad i \in I, y \in U^+.$$

Twisting by $\iota$ we obtain an action of $U^-$ on $U^+$

$$x \cdot y := \iota(x) \bullet y, \quad x \in U^-, y \in U^+.$$

Specifically, for $F \in U^-$, $u \in U^+$, $F \bullet u \in U^+$ is the unique element such that

$$(F \bullet u, v)' = (u, \omega(\iota(F))v)', \quad v \in U^+.$$

By (4.2.11), $U^+$ is a locally nilpotent $U^-$-module. Hence, for any sequence $\mathbf{i} \in I^r$ we can consider the string map $c_{\mathbf{i}}$ associated to $U^+$ with respect to this locally nilpotent $U^-$-structure.

**Theorem 4.2.23** (Littlemann [95], Berenstein-Zelevinsky [14, Proposition 3.5])**.** *Let $\mathbf{i} \in R(w_0)$. Then, the string map $c_{\mathbf{i}}$ associated to $U^+$ defines a bijection from $\mathcal{B}$ onto the set of all lattice points $C_{\mathbf{i}}(\mathbb{Z})$ of some rational polyhedral cone $C_{\mathbf{i}} \subseteq \mathbb{R}^{\ell(w_0)}$.*

**Definition 4.2.24.** Let $\mathbf{i} \in R(w_0)$. We define the *string parameterisation of $\mathcal{B}$ in the direction* $\mathbf{i}$ to be the function

$$\begin{aligned} c_{\mathbf{i}} : \ \mathcal{B} &\longrightarrow C_{\mathbf{i}}(\mathbb{Z}) \subseteq \mathbb{Z}_{\geq 0}^{\ell(w_0)} \\ b &\longmapsto c_{\mathbf{i}}(b). \end{aligned} \tag{4.2.12}$$

The cone $C_{\mathbf{i}} \subseteq \mathbb{R}_{\geq 0}^{\ell(w_0)}$ spanned by the image of $c_{\mathbf{i}}$ is called the *string cone in the direction* $\mathbf{i}$.

**Remark 4.2.25.** In [95] and [14], the authors define string maps for locally nilpotent $U^+$-modules and obtain string parameterisations for the *dual canonical basis* $\mathcal{B}^*$. We briefly describe the construction of string maps from [14] and explain why it is equivalent to the definition given above.

Certainly, for any locally nilpotent $U^+$-module $V$ and $\mathbf{i} \in I^r$, there is an analogous notion of string maps $c_{\mathbf{i}}$ (replace $F_i$ by $E_i$ in the construction). Let $U_*^+$ be the graded dual of $U^+$ (recall (4.1.9)),

$$U_*^+ := \bigoplus_{\alpha \geq 0} U_\alpha^*, \qquad \text{where } U_\alpha^* = \mathrm{Hom}_{\mathbb{C}(q)}(U_\alpha, \mathbb{C}(q)).$$

By Proposition 4.2.3 and Remark 4.2.4, for any $\alpha > 0$, we have $\dim U_\alpha = \mathcal{P}(\alpha) < \infty$, where $\mathcal{P}$ is Kostant's partition function ([68, Section 24]), so that $\dim U_\alpha^+ < \infty$, for all $\alpha$. Recall that the grading on $U^+$ is given by the $U^0$-action $u \mapsto K_h u K_{-h}$, $u \in U^+$, $h \in X^\vee$. Therefore, $U_*^+$ is a $Q_-$-graded vector space and $U_{*,\alpha}^+ = U_{-\alpha}^*$.

The *dual canonical basis* $\mathcal{B}^*$ is the basis of $U_*^+$ dual to $\mathcal{B} \subseteq U^+$: for $b \in \mathcal{B}$, we define $b^* \in \mathcal{B}^*$ by

$$b^*(b') = \delta_{b,b'}, \quad b' \in \mathcal{B}.$$

There is an action of $U^+$ on $U_*^+$: for $E \in U^+$, $f \in U_*^+$, we have $E \cdot f \in U_*^+$ determined by

$$(E \cdot f)(u) = f(\iota(E)u), \quad u \in U^+. \tag{4.2.13}$$

With this definition $U^+$ acts locally nilpotently on $U_*^+$. Hence, for any sequence $\mathbf{i} = (i_1, \dots, i_r) \in I^r$, we can consider the string map $c_{\mathbf{i}}$ associated to $U_*^+$. It is this string map that appears in [14]: the authors show that $c_{\mathbf{i}}$ is a bijection between $\mathcal{B}^*$ and $C_{\mathbf{i}}(\mathbb{Z})$.

Twisting the $U^{\geq 0}$-module $U_*^+$ by $\omega$ we obtain a $Q_+$-graded $U^{\leq 0}$-module. The form (4.2.9) identifies the $U^{\leq 0}$-modules $U^+ \cong {}^{\omega}U_*^+$. In particular, the string cones defined via either construction are equal.

In [14], Berenstein-Zelevinsky gave an explicit description of $C_{\mathbf{i}}$ by describing a set of defining inequalities. A remarkable feature of this description is that the defining inequalities are defined in terms of the representation theory of the Langlands dual ${}^L\mathfrak{g}$. We recall their result.

Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. An $\mathbf{i}$-trail from $\gamma$ to $\delta$ in $V(\omega_i^\vee)$, where $V(\omega_i^\vee)$ is the irreducible representation of ${}^LG$ having highest weight $\omega_i^\vee$, is a sequence of weights $\pi = (\gamma = \gamma_0, \gamma_1 \ldots, \gamma_m = \delta)$, $\gamma_i \in \mathrm{wt}(V(\omega_i^\vee)) \subseteq X^\vee$, such that

(i) for $k = 1, \ldots, m$ we have $\gamma_{k-1} - \gamma_k = c_k \alpha_{i_k}^\vee$, for some $c_k \in \mathbb{Z}_{\geq 0}$, and

(ii) $e_{i_1}^{c_1} \cdots e_{i_m}^{c_m}$ is a nonzero linear map from $V(\omega_i^\vee)_\delta$ to $V(\omega_i^\vee)_\gamma$.

Given an $\mathbf{i}$-trail $\pi = (\gamma_0, \ldots, \gamma_m)$ in $V(\omega_i^\vee)$, define (recall that $\gamma_i \in X^\vee$)

$$d_k^{(i)}(\pi) := \frac{1}{2} \langle \alpha_{i_k}, \gamma_{k-1} + \gamma_k \rangle, \quad k = 1, \ldots, m. \tag{4.2.14}$$

Condition (i) implies that $d_k(\pi) \in \mathbb{Z}$, for all $k$.

**Theorem 4.2.26** ([14, Theorem 3.10]). *Let $\mathbf{i} \in R(w_0)$, $m = \ell(w_0)$. Then, the string cone $C_{\mathbf{i}}$ is the cone in $\mathbb{R}^m$ consisting of all $(t_1, \ldots, t_m)$ such that $\sum_k d_k^{(i)}(\pi) t_k \geq 0$, for any $i \in I$ and any $\mathbf{i}$-trail from $\omega_i^\vee$ to $w_0 s_i \omega_i^\vee$ in $V(\omega_i^\vee)$.*

We describe the relationship between string cones and the representation theory of $U$. Let $\lambda \in X_-$ be an antidominant weight. There is an exact sequence of $U$-modules

$$0 \longrightarrow I_\lambda \longrightarrow \nabla(\lambda) \longrightarrow V_q(\lambda) \longrightarrow 0 \tag{4.2.15}$$

where $I_\lambda = \sum_{i \in I} U(E_i^{-\langle \lambda, \alpha_i^\vee \rangle + 1} + J_\lambda^+) \subseteq \Delta(\lambda)$ and $V_q(\lambda)$ is the finite-dimensional irreducible $U$-module having lowest weight $\lambda$ (see (4.1.11) and Definition 4.1.12).

In Theorem 4.2.15 we saw that the canonical basis $\mathcal{B} \subseteq U^+$ gives rise to a canonical basis of $V_q(\lambda) = V^q(w_0(\lambda))$. Recall the corresponding subset $\mathcal{B}(w_0(\lambda)) \subseteq \mathcal{B}$ (Definition 4.2.16). Therefore, we have the following result.

**Proposition 4.2.27.** *Let $\mathbf{i} \in R(w_0)$. Then, for any $\lambda \in X_+$, $\mathcal{B}(\lambda)$ is parameterised by a subset of $C_{\mathbf{i}}(\mathbb{Z})$ via the string map in the direction $\mathbf{i}$.*

**Definition 4.2.28.** Let $\mathbf{i} \in R(w_0)$. Define the *weighted string cone* $\underline{C}_{\mathbf{i}} \subseteq \mathbb{R}X \times \mathbb{R}^{\ell(w_0)}$ to be the $\mathbb{R}_{\geq 0}$-span of

$$\underline{C}_{\mathbf{i}}(\mathbb{Z}) := \{(\lambda, t) \in X \times \mathbb{Z}^{\ell(w_0)} \mid t = c_{\mathbf{i}}(b), \text{ for some } b \in \mathcal{B}(\lambda)\}$$

Observe that, for any $\lambda \in X_+$, $1 \in \mathcal{B}(\lambda) \subseteq U^+$: by construction, 1 corresponds to a lowest weight vector in $V^q(\lambda)$. Hence, for any $\lambda, \lambda' \in X_+$, $\mathcal{B}(\lambda) \cap \mathcal{B}(\lambda') \neq \varnothing$. The weighted string cone 'separates' the subsets $\mathcal{B}(\lambda)$: consider the projection onto the first factor

$$\begin{array}{rccc} p: & \underline{C}_{\mathbf{i}} & \longrightarrow & \mathbb{R}X \\ & (\lambda, t) & \longmapsto & \lambda \end{array} \tag{4.2.16}$$

Then, for any $\lambda \in X_+$, we can identify $p^{-1}(\lambda) \cap \underline{C}_{\mathbf{i}}(\mathbb{Z}) = \mathcal{B}(\lambda)$, and we obtain an identification

$$\underline{C}_{\mathbf{i}}(\mathbb{Z}) = \bigsqcup_{\lambda \in X_+} \mathcal{B}(\lambda). \tag{4.2.17}$$

We make the following definitions.

**Definition 4.2.29.** Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. Define the *highest weight map*

$$\begin{array}{rccc} \mathrm{hw}: & \underline{C}_{\mathbf{i}} & \longrightarrow & \mathbb{R}X \\ & (\lambda, t) & \longmapsto & \lambda \end{array} \tag{4.2.18}$$

and the *weight map*

$$\begin{array}{rccc} \mathrm{wt}: & \underline{C}_{\mathbf{i}} & \longrightarrow & \mathbb{R}X \\ & (\lambda, t) & \longmapsto & w_0(\lambda) + \sum_{j=1}^m a_j \alpha_{i_j} \end{array} \tag{4.2.19}$$

**Remark 4.2.30.** The weight map (4.2.19) also appears in [1], albeit in slightly different form. Observe that, for $\mu \in X$, $\lambda \in X_+$, the intersection of fibres $\mathrm{wt}^{-1}(\mu)$ can be identified with the $\mu$-weight space in $V(\lambda)$.

A description of the inequalities defining $\underline{C}_{\mathbf{i}} \subseteq \mathbb{R}X \times \mathbb{R}^{\ell(w_0)}$ first appeared in [95] for particular $\mathbf{i}$. An implicit description can be found in later work of Berenstein-Zelevinsky [14] for arbitrary $\mathbf{i} \in R(w_0)$, and was explicitly given in [1, Theorem 1.1]).

**Theorem 4.2.31.** *Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. The weighted string cone $\underline{C}_{\mathbf{i}}$ is the intersection of $\mathbb{R}X \times C_{\mathbf{i}}$ with the $\ell(w_0)$ half spaces defined by the inequalities*

$$t_k + \sum_{l=k+1}^{m} \langle \alpha_{i_l}, \alpha_{i_k}^\vee \rangle t_l \leq \langle \lambda^*, \alpha_{i_k}^\vee \rangle, \quad k = 1, \ldots, m. \tag{4.2.20}$$

*Here $\lambda^* = -w_0(\lambda)$. The inequalities in (4.2.20) are called* $\lambda$-inequalities.

We provide some the motivation for the above definitions. The definition of the weight map and the appearance of the $\lambda$-inequalities can be understood in terms of the combinatorics of the representation theory of $U$. By [99, Theorem 14.3.2(c)], we have

$$L_i^{(c_i(b))}(b) \in \mathcal{B}, \quad \text{for any } b \in \mathcal{B}.$$

Here $L_i^{(r)} := \frac{L_i^r}{[r]_q!}$ is the associated $q$-divided power.

Hence, if $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ and $c_{\mathbf{i}}(b) = (a_1, \ldots, a_m)$, $b \in \mathcal{B}$, then

$$b_j := L_{i_j}^{(a_j)} \cdots L_{i_1}^{(a_1)}(b) \in \mathcal{B}, \quad j = 1, \ldots, m.$$

Littelmann [95, Section 1] has shown that, for any $b \in \mathcal{B}$,

$$L_{i_m}^{(a_m)} \cdots L_{i_1}^{(a_1)}(b) = 1 \in \mathcal{B}.$$

Let $\lambda \in X_+$ and choose $v_\lambda \in V^q(\lambda)$ a lowest weight vector. Then, $\{bv_\lambda \mid b \in \mathcal{B}(\lambda)\}$ is a basis of $V^q(\lambda)$. In particular, if $b \in \mathcal{B}(\lambda)$ with $c_{\mathbf{i}}(b) = (a_1, \dots, a_m)$, then $bv_\lambda$ is a weight vector having weight

$$w_0(\lambda) + \sum_{j=1}^{m} a_j \alpha_{i_j} = \mathrm{wt}(\lambda, a_1, \dots, a_m).$$

This is where the definition for the weight map comes from.

Identify $b_j \in \mathcal{B}$ with the basis element $b_j v_\lambda$ so that $b_j$ has weight $\mathrm{wt}(\lambda, a)$. Recall from (4.2.8) the definition of $c_i$, $i \in I$, the string in the direction $i$. Consider the $i_m$-string in $V^q(\lambda)$ through 1. Then, $a_m = c_{i_m}(b_{m-1})$ implies

$$a_m \leq -\langle w_0(\lambda), \alpha_{i_m}^\vee \rangle.$$

Next, consider the $i_{m-1}$-string in $V^q(\lambda)$ through $b_{j-1}$. Since $a_{m-1} = c_{i_{m-1}}(b_{j-2})$ we must have

$$a_{m-1} \leq -\langle w_0(\lambda) + a_m \alpha_{i_m}, \alpha_{i_{m-1}}^\vee \rangle \quad \Longrightarrow \quad a_{m-1} + a_m \langle \alpha_{i_m}, \alpha_{m-1}^\vee \rangle \leq \langle -w_0(\lambda), \alpha_{i_{m-1}} \rangle.$$

Continuing in this fashion we recover the $\lambda$-inequalities (4.2.20): consider the $i_k$-string in $V^q(\lambda)$ through $b_k$. Then, $a_k = c_{i_k}(b_{k-1})$, implies

$$a_k \leq -\langle w_0(\lambda) + \sum_{j=k+1}^{m} a_j \alpha_{i_j}, \alpha_{i_k} \rangle \quad \Longrightarrow \quad a_k + \sum_{j=k+1}^{m} a_j \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle \leq \langle -w_0(\lambda), \alpha_{i_k}^\vee \rangle.$$

The content of the Littlemann, Berenstein-Zelevinsky results cited aboved is that these necessary conditions are sufficient.

**Remark 4.2.32.** As Lusztig's canonical basis $\mathcal{B}$ is determined by $U_q(\mathfrak{g})$, it does not depend on the isogeny class of a semisimple complex algebraic group. Therefore, the string cone is independent of the isogeny class of a semisimple complex algebraic group. The explicit description of the string cone given by Berenstein-Zelevinsky (see Theorem 4.2.26) requires $G$ to be simply-connected in order to define the appropriate $\mathbf{i}$-trails. However, this is not a problem when we are considering parameterisations of bases of irreducible representations as parameterisations of bases for representations extend across isogeny classes. If $G$ is reductive then $G$ is an extension of a semisimple algebraic group $G^{ss}$ by a central torus and a similar argument allows us to consider parameterisations of bases of irreducible representations of $G$ via $G^{ss}$. Of course, if we want to keep track of weights then we must remember how the central torus acts.

## 4.3 Combinatorial and geometric crystals

*'In some sense the $q \to 0$ limit strips a module of its linear structure, so we are reduced to combinatorics.'* A. Joseph, [71, p.26]

Let $U = U_q(\mathfrak{g})$ be the quantised universal enveloping algebra associated to the Lie algebra $\mathfrak{g}$ of a reductive complex algebraic group. In this section we will describe discrete combinatorial models of the representation theory of $U$ discovered by Kashiwara, called *crystals*. The theory of crystals developed from Kashiwara's investigations into bases of $U^-$ at the specialisation $q = 0$. A purely combinatorial construction of crystals, not relying on $U^-$, and using an arbitrary (not necesarily symmetrisable) Cartan datum $(\Pi, S, \Pi^\vee, S^\vee)$, was given by Littelmann soon thereafter in [96] using his path model.

Since this early development, crystal structures have been discovered throughout mathematics: using the geometry of the affine Grassmannian of ${}^L\mathfrak{g}$ [19], [72]; using the symplectic geometry of quiver varieties [79]; in the study of the generalisation of the Casselmann-Shalika formula to the metaplectic group and associated Eisenstein series[22].

For a reductive complex algebraic group $G$, Berenstein-Kazhdan described a general geometric framework to obtain crystal structures [12], [13]. Using only the geometry and representation theory of $G$ they recovered the crystal structures obtained by Kashiwara and Littelmann. The tool that they used to construct Kashiwara crystals was the tropicalisation functor Trop. We provide a construction of Trop on the category of algebraic tori and extend its domain to the category of positive varieties.

### Kashiwara crystals

In this section we briefly recall Kashiwara's notion of an *(abstract) crystal*. Crystals provide combinatorial models of the crystal bases of (specialisations of) integrable $U$-modules, and therefore for the representation theory of $G$. We will also indicate Kashiwara's original introduction of the string parameterisation. For further details on crystal bases see [76]; for further details on the category of abstract crystals see [77], [75].

A Kashiwara crystal encodes the combinatorial data of the crystal base of an integrable $U$-module $V$. As a first approximation, and sufficient for our considerations, a crystal base is a basis $B$ of $V$ *at* $q = 0$ satisfying the following property: for any $i \in I$, the $\mathbb{C}(q^{d_i})$-subalgebra $U_i$ of $U$ generated by $E_i, F_i, K_{\pm d_i \alpha_i^\vee}$ is isomorphic to $U_q(\mathfrak{sl}_2)$ and $V$ decomposes as a $U_i$-module

$$V \cong \bigoplus_j V(l_j^{(i)}).$$

Here $V(l_j^{(i)})$ is the $(l_j^{(i)} + 1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module. Then, $B$ is a basis of the specialisation at $q = 0$ of $V$ such that, for any $i \in I$, $B$ induces an isomorphism $V_0 \cong \bigoplus_j V(l_j^{(i)})_0$. Here $W_0$ is the specialisation at $q = 0$ of a $U_i$-module $W$.

In [76] Kashiwara showed the existence of a crystal basis for any integrable $U$-module $V$ (more generally, he proved the existence of a crystal basis for the negative part $U^-$

of the quantised universal enveloping algebra associated to any symmetrisable Kac-Moody algebra). Moreover, Kashiwara shows that a crystal basis could be 'melted' (i.e. lifted from the specialisation at $q = 0$) to provide a basis of the $U$-module $V$, called a *global basis*, and that the global basis for the irreducible $U$-modules $V^q(\lambda)$, $\lambda \in X_+$, can be obtained from a *global basis* of $U^-$. Grojnowski-Lusztig later showed in [56] that Kashiwara's global basis was equal to Lusztig's canonical basis in $U^-$ (in the symmetrisable Kac-Moody setting): the composition of $\omega$ with the bar involution $^-$ takes $\mathcal{B}$ to Kashiwara's global basis.

We now define the abstract notion of a crystal and restrict ourselves to those crystals associated to the Lie algebra $\mathfrak{g}$. Essentially all of the definitions and constructions extend to symmetrisable Kac-Moody type. For the further details on crystals coming from symmetrisable Kac-Moody algebras see [66]; for details on the theory of abstract crystals associated to arbitrary Cartan datum (without requiring recourse to quantised universal enveloping algebras) see [96], [71], [23].

Extend the standard order on $\mathbb{Z}$ to a linear ordering on $\mathbb{Z}_{-\infty} \coloneqq \mathbb{Z} \cup \{-\infty\}$ so that $-\infty$ is the smallest element. Define

$$-\infty + x = -\infty, \quad \text{for any } x \in \mathbb{Z}_{-\infty}.$$

Let $G$ be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$.

**Definition 4.3.1.** An *abstract (Kashiwara) crystal of type* $(R, X)$ is a (nonempty) set $B$ together with maps

$$\mathrm{wt} : B \to X, \quad \varepsilon_i, \varphi_i : B \to \mathbb{Z}_{-\infty}, \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \quad (i \in I). \tag{4.3.1}$$

Here 0 is a ghost element not contained in $B$. We call the maps $\mathrm{wt}, \varepsilon_i, \varphi_i$, $i \in I$, the structure maps. The collection $(B, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)_{i \in I}$ is subject to the following axioms:

(C1) $\varphi_i(b) - \epsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle$, for each $i \in I$;

(C2) if $b \in B$ satisfies $\tilde{e}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i$, $\epsilon(\tilde{e}_i(b)) = \epsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$;

(C2)' if $b \in B$ satisfies $\tilde{f}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i$, $\epsilon(\tilde{f}_i(b)) = \epsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$;

(C3) for $b, b' \in B$, $b' = \tilde{f}_i(b)$ if and only if $\tilde{e}_i(b') = b$;

(C4) if $\varphi_i(b) = -\infty$ then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

For $\mu \in X$, define $B_\mu \coloneqq \{b \in B \mid \mathrm{wt}(b) = \mu\}$. The functions $\tilde{e}_i, \tilde{f}_i$, $i \in I$, are called *crystal operators*. Given a crystal $B$ one may associate an $I$-coloured directed graph called the *crystal graph*: the set of vertices is $B$ and there exists a directed arrow $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i(b) = b'$. We say that $B$ is *connected* if its crystal graph is connected. A union of connected components of the crystal graph determines a subcrystal $B' \subseteq B$. $B$ is a *highest weight crystal* if there exists unique $b \in B$ such that $\tilde{e}_i b = 0$, for all $i \in I$; $B$ is a *lowest weight crystal* if there exists unique $b \in B$ such that $\tilde{f}_i b = 0$, for all $i \in I$.

Let $B_1$ and $B_2$ be crystals of the same type. A morphism of crystals is a function $\psi : B \sqcup \{0\} \to B' \sqcup \{0\}$ such that

(CM1) $\psi(0) = 0$;

(CM2) if $\psi(b) \neq 0$ then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\epsilon_i(\psi(b)) = \epsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$, for all $i \in I$;

(CM3) for $b \in B$ such that $\psi(b) \neq 0$ and $\psi(\tilde{e}_i(b)) \neq 0$, we have $\psi(\tilde{e}_i(b)) = \tilde{e}_i(\psi(b))$;

(CM3)' for $b \in B$ such that $\psi(b) \neq 0$ and $\psi(\tilde{f}_i(b)) \neq 0$, we have $\psi(\tilde{f}_i(b)) = \tilde{f}_i(\psi(b))$.

Crystals together with crystal morphisms define a category $\mathcal{K}$. If $B_1$ and $B_2$ are crystals of the same type then their disjoint union $B_1 \sqcup B_2$ is a crystal in the obvious way: this provides $\mathcal{K}$ with a coproduct. Moreover, $\mathcal{K}$ can be equipped with a tensor structure, defined as follows: if $B_1, B_2 \in \mathcal{K}$ are crystals of the same type then define the crystal $B_1 \otimes B_2$, where

(i) $B_1 \otimes B_2 = B_1 \times B_2$ as a set. We write $b_1 \otimes b_2$ for the pair $(b_1, b_2)$.

(ii) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$.

(iii)
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1), \\ b_1 \otimes \tilde{f}_i(b_2), & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1). \end{cases}$$

(iv)
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \\ b_1 \otimes \tilde{e}_i(b_2), & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1). \end{cases}$$

We understand $b \otimes 0 = 0 \otimes b = 0$.

(v) $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1), \varphi_i(b_2) + \langle \mathrm{wt}(b_1), \alpha_i^\vee \rangle)$, and

(vi) $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_2), \varepsilon_i(b_1) \langle \mathrm{wt}(b_2), \alpha_i^\vee \rangle)$.

**Remark 4.3.2.** If $B_1, \ldots, , B_k$ are crystals such that

$$\varepsilon_i(b) = \max\{r \mid \tilde{e}_i^r(b) \neq 0\}, \quad \varphi_i(b) = \max\{r \mid \tilde{f}_i(b) \neq 0\}, \quad b \in B_1 \cup \ldots \cup B_k, i \in I, \tag{4.3.2}$$

then the action of the crystal operators $\tilde{e}_i, \tilde{f}_i$ on a tensor $b_1 \otimes \cdots \otimes b_k$ can be computed using the *signature rule*: decorate each tensorand $b_j$ with $\varphi_i(b_j)$ '$-$' signs followed by $\varepsilon_i(b_j)$ '$+$' signs. This gives rise to a sequence in the alphabet $\{-, +\}$. Successively cancel all adjacent pairs $+-$ to obtain a sequence having $a$ '$-$' signs followed by $b$ '$+$' signs. Then,

$$\varphi_i(b_1 \otimes \cdots \otimes b_k) = a, \quad \text{and} \quad \varepsilon_i(b_1 \otimes \cdots \otimes b_k) = b,$$

and $\tilde{f}_i$ acts on the tensor factor associated to the rightmost remaining $-$, and $\tilde{e}_i$ acts on the tensor fact associated to the leftmost $+$.

**Example 4.3.3.** (1) Let $G$ be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$, $\mathfrak{g}$ its Lie algebra. Let $U = U_q(\mathfrak{g})$ be the associated quantised universal enveloping algebra. Let $\lambda \in X_-$ be antidominant and $V_q(\lambda) = U^+/I_{-\lambda}$, where $I_{-\lambda} = \sum_i U^+ E_i^{\langle -\langle \lambda, \alpha_i^\vee \rangle + 1}$. Let $\lambda' = w_0(\lambda)$. Hence, $V_q(\lambda)$ is the irreducible finite dimensional $U$-module having lowest weight $\lambda$ and highest weight $\lambda'$. Let $\mathcal{B}(\lambda') \subseteq \mathcal{B}$ be the subset from Definition 4.2.16. Then, $\mathcal{B}(\lambda')$ maps onto a basis of $V_q(\lambda)$. Let $\mathcal{V}(\lambda')$ be the $\mathbb{Z}[q^{-1}]$-span of this base. By Theorem 4.2.10, we obtain a homogenous basis $B(\lambda')$ of the $\mathbb{Z}$-module $\mathcal{V}(\lambda')/q^{-1}\mathcal{V}(\lambda')$. The pair $(B(\lambda'), \mathcal{V}(\lambda'))$ is an example of a crystal base. Recall the $\mathbb{Z}[q^{-1}]$-submodule $\mathcal{L}$ from Theorem 4.2.10. If $b \in B(\lambda')$, let $\mathbf{b} \in \mathcal{B}(\lambda')$ be the unique element in $\mathcal{B}$ that maps to $b$. Let $\pi_\lambda : \mathcal{L} \to \mathcal{V}(\lambda')/q^{-1}\mathcal{V}(\lambda')$ be the composition of the above projections. We define a crystal structure of type $(R, X)$ on $B(\lambda')$:

(i) if $b \in B(\lambda')_\mu$ then $\mathrm{wt}(b) = \mu$;

(ii) if $b \in B(\lambda')$ then $\tilde{e}_i b := \pi_\lambda(E_i \mathbf{b})$, $\tilde{f}_i b := \pi_\lambda(L_i(\mathbf{b}))$, $i \in I$;

(iii) $\varepsilon_i(b) = \max\{r \mid \tilde{e}_i^r(b) \neq 0\}$, $\varphi_i(b) = \max\{r \mid \tilde{f}_i^r(b) \neq 0\}$.

For details regarding the well-definedness of these definitions see [99, Part III].

(2) More generally, given any integrable $U$-module $V$, one can define a crystal $B(V)$ in an analogous manner.

(3) Let $B(-\infty) \subseteq \mathcal{L}/q^{-1}\mathcal{L}$ be the image of the canonical basis $\mathcal{B}$ (recall the lattice $\mathcal{L}$ fom Theorem 4.2.10). Define the weight map wt, the crystal operators and $\varphi_i$, $i \in I$, as in the previous example. Define $\varepsilon_i(b) = \varphi_i(b) + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$. Ths endows $B(-\infty)$ with a crystal structure.

(4) If $B$ is a crystal then we define $B^\vee$ as follows: as a set $B^\vee = \{b^\vee \mid b \in B\}$ and

$$\mathrm{wt}(b^\vee) = -\mathrm{wt}(b),\ \varepsilon_i(b^\vee) = \varphi_i(b),\ \varphi_i(b^\vee) = \varepsilon_i(b),\ \tilde{e}_i(b^\vee) = (\tilde{f}_i(b))^\vee,\ \tilde{f}_i(b^\vee) = (\tilde{e}_i(b))^\vee.$$

Here $0^\vee = 0$. $B^\vee$ is called the dual of $B$ and $B^{\vee\vee} = B$.

(5) $B(\infty) := B(-\infty)^\vee$. This is the crystal associated to the crystal basis of $U^-$ constructed by Kashiwara [76].

(6) Let $\lambda \in X$. Then, there is a crystal $T_\lambda = \{t_\lambda\}$ with $\mathrm{wt}(t_\lambda) = \lambda$ and $\varphi_i(t_\lambda) = -\infty$, for all $i \in I$.

(7) Let $\lambda \in X_-$ be antidominant. There is an isomorphism of crystals $B(\lambda)^\vee \cong B(-w_0(\lambda))$.

**Remark 4.3.4.** Let $B_1, B_2$ be crystals associated to crystal bases of integrable $U$-modules $V_1, V_2$. The crystal $B_1 \cup B_2$ is the crystal associated to the integrable $U$-module $V_1 \oplus V_2$, and the crystal $B_1 \otimes B_2$ is the crystal associated to the integrable $U$-module $V_1 \otimes V_2$. It is shown in [75] that there are isomorphisms $B \otimes T_0 \cong B$, $T_0 \otimes B \cong B$, where $T_0$ is the crystal from Example 4.3.3, and that $\mathcal{K}$ is a monoidal category.

The following result indicates the relationship between crystals and representation theory.

**Proposition 4.3.5.** *Let $B = B(V)$ be a finite crystal of type $(R, X)$ associated to an integrable $U_q(\mathfrak{g})$-module $V$. Then, if $V \cong \bigoplus_{\lambda \in X_+} V^q(\lambda)^{c_\lambda}$ then $B$ decomposes as a disjoint union*

$$B = \sqcup_{\lambda \in X_+} B(\lambda)^{c_\lambda}.$$

Therefore, crystals provide a tool to approach tensor product decomposition computations: if $\lambda_1, \ldots, \lambda_k \in X_-$ are antidominant then the connected components of the crystal $B(\lambda_1) \otimes \cdots \otimes B(\lambda_k)$ correspond precisely to the irreducible summands of $V^q(\lambda_1) \otimes \cdots \otimes V^q(\lambda_k)$. Upon specialisation at $q = 1$ this provides an effective computational model for computing tensor product multiplicities for $\mathfrak{g}$.

An important problem is to determine combinatorially accessible examples of crystals. We provide a construction for all crystals $B(\lambda)$, $\lambda \in X_+$, in type $A$

Let $\mathfrak{g} = \mathfrak{gl}_n$ be the Lie algebra of the general linear group $\mathrm{GL}_n(\mathbb{C})$. Let $I = \{1, \ldots, n-1\}$, $X = \mathbb{Z}^n$ with standard basis $\epsilon_1, \ldots, \epsilon_n$, $S = \{\alpha_i\}_{i=1}^{n-1}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, \ldots, n-1$. Let $\varpi_i = \sum_{j=1}^{i} \epsilon_j$, for $i = 1, \ldots, n$. Recall that, for $\mathfrak{gl}_n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in X_+$ if and only if $\lambda_1 \geq \cdots \geq \lambda_n$.

Define the crystal $\mathbb{B} = \{\mathbf{1}, \ldots, \mathbf{n}\}$ as follows: its crystal graph is

$$\mathbf{1} \xrightarrow{1} \mathbf{2} \xrightarrow{2} \mathbf{3} \xrightarrow{3} \cdots \xrightarrow{n-2} \mathbf{n-1} \xrightarrow{n-1} \mathbf{n}$$

This specifies how the crystal operators act. We set

$$\varphi_i(\mathbf{j}) = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i. \end{cases} \qquad \text{and} \quad \varepsilon_i(\mathbf{j}) = \begin{cases} 0, & \text{if } j \neq i+1, \\ 1, & \text{if } j = i+1, \end{cases}$$

and $\mathrm{wt}(\mathbf{1}) = \varpi_1$. Observe that $\mathbb{B}$ satisfies the condition in (4.3.2). Axiom (C4)' and the crystal graph imply that $\mathrm{wt}(\mathbf{k}) = \varpi_1 - \sum_{j=1}^{k-1} \alpha_j$. In particular, $\mathrm{wt}(\mathbf{n}) = -\varpi_{n-1}$. $\mathbb{B} = B(\varpi_1)$ is the crystal graph associated to the irreducible $U_q(\mathfrak{sl}_n)$-module $V^q(\varpi_1)$. In the specialisation $q = 1$ this is the defining representation $\mathbb{C}^n$ of $\mathfrak{gl}_n$.

Fix $n = 3$. Using the signature rule (Remark 4.3.2), we can determine the crystal graph of $\mathbb{B} \otimes \mathbb{B}$:

$$\begin{tikzcd}
\mathbf{1}\otimes\mathbf{1} \arrow[r, "1"] & \mathbf{1}\otimes\mathbf{2} \arrow[r, "2"] \arrow[d, "1"'] & \mathbf{1}\otimes\mathbf{3} \arrow[d, "1"] \\
\mathbf{2}\otimes\mathbf{1} \arrow[d, "2"'] & \mathbf{2}\otimes\mathbf{2} \arrow[r, "2"'] & \mathbf{2}\otimes\mathbf{3} \arrow[d, "2"] \\
\mathbf{3}\otimes\mathbf{1} \arrow[r, "1"'] & \mathbf{3}\otimes\mathbf{2} & \mathbf{3}\otimes\mathbf{3}
\end{tikzcd}$$

The connected component

$$\mathbf{2}\otimes\mathbf{1} \xrightarrow{2} \mathbf{3}\otimes\mathbf{1} \xrightarrow{1} \mathbf{3}\otimes\mathbf{2}$$

is isomorphic to the dual crystal $\mathbb{B}^\vee = B(\varpi_2)$ and the large connected component

$$\begin{tikzcd}
\mathbf{1}\otimes\mathbf{1} \arrow[r, "1"] & \mathbf{1}\otimes\mathbf{2} \arrow[r, "2"] \arrow[d, "1"'] & \mathbf{1}\otimes\mathbf{3} \arrow[d, "1"] \\
& \mathbf{2}\otimes\mathbf{2} \arrow[r, "2"'] & \mathbf{2}\otimes\mathbf{3} \arrow[d, "2"] \\
& & \mathbf{3}\otimes\mathbf{3}
\end{tikzcd}$$

is isomorphic to the crystal $B(2\varpi_1)$. We find $\mathbb{B}\otimes\mathbb{B} = B(\varpi_2)\sqcup B(2\varpi_1)$, corresponding to the decomposition of $\mathfrak{gl}_3$-modules $V\otimes V \cong V(\varpi_2)\oplus V(2\varpi_2)$. In particular, using the crystal $\mathbb{B} = B(\varpi_1)$ we've obtained $B(2\varpi_1)$. More generally, we can obtain the crystal $B(k\varpi_1)$ as a subcrystal of $\mathbb{B}^{\otimes k}$. This is a special instance of the following result (see [66, Chapter 7]).

**Proposition 4.3.6.** *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in X_+$. Assume $\lambda_n \geq 0$. Then, $B(\lambda)$ is isomorphic to the connected component of $\mathbb{B}^{\otimes|\lambda|}$ containing the highest weight element*

$$\underbrace{\mathbf{n}\otimes\cdots\otimes\mathbf{n}}_{\lambda_n}\otimes\cdots\otimes\underbrace{\mathbf{2}\otimes\cdots\otimes\mathbf{2}}_{\lambda_2}\underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{\lambda_1}$$

When $(R, X)$ is of classical type, Kashiwara-Nakashima [78] obtained models of the crystals $B(\lambda)$, $\lambda \in X_+$, using Young tableaux. See [66] for further details and examples.

**Example 4.3.7.** The crystal graph for the crystal $B(\varpi_1 + \varpi_2)$ is given below.

$$\begin{tikzcd}
 & & \mathbf{2}\otimes\mathbf{1}\otimes\mathbf{1} \arrow[dll, "1"'] \arrow[drr, "2"] & & \\
\mathbf{2}\otimes\mathbf{1}\otimes\mathbf{2} \arrow[drrr, "2"] & & & & \mathbf{3}\otimes\mathbf{1}\otimes\mathbf{1} \arrow[dlll, "1"'] \\
 & \mathbf{3}\otimes\mathbf{1}\otimes\mathbf{2} \arrow[dl, "1"'] & & \mathbf{2}\otimes\mathbf{1}\otimes\mathbf{3} \arrow[dr, "2"] & \\
\mathbf{3}\otimes\mathbf{2}\otimes\mathbf{2} \arrow[drr, "2"'] & & & & \mathbf{3}\otimes\mathbf{1}\otimes\mathbf{3} \arrow[dll, "1"] \\
 & & \mathbf{3}\otimes\mathbf{2}\otimes\mathbf{3} & &
\end{tikzcd}$$

## Tropicalisation and positivity

In this section we will describe a 'geometrisation' of crystals, following Berenstein-Kazhdan [12], [13]. We will describe a birational model of Kashiwara crystals called a *geometric crystal*. Geometric crystals are varieties birational to algebraic tori, together with a collection of birational maps that 'model' the combinatorial data of a Kashiwara crystal. Through the process of *tropicalisation* (or *ultra-discretization* [111]), the geometry of the geometric crystal is stripped away revealing the data of a Kashiwara crystal (Definition 4.3.1).

Let $G$ be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$ and simple roots $S$ and simple coroots $S^\vee$. Let $T \subseteq G$ be a maximal torus.

**Definition 4.3.8.** A decorated geometric crystal is the data $(X, \gamma, \varphi_i, \varepsilon_i, e_i^{\cdot}, f \mid i \in I)$, where $X$ is an irreducible variety, $\gamma$ is a rational morphism $X \to T$, called the *weight map*, $\varphi_i, \varepsilon_i : X \to \mathbb{A}^1$ are rational functions, and each $e_i^{\cdot} : \mathbb{G}_m \times X \to X$ is a unital rational action (denoted $(c, x) \mapsto e_i^c(x)$) such that, for each $i \in I$, one has either

(i) $\varphi_i = \varepsilon_i = 0$ and the action is trivial, or,

(ii) $\varphi_i \neq 0$ and $\varepsilon_i \neq 0$, and

$$\gamma(e_i^c(x)) = \alpha_i^\vee(c)\gamma(x), \quad \varepsilon_i(x) = \alpha_i(\gamma(x))\varphi_i(x),$$

$$\varepsilon_i(e_i^c(x)) = c\varepsilon_i(x), \quad \varphi_i(e_i^c(x)) = c^{-1}\varphi_i(x).$$

Moreover, $f : X \to \mathbb{A}^1$ is a rational function, called the *decoration*, on $X$ such that

$$f(e_i^c(x)) = f(x) + \frac{c-1}{\varphi_i(x)} + \frac{c^{-1}-1}{\varepsilon_i(x)}.$$

**Remark 4.3.9.** Observe the analogy between the conditions defining a geometric crystal and a Kashiwara crystal (Definition 4.3.1). For our purposes we will only be interested in the data $(X, \gamma, f)$. For general results and examples see [12], [13].

Our interest in geometric crystals is the process by which we can recover Kashiwara crystals. This is the process of tropicalisation, which we now describe. Basically, we want to obtain a discretisation of the data $(X, \gamma, f)$ coming from a geometric crystal.

First, given a real vector space $E$, we construct the *semi-field of polytopes in* $E$. This construction will be the foundation of our notion of *tropicalisation*. Further details and proofs can be found in [108], [13, Section 4].

Let $E$ be a finite-dimensional real vector space, $E^*$ the dual vector space. Define $\mathcal{P}_E$ to be the set of all convex polytopes in $E$, and define the Minkowski sum

$$P + Q \coloneqq \{p + q \mid p \in P, q \in Q\}, \quad P, Q \in \mathcal{P}_E.$$

Then, $(\mathcal{P}_E, +)$ is a monoid with unit $\{0\}$. For $P \in \mathcal{P}_E$, define the *support function of* $P$, to be

$$\begin{aligned} \chi_P : \quad E^* &\longrightarrow \mathbb{R} \\ \xi &\longmapsto \min\{\xi(p) \mid p \in P\}. \end{aligned}$$

If $\mathrm{vert}(P)$ is the set of vertices of $P$ then we have $\chi_P(\xi) = \min\{\xi(p) \mid p \in \mathrm{vert}(P)\}$. We have the following elementary result.

**Lemma 4.3.10.** *The assignment*

$$\begin{aligned} \chi : \quad \mathcal{P}_E &\longrightarrow \mathrm{Fun}(E^*, \mathbb{R}) \\ P &\longmapsto \chi_P \end{aligned}$$

*is an injective homomorphism of monoids. Here* $\mathrm{Fun}(E^*, \mathbb{R}) = \{f : E^* \to \mathbb{R}\}$ *is the set of* $\mathbb{R}$*-valued functions on* $E^*$*, considered as a monoid under pointwise addition.*

**Corollary 4.3.11.** *Let* $P, Q, R \in \mathcal{P}_E$*. Then,* $\mathcal{P}_E$ *admits the canellation property:* $P + R = Q + R$ *if and only if* $P = Q$*.*

For $P, Q \in \mathcal{P}_E$ we define the *join of* $P$ *and* $Q$ to be

$$P \vee W \coloneqq \mathrm{conv}(P \cup Q),$$

the convex hull of $P \subseteq Q$. Hence, $\chi_{P \vee Q} = \min(\chi_P, \chi_Q)$ and, since

$$\min(\chi_P, \chi_Q) + \chi_R = \min(\chi_P + \chi_R, \chi_Q + \chi_R),$$

we obtain the following identity in $\mathcal{P}_E$,

$$(P \vee Q) + R = (P + R) \vee (Q + R).$$

This shows that $(\mathcal{P}_E, +, \vee)$ is a semi-ring with addition $\vee$, and multiplication $+$.

Define $\mathcal{P}_E^+$ to be the Grothendieck group of the monoid $(\mathcal{P}_E, +)$, with generators $[P]$, $P \in \mathcal{P}_E$, subject to the relation $[P+Q] = [P] + [Q]$. Then, the join operation can be uniquely extended to $\mathcal{P}_E^+$ using the 'quotient' rule

$$([P] - [Q]) \vee ([P'] - [Q']) \coloneqq [(P + Q') \vee (P' + Q)] - [Q + Q'].$$

Hence, $\mathcal{P}_E^+$ is a semi-field. Moreover, the homomorphism $\chi : (\mathcal{P}_E, +) \to (\mathrm{Fun}(E^*, \mathbb{R}), +)$ extends uniquely to an injective homomorphism of semi-fields

$$\begin{array}{rccc} \tilde{\chi} : & \mathcal{P}_E^+ & \longrightarrow & \mathrm{Fun}(E^*, \mathbb{R}) \\ & P & \longmapsto & \chi_P \end{array}$$

Here $\mathrm{Fun}(E^*, \mathbb{R})$ is a semi-field with the operation of 'addition' $(f, g) \mapsto \min(f, g)$ and 'multiplication' $(f, g) \mapsto f + g$. By abuse of notation we will simply write $\chi$ instead of $\tilde{\chi}$.

We apply the above polytope algebra construction in the category of rational tori. Let $S$ be an algebraic torus split over $\mathbb{Q}$, and define

$$X(S) \coloneqq \mathrm{Hom}(S, \mathbb{G}_m), \quad X^\vee(S) \coloneqq \mathrm{Hom}(\mathbb{G}_m, S),$$

with canonical pairing

$$\langle , \rangle : \; X(S) \times X^\vee(S) \longrightarrow \mathbb{Z}$$

Denote the group algebra of $X(S)$ over $\mathbb{Q}$ by $\mathbb{Q}[X(S)]$. The elements in $\mathbb{Q}[X(S)]$ can be canonically identified with the algebra of regular functions on $S$. For $f \in \mathbb{Q}[X(S)]$, say $f = \sum_{\mu \in X(S)} a_\mu e^\mu$, define the *Newton polytope of $f$*

$$N(f) \coloneqq \mathrm{conv}\{\mu \mid a_\mu \neq 0\} \subseteq \mathbb{R}X(S).$$

The *support of $f$* is the set

$$\mathrm{supp}(f) \coloneqq \{\mu \mid f = \sum_{\mu \in X(S)} a_\mu e^\mu, \; a_\mu \neq 0\} \subseteq \mathrm{vert}(N(f)).$$

We have the following consequence of the definition.

**Lemma 4.3.12.** *Let $f, g : S \to \mathbb{A}^1$ be regular functions, both nonzero. Then,*

*(a) $N(fg) = N(f) + N(g)$, and*

*(b) $N(f + g) \subseteq N(f) \vee N(g)$.*

Lemma 4.3.12 gives the following result.

**Proposition 4.3.13.** *The assignment*

$$\begin{aligned} N:\quad \mathbb{Q}[X(S)]^\times &\longrightarrow \mathcal{P}_E \\ f &\longmapsto N(f) \end{aligned}$$

*is a homomorphism of monoids. Moreover, $N$ extends to a well-defined homomorphism of abelian groups*

$$\begin{aligned} \tilde{N}:\quad \mathrm{Frac}(S)^\times &\longrightarrow \mathcal{P}_E^+ \\ \tfrac{f}{g} &\longmapsto [N(f)]-[N(g)] \end{aligned}$$

*By abuse of notation we will simply write $N$ instead of $\tilde{N}$.*

We will now define the notion of *tropicalisation* for algebraic tori defined over $\mathbb{Q}$. Let $S$ be an algebraic torus defined over $\mathbb{Q}$, and set $E=\mathbb{R}X(S)$. We identify $E^*=\mathbb{R}X^\vee(S)$. Consider the following modification of the homomorphism $\chi:\mathcal{P}_E^+\to \mathrm{Fun}(E^*,\mathbb{R})$:

$$\begin{aligned} \chi^0:\quad \mathcal{P}_E^+ &\longrightarrow \mathrm{Fun}(X^\vee(S),\mathbb{Z}) \\ P &\longmapsto \chi_P^0 \end{aligned} \tag{4.3.3}$$

where

$$\chi_P^0(\xi) := \min\{\langle p,\xi\rangle \mid p\in P\cap X(S)\},\quad \xi\in X^\vee(S).$$

**Definition 4.3.14.** Define *tropicalisation (with respect to $S$)* to be the composition

$$\mathrm{Trop}_S := \chi^0\circ N:\quad \mathrm{Frac}(S)^\times \longrightarrow \mathrm{Fun}(X^\vee(S),\mathbb{Z}).$$

If $S=\mathbb{G}_m^k$ is the standard torus then we simply write $\mathrm{Trop}_k$.

If $f:S\to S'$ is a rational morphism of tori, define the *tropicalisation of $f$*

$$\mathrm{Trop}(f):\quad X^\vee(S)\longrightarrow X^\vee(S')$$

to be the unique function such that the following diagram commutes

$$\begin{tikzcd} X(S)\subseteq \mathrm{Frac}\,S^\times \arrow[r, "\mathrm{Trop}_S"] & \mathrm{Fun}(X^\vee(S),\mathbb{Z}) \\ X(S')\subseteq \mathrm{Frac}(S')^\times \arrow[u, "f^*"] \arrow[r, "\mathrm{Trop}_{S'}"'] & \mathrm{Fun}(X^\vee(S'),\mathbb{Z}) \arrow[u, "\mathrm{Trop}(f)^*"'] \end{tikzcd}$$

That is, $\mathrm{Trop}(f):X^\vee(S)\to X^\vee(S')$ is the unique function such that, for every $\lambda'\in X(S')$, we have an equality of functions

$$\mathrm{Trop}_S(\lambda'\circ f)=\mathrm{Trop}_{S'}\circ \mathrm{Trop}(f):X^\vee(S)\to\mathbb{Z}.$$

Equivalently, $\mathrm{Trop}(f):X^\vee(S)\to X^\vee(S')$ is the unique function such that

$$\mathrm{Trop}_S(\lambda'\circ f)(\mu)=\langle\lambda',\mathrm{Trop}(f)(\mu)\rangle,\quad \text{for every } \mu\in X^\vee(S),\ \lambda'\in X(S').$$

**Example 4.3.15.** Consider the rational morphism

$$\begin{array}{rccc} f: & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 \\ & (x,y) & \longmapsto & \left(\frac{x}{x^2+y}, xy^{-1}\right) \end{array}$$

Write $f = (f_1, f_2)$, and let $e_1, e_2 \in \mathbb{Z}^2$ be the standard basis with dual basis $e_1^*, e_2^*$. Hence,

$$\mathrm{Trop}(f)(a_1e_1^* + a_2e_2^*) = b_1e_1^* + b_2e_2^*$$

must satisfy

$$\begin{aligned} b_1 &= \langle e_1, \mathrm{Trop}(f)(a_1e_1^* + a_2e_2^*)\rangle \\ &= \mathrm{Trop}_{\mathbb{G}_m^2}(f_1)(a_1e_1^* + a_2e_2^*) \\ &= \chi_{N(f_1)}(a_1e_1^* + a_2e_2^*) \\ &= \langle e_1, a_1e_1^* + a_2e_2^*\rangle - \min\left(\langle 2e_1, a_1e_1^* + a_2e_2^*\rangle, \langle e_2, a_1e_1^* + a_2e_2^*\rangle\right) \\ &= a_1 - \min(2a_1, a_2). \end{aligned}$$

Similarly, we compute

$$b_2 = a_1 - a_2.$$

Hence,

$$\begin{array}{rccc} \mathrm{Trop}(f): & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ & (a_1,a_2) & \longmapsto & (a_1 - \min(2a_1, a_2), a_1 - a_2). \end{array}$$

**Remark 4.3.16.** Observe that if we define

$$\begin{array}{rccc} g: & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 \\ & (x,y) & \longmapsto & \left(\frac{x}{x^2-y}, xy^{-1}\right) \end{array}$$

then $\mathrm{Trop}(g) = \mathrm{Trop}(f)$, where $f$ is from Example 4.3.15.

Example 4.3.15 indicates that the notion of tropicalisation we have defined is the same as the usual notion of tropicalisation appearing in tropical geometry (see [103]). We verify this observation with the following result.

**Proposition 4.3.17.** *Let $f = (f_1, \ldots, f_l) : \mathbb{G}_m^k \to \mathbb{G}_m^l$ be a rational morphism. Then,*

$$\begin{array}{rccc} \mathrm{Trop}(f): & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^l \\ & (a_1,\ldots,a_k) & \longmapsto & (\mathrm{Trop}_k(f_1)(a_1,\ldots,a_k), \ldots, \mathrm{Trop}_k(f_l)(a_1,\ldots,a_k)) \end{array}$$

*where $\mathrm{Trop}_k(f_j)(a_1, \ldots, a_k)$ is considered to be a tropical rational function taking values in the tropical semi-ring (see [103]). Namely, $\mathrm{Trop}_k(f_j)(a_1, \ldots, a_k)$ is the piecewise linear function* *obtained from $f_j(x_1, \dots, x_k)$ by replacing*

$$\begin{array}{ccc} x_i & \longleftrightarrow & a_i \\ + & \longleftrightarrow & \min \\ \times & \longleftrightarrow & + \\ \div & \longleftrightarrow & - \end{array}$$

*Proof.* It suffices to show that $\mathrm{Trop}_k(f)$ is the claimed piecewise linear function, for $f \in \mathrm{Frac}(\mathbb{G}_m^k) = \mathbb{Q}(x_1, \dots, x_k)$.

Let $f = \frac{g}{h}$ with $g, h \in \mathbb{Q}[x_1, \dots, x_k]$. By definition, we have $\mathrm{Trop}_k(f) \in \mathrm{Fun}(\mathbb{Z}^k, \mathbb{Z})$ and $\mathrm{Trop}_k(f) = \chi^0 \circ N(f)$, where $\chi^0$ is given in (4.3.3) and $N(f) = [N(g)] - [N(h)]$ is the (virtual) Newton polytope of $f$. Since $\chi^0$ is a morphism of semi-fields, it suffices to consider the case when $f \in \mathbb{Q}[x_1, \dots, x_k]$.

Now, let $f \in \mathbb{Q}[x_1, \dots, x_k]$. Then, by definition

$$\mathrm{Trop}_k(f)(a_1, \dots, a_l) = \chi^0(N(f)) = \min\{(b_1, \dots, b_k) \cdot (a_1, \dots, a_k) \mid (b_1, \dots, b_k) \in \mathrm{supp}(f)\}$$

and this expression is what we are looking for. Here $\cdot$ is the standard dot product on $\mathbb{Z}^k$. □

The tropicalisation of the homomorphisms between algebraic tori will be of most interest to us. We record some elementary consequences from the definitions.

**Proposition 4.3.18.** *Let $S$ be an algebraic torus defined over $\mathbb{Q}$.*

*(a) Let $\lambda \in X(S)$, considered as a rational function $\lambda : S \to \mathbb{G}_m$. Recall the canonical identification $X(S) \cong \mathrm{Hom}_{\mathbb{Z}}(X^\vee(S), \mathbb{Z})$. Then,*

$$\mathrm{Trop}_S(\lambda) = \lambda \in \mathrm{Hom}(X^\vee(S), \mathbb{Z}) \subseteq \mathrm{Fun}(X^\vee, \mathbb{Z}).$$

*(b) Let $\xi \in X^\vee(S)$, considered as a rational function $\xi : \mathbb{G}_m \to S$. Then, for any $\mu \in X^\vee(\mathbb{G}_m)$,*

$$\mathrm{Trop}(\xi)(\mu) = \xi \circ \mu \in X^\vee(S).$$

*Identifying $X^\vee(\mathbb{G}_m) \cong \mathbb{Z}$, $\mathrm{id}_{\mathbb{G}_m} \mapsto 1 \in \mathbb{Z}$, we have*

$$\mathrm{Trop}(\xi)(n) = n\xi \in X^\vee(S), \quad n \in \mathbb{Z}.$$

*(c) More generally, if $f : S \to T$ is an homomorphism of algebraic tori then, for any $\mu \in X^\vee(S)$,*

$$\mathrm{Trop}(f)(\mu) = f \circ \mu \in X^\vee(T)$$

(d) *Let $f : S_1 \to T$ and $g : S_2 \to T$ be rational morphisms, and*

$$\begin{array}{rccc} fg: & S_1 \times S_2 & \longrightarrow & T \\ & (x, y) & \longmapsto & f(x)g(y) \end{array}$$

*Then,* $\mathrm{Trop}(fg) = \mathrm{Trop}(f) + \mathrm{Trop}(g)$.

*Proof.* (a) Let $\lambda \in X(S)$. By definition, for any $\mu \in X^\vee(S)$,

$$\begin{aligned} \mathrm{Trop}_S(\lambda)(\mu) &= \chi^0_{N(\lambda)}(\mu) \\ &= \min\{\langle p, \mu\rangle \mid p \in N(\lambda) \cap X(S)\} \\ &= \langle \lambda, \mu\rangle, \quad \text{since } N(\lambda) = \{\lambda\}. \end{aligned}$$

(b) Let $\xi \in X^\vee(S)$. Then, $\mathrm{Trop}(\xi) : X^\vee(\mathbb{G}_m) \to X^\vee(S)$ is the unique function such that, for every $\mu \in X^\vee(\mathbb{G}_m)$, $\lambda \in X(S)$,

$$\mathrm{Trop}_1(\lambda \circ \xi)(\mu) = \langle \lambda, \mathrm{Trop}(\xi)(\mu)\rangle.$$

Then,

$$\begin{aligned} \mathrm{Trop}_1(\lambda \circ \xi)(\mu) &= \langle \lambda \circ \xi, \mu\rangle, \quad \text{by (a)}, \\ &= \langle \lambda, \xi \circ \mu\rangle, \quad \text{by definition of the pairing } \langle , \rangle. \end{aligned}$$

Now, let $\mu_n \in X^\vee(\mathbb{G}_m)$, $\mu_n(z) = z^n$. Then,

$$\mathrm{Trop}(\xi)(\mu_n) = \xi \circ \mu_n = \xi^n,$$

because $\xi$ is a homomorphism.

(c) The argument is similar.

(d) Let $f : S_1 \to T$, $g : S_2 \to T$. There is a canonical identification $X^\vee(S_1 \times S_2) = X^\vee(S_1) \oplus X^\vee(S_2)$. For any $\mu = (\mu_1, \mu_2) \in X^\vee(S_1) \oplus X^\vee(S_2)$, $\lambda \in X(T)$, we must have

$$\mathrm{Trop}_{S_1 \times S_2}(\lambda \circ fg)(\mu) = \langle \lambda, \mathrm{Trop}(fg)(\mu)\rangle.$$

Then,

$$\begin{aligned} \mathrm{Trop}_{S_1 \times S_2}(\lambda \circ fg)(\mu) &= \langle \lambda \circ fg, \mu\rangle, \quad \text{by (a)}, \\ &= \langle (\lambda \circ f)(\lambda \circ g), \mu\rangle, \quad \text{since } \lambda \text{ is a homomorphism}, \\ &= \langle (\lambda \circ f), \mu\rangle + \langle (\lambda \circ g), \mu_2\rangle, \quad \text{by definition of the pairing } \langle , \rangle, \\ &= \langle \lambda, \mathrm{Trop}(f)(\mu_1)\rangle + \langle \lambda, \mathrm{Trop}(g)(\mu_2)\rangle, \\ &= \langle \lambda, \mathrm{Trop}(f)(\mu_1) + \mathrm{Trop}(g)(\mu_2)\rangle. \end{aligned}$$

Hence, $\mathrm{Trop}(fg) = \mathrm{Trop}(f) + \mathrm{Trop}(g)$.

∎

Consider the rational morphism

$$\begin{array}{rccc} h: & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 \\ & (x,y) & \longmapsto & (x, x+y) \end{array}$$

This map is a birational isomorphism with rational inverse $(x,y) \mapsto (x, y-x)$. However,

$$\begin{array}{rccc} \mathrm{Trop}(h): & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ & (a,b) & \longmapsto & (a, \min(a,b)) \end{array}$$

is not a bijection. In light of this example, if we want to define a *tropicalisation functor* on some category of split algebraic tori (defined over $\mathbb{Q}$), we need to restrict morphisms.

In [12], Berenstein-Kazhdan determined an appropriate category on which to define a tropicalisation functor. We recall their results.

**Definition 4.3.19.** Let $S$ be an algebraic torus, split over $\mathbb{Q}$.

(a) Let $f \in \mathrm{Frac}(S)^\times$. We say $f$ is *positive* if it can be written as a quotient $f = g/h$, where $g, h \in \mathbb{Z}_{>0}[X(S)]$. Denote the semi-field of positive rational functions on $S$ by $\mathrm{Frac}_+(S)$.

(b) A rational morphism $f : S \to S'$ is *positive* if the pullback map

$$f^* : \quad X(S') \subseteq \mathrm{Frac}(S') \longrightarrow \mathrm{Frac}_+(S)$$

is well-defined. Denote the set of positive rational morphisms $S \to S'$ by $\mathrm{Mor}_+(S, S')$.

**Remark 4.3.20.** It can be shown that $f \in \mathrm{Mor}_+(S, S')$ if and only if $f : S(\mathbb{Q}_{>0}) \to S'(\mathbb{Q}_{>0})$ is well-defined (see [13, Section 4]).

**Theorem 4.3.21** (Berenstein-Kazhdan, [12, Section 2.4])**.** *Let $\mathcal{T}_+$ be the monoidal category of algebraic tori split over $\mathbb{Q}$ with positive rational morphisms. Then,*

$$\begin{array}{rccc} \mathrm{Trop}: & \mathcal{T}_+ & \longrightarrow & \mathrm{Set} \\ & S & \longmapsto & X^\vee(S) \\ & f & \longmapsto & \mathrm{Trop}(f) \end{array}$$

*is a (covariant) functor. If we equip $\mathcal{T}_+$ and* Set *with their standard monoidal structure then* Trop *is monoidal.*

Tropicalisation, as we have defined it, applies to the positive rational functions on a split algebraic torus. Therefore, we might expect to be able to apply tropicalisation to those varieties birational to an algebraic torus. As we will see, this will be possible, but will require our introducing a notion of *positivity* for varieties.

**Definition 4.3.22.** Let $X$ be a variety defined over $\mathbb{Q}$.

(a) A *toric chart on* $X$ is a birational isomorphism $\theta : S \to X$, for some algebraic torus $S$ split over $\mathbb{Q}$.

(b) We say that two toric charts

$$\theta : S \to X, \quad \theta' : S' \to X$$

are *positively equivalent* if $\theta^{-1} \circ \theta'$ is an isomorphism in $\mathcal{T}_+$. In particular,

$$\theta^{-1} \circ \theta' \in \mathrm{Mor}_+(S', S), \quad \text{and} \quad (\theta')^{-1} \circ \theta \in \mathrm{Mor}_+(S, S').$$

(c) A *positive atlas on* $X$ is an positive equivalence class of toric charts on $X$. We will also call a positive atlas a *positive structure on* $X$.

(d) A *positive variety* $(X, \Theta_X)$ is a variety together with a choice of positive atlas. A *morphism of positive varieties* $(X, \Theta_X)$, $(Y, \Theta_Y)$ is a rational morphism

$$f : \ X \longrightarrow Y$$

such that

$$f : \ X(\mathbb{Q}_{>0}) \longrightarrow Y(\mathbb{Q}_{>0})$$

is a well-defined function, and $\mathrm{range}(f) \cap \mathrm{dom}(\theta^{-1}) \neq \varnothing$, for any $\theta \in \Theta_Y$.

With these objects and morphisms, we define the *category of positive varieties* $\mathcal{V}_+$.

**Example 4.3.23.** (1) Let $S$ be a split algebraic torus, defined over $\mathbb{Q}$. The *standard positive structure on* $S$ is the positive structure containing $\mathrm{id}_S : S \to S$, which we will denote $\Theta_S$. The *standard positive structure on* $\mathbb{A}^k$ is the positive structure containing the canonical open embedding $\mathbb{G}_m^k \to \mathbb{A}^k$, which we will denote $\Theta_k$. It's straightforward to see that, as positive varieties, $\mathbb{G}_m^k \cong \mathbb{A}^k$.

(2) More generally, let $(X, \Theta_X)$ be a positive variety, $\theta : S \to X \in \Theta_X$. Then, $\theta$ defines an isomorphism of positive varieties $(S, \Theta_S) \xrightarrow{\sim} (X, \Theta_X)$.

(3) Any homomorphism of algebraic tori is positive.

**Remark 4.3.24.** We will always consider an algebraic torus as a positive variety equipped with the standard positive structure.

The importance of the notion of positivity is that it gives the correct condition to construct a tropicalisation functor. As we are motivated to reconstruct Kashiwara crystals via tropicalisation, we introduce the following definition.

**Definition 4.3.25.** A *positive decorated geometric crystal* is a decorated geometric crytal $(X, \gamma, \varepsilon_i, \varphi_i, e_i^{\cdot}, f \mid i \in I)$ such that $X$ is equipped with a positive structure $\Theta$, with respect to which all of the maps appearing in the definition are positive (with the respect to the appropriate standard positive structures). We will write simply $(X, \Theta, f)$ for the data of a positive decorated geometric crystal.

The following result is an imediate consequence of the definition of a positive structure.

**Lemma 4.3.26.** *Let $(X, \Theta_X)$ and $(Y, \Theta_Y)$ be positive varieties. Then, $(X \times Y, \Theta_X \times \Theta_Y)$, where*

$$\Theta_X \times \Theta_Y := \{\theta \times \theta' \mid \theta \in \Theta_X, \theta' \in \Theta_Y\},$$

*is a positive variety. Therefore, $\mathcal{V}_+$ is a monoidal category.*

Consider the natural inclusion functor

$$\begin{array}{ccc} \mathcal{T}_+ & \longrightarrow & \mathcal{V}_+ \\ S & \longmapsto & (S, \Theta_S) \end{array} \tag{4.3.4}$$

Clearly, this functor is fully faithful and monoidal. Moreover, by Example 4.3.23, we have the following result.

**Proposition 4.3.27.** *The inclusion functor* (4.3.4) *is an equivalence of monoidal categories.*

Consider the category $\mathcal{V}_{++}$ with objects $(X, \Theta_X, \theta)$, where $(X, \Theta_X) \in \mathcal{V}_+$ and $\theta \in \Theta_X$, and morphisms being morphisms of the underlying positive varieties. By definition, the forgetful functor

$$\begin{array}{ccc} \mathcal{V}_{++} & \longrightarrow & \mathcal{V}_+ \\ (X, \Theta_X, \theta) & \longmapsto & (X, \Theta_X) \end{array} \tag{4.3.5}$$

is an equivalence of monoidal categories. Any adjoint to this functor corresponds to a simultaneous choice of toric chart $\theta : S \xrightarrow{\sim} X \in \Theta_X$, for every $(X, \theta_X)$. All such adjoints are isomorphic.

Define the functor

$$\begin{array}{cccc} \tau : & \mathcal{V}_{++} & \longrightarrow & \mathcal{T}_+ \\ & (X, \Theta_X, \theta) & \longmapsto & \mathrm{dom}(\theta) \\ & (X, \Theta_X, \theta) \xrightarrow{f} (Y, \Theta_Y, \theta') & \longmapsto & (\theta')^{-1} \circ f \circ \theta \end{array} \tag{4.3.6}$$

Then, $\tau$ is an equivalence of monoidal categories.

We will now extend the tropicalisation functor to the category of positive varieties. Let $\mathcal{G} : \mathcal{V}_+ \to \mathcal{V}_{++}$ be an adjoint to the forgetful functor. By the discussion above, all such adjoints are isomorphic to each other.

**Definition 4.3.28.** The composition

$$\mathrm{Trop}_{\mathcal{G}} \coloneqq \mathrm{Trop} \circ \tau \circ \mathcal{G} : \quad \mathcal{V}_+ \quad \longrightarrow \quad \mathrm{Set} \tag{4.3.7}$$

will be called a *tropicalisation functor*.

**Remark 4.3.29.** All tropicalisation functors $\mathrm{Trop}_{\mathcal{G}} : \mathcal{V}_+ \to \mathrm{Set}$ are isomorphic to each other. For the remainder of this thesis we assume that we have fixed a choice of tropicalisation functor and write $\mathrm{Trop} : \mathcal{V}_+ \to \mathrm{Set}$ (by abuse of notation).

Moreover, if $f : X \to \mathbb{A}^1$ is a rational function then $\mathrm{Trop}(f) : X^\vee(S) \to \mathbb{Z}$ depends only only the positive equivalence class of $f$. Here, rational functions $f, f' \in \mathrm{Frac}_+(X)$ are *positively equivalent* if there is an isomorphism of positive varieties $h : (X, \Theta_X) \xrightarrow{\sim} (X, \Theta_X)$ such that $f' = f \circ h$ (see [13, Section 6.1]).

We finish this section with the main result in [13], which states that the tropicalisation of a positive decorated geometric crystal is a Kashiwara crystal. For more details see [13].

**Theorem 4.3.30** (Berenstein-Kazhdan, [13])**.** *Let $(X, \Theta, f)$ be a positive decorated geometric crystal. Define the* tropical locus of $f$ (with respect to $\Theta$) *to be*

$$B_{f,\Theta} \coloneqq \{x \in \mathrm{Trop}(X) \mid \mathrm{Trop}(f)(x) \geq 0\}.$$

*Then, the data $(B_{f,\Theta}, \mathrm{Trop}(\gamma), \mathrm{Trop}(\varepsilon_i), \mathrm{Trop}(\varphi_i), \mathrm{Trop}(e_i^{\cdot}) \mid i \in I)$ is a Kashiwara crystal, where we consider the tropicalised data as being restricted to $B_{f,\Theta}$. The map* $\mathrm{Trop}(\gamma) = \mathrm{wt}$ *for the crystal.*

## 4.4 Crystal structures in mirror symmetry

In this section we will demonstrate the appearance of Kashiwara crystal structures in the Rietsch mirror family $(M_B, f_B)$ to the complete flag variety ${}^LG/{}^LB_+$. We will see that the quantum structure map $q$ and the equivariant structure map $e$ from Definition 3.1.3 play an essential role in these structures. Specifically, we identify the extended string cone $\underline{C}_{\mathbf{i}}(\mathbb{Z})$ (Definition 4.2.28) as the tropicalisation of the Rietsch mirror family $(M_P, f_P)$, explicitly recovering the inequalities defining the string cone and the $\lambda$-inequalities. The tropicalisation $\mathrm{Trop}(q)$ is the highest weight map hw (4.2.18) and $\mathrm{Trop}(e)$ (4.2.19) is the weight map hw. These results are inspired by, and similar to, [13]. However, our approach makes use of a family of non-standard parameterisations $\theta_{\mathbf{i}}$ of $M_B$.

As usual, $G$ is a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$, and we use the notation and conventions from Section 1.3.

For $w \in W$, define the varieties

$$B_-^w \coloneqq B_- \cap N_+\overline{w}N_+, \quad N_+^w \coloneqq B_-\overline{w}B_- \cap N_+. \tag{4.4.1}$$

Let $M \coloneqq B_- \cap N_+ T \overline{w}_0 N_+$ be the mirror family of ${}^L G / {}^L B_+$. Recall the quantum and equivariant structure maps (Definition 3.1.3),

$$\begin{tikzcd} T & M \arrow[l, "q"'] \arrow[d, "e"] \\ & T \end{tikzcd} \tag{4.4.2}$$

where

$$\begin{array}{rccc} q: & M = B_- \cap N_+ T \overline{w}_0 N_+ & \longrightarrow & T \\ & b = zt\overline{w}_P u & \longmapsto & t \end{array} \tag{4.4.3}$$

and

$$\begin{array}{rccc} e: & M \subseteq B- = N_- T & \longrightarrow & T \\ & b = vs & \longmapsto & s \end{array} \tag{4.4.4}$$

By Section 3.1, $q$ is a smooth trivial fibration with fibre $B_-^{w_0} \coloneqq B_- \cap N_+ \overline{w}_0 N_+$. We fix the following trivialisation

$$\begin{array}{rccc} j: & T \times B_-^{w_0} & \longrightarrow & M \\ & (t,x) & \longmapsto & \pi^+ \left( \left( \overline{w}_0 x^T \right)^{-1} \right) \overline{w}_0 x^T \pi^0 \left( x^{-T} \right) t^{w_0} \end{array} \tag{4.4.5}$$

**Lemma 4.4.1.** *The trivialisation $j$ is well-defined.*

*Proof.* We must show that

(i) if $j_t(x) = j(t,x)$ then $j_t(x) \in B_- \cap N_+ t \overline{w}_0 N_+$, and

(ii) $j$ is an isomorphism.

Write $x = z\overline{w}_0 u$. Then,

$$\begin{aligned} (\overline{w}_0 x^T)^{-1} &= x^{-T} \overline{w}_0^{-1} \\ &= z^{-T} \overline{w}_0 u^{-T} \overline{w}_0^{-1} \end{aligned}$$

Hence, $\pi^+((\overline{w}_0 x^T)^{-1}) = \overline{w}_0 u^{-T} \overline{w}_0^{-1}$ and we have

$$j_t(z\overline{w}_0 u) = z^T \pi^0 (z^T \overline{w}_0 u^T) t^{\overline{w}_0} \in B_-.$$

Also, we have $x = vs \in B_- = N_- T$ and $\pi^0(x^{-T}) = s^{-1}$, Hence,

$$j_t(x) = j_t(vs) = \pi^+ \left( \left( \overline{w}_0 x^T \right)^{-1} \right) \overline{w}_0 s v^T s^{-1} t^{\overline{w}_0} \in N_+ \overline{w}_0 N_+ t^{\overline{w}_0} = N_+ t \overline{w}_0 N_+.$$

The inverse to $j$ is seen to be

$$\begin{array}{rcl} M & \longrightarrow & T \times B_-^{w_0} \\ b = zt\overline{w}_0 u & \longmapsto & \left( \pi^0(\overline{w}_0^{-1} b), \pi^{\geq 0} \left( \overline{w}_0^{-1} \pi^-(b) \right)^T \right) \end{array} \tag{4.4.6}$$

□

**Remark 4.4.2.** Observe that the trivialisation $j$ is not the obvious choice of trivialisation induced by multiplication. A similar trivialisation appears in [28].

For any $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$, define

$$\begin{array}{rccc} x_{-\mathbf{i}}: & \mathbb{G}_m^m & \longrightarrow & B_-^{w_0} \\ & (a_1, \ldots, a_m) & \longmapsto & x_{-i_1}(a_1) \cdots x_{-i_m}(a_m), \end{array} \tag{4.4.7}$$

where, for $i \in I$,

$$\begin{array}{rccc} x_{-i}: & \mathbb{G}_m & \longrightarrow & B_-^{w_0} \\ & c & \longmapsto & y_i(c)\alpha_i^\vee(c^{-1}). \end{array}$$

Here $y_i : \mathbb{A}^1 \to N_-$ is the root subgroup corresponding to the (simple) negative root $-\alpha_i$, and $\alpha_i^\vee \in X^\vee(T)$.

**Definition 4.4.3.** For $\mathbf{i} \in R(w_0)$, define

$$\begin{array}{rccc} \underline{\theta}_{\mathbf{i}}: & T \times \mathbb{G}_m^{\ell(w_0)} & \longrightarrow & M \\ & (t, a) & \longmapsto & j(t, x_{-\mathbf{i}}(a)) \end{array}$$

By Fomin-Zelevinsky [36], we have the following result.

**Proposition 4.4.4.** *(a) For any $\mathbf{i} \in R(w_0)$, $\underline{\theta}_{\mathbf{i}}$ is a toric chart.*

*(b) For $\mathbf{i}, \mathbf{i}' \in R(w_0)$, the birational isomorphism*

$$\underline{\theta}_{\mathbf{i}}^{\mathbf{i}'} := \underline{\theta}_{\mathbf{i}'}^{-1} \circ \underline{\theta}_{\mathbf{i}} : \quad T \times \mathbb{G}_m^{\ell(w_0)} \longrightarrow T \times \mathbb{G}_m^{\ell(w_0)}$$

*is a positive morphism. In other words, $\underline{\theta}_{\mathbf{i}}$ and $\underline{\theta}_{\mathbf{i}'}$ are positively equivalent.*

Proposition 4.4.4 implies that we can equip $M$ with the structure of a positive variety $(M, \Theta_0)$, where we define $\Theta_0$ to be the positive equivalence class of toric charts on $M$ containing $\{\underline{\theta}_{\mathbf{i}} \mid \mathbf{i} \in R(w_0)\}$.

Recall from Definition 3.1.9 the *superpotential* $f_B$

$$\begin{array}{rccc} f_B: & M & \longrightarrow & \mathbb{C} \\ & b = zt\overline{w}_0u & \longmapsto & \chi(z) + \chi(u) \end{array}$$

where $\chi = \sum_{i \in I} \chi_i \in \mathrm{Hom}(N_+, \mathbb{A}^1)$ and $\chi_i$, $i \in I$, is the character of $N_+$ uniquely determined by

$$\chi_i(x_j(a)) = \delta_{ij}a, \quad a \in \mathbb{A}^1.$$

Define

$$\begin{array}{rccc} f_B^{(1)}: & M & \longrightarrow & \mathbb{C} \\ & b = zt\overline{w}_0u & \longmapsto & \chi(z) \end{array} \tag{4.4.8}$$

and

$$\begin{array}{rccc} f_B^{(2)}: & M & \longrightarrow & \mathbb{C} \\ & b = zt\overline{w}_0 u & \longmapsto & \chi(u) \end{array} \tag{4.4.9}$$

For $\mathbf{i} \in R(w_0)$, we define

$$f_{B,\mathbf{i}} \coloneqq f_B \circ \underline{\theta}_{\mathbf{i}}, \quad f_{B,\mathbf{i}}^{(1)} \coloneqq f_B^{(1)} \circ \underline{\theta}_{\mathbf{i}}, \quad f_{B,\mathbf{i}}^{(2)} \coloneqq f_B^{(2)} \circ \underline{\theta}_{\mathbf{i}}, \tag{4.4.10}$$

We now state the main result of this section.

**Theorem 4.4.5.** *Let $\mathbf{i} \in R(w_0)$. Then, the subset*

$$\{(\lambda, a) \in X^\vee \times \mathbb{Z}^{\ell(w_0)} \mid \operatorname{Trop}(f_{B,\mathbf{i}})(\lambda, a) \geq 0\} \subseteq X^\vee(T \times \mathbb{G}_m^{\ell(w_0)}) = X({}^LT) \times \mathbb{Z}^{\ell(w_0)} \tag{4.4.11}$$

*equals the set of lattice points in the weighted string cone $\underline{C}_{\mathbf{i}}(\mathbb{Z})$ of the Langlands dual group ${}^LG$. More precisely,*

*(a)* $\operatorname{Trop}(f_{B,\mathbf{i}}^{(1)})(\lambda, a) \geq 0$ *are the inequalites defining $\mathbb{R}X({}^LT) \times C_{\mathbf{i}}$ in $\mathbb{R}X({}^LT) \times \mathbb{R}^{\ell(w_0)}$, and*

*(b)* $\operatorname{Trop}(f_{B,\mathbf{i}}^{(2)})(\lambda, a) \geq 0$ *are the $\lambda$-inequalities (see (4.2.20)).*

*Moreover, the maps $q$ and $e$ are positive morphisms and*

$$\operatorname{Trop}(q) = \mathrm{hw}, \quad \operatorname{Trop}(e) = \mathrm{wt}\,. \tag{4.4.12}$$

*Proof.* First, we show the statements on the description of the weighted string cone. Let $\mathbf{i} = (i_1, \dots, i_m) \in R(w_0)$. Then, there exists $z, u \in N_+$ such that

$$j(t, x_{-\mathbf{i}}(a)) = zt\overline{w}_0 u \in N_+ t\overline{w}_0 N_+ \cap B_-.$$

Observe that

$$u = \pi^+\left(\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))\right).$$

Hence, we have

$$f_{B,\mathbf{i}}^{(2)}(t, a) = \chi\left(\pi^+\left(\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))\right)\right).$$

Using the definition of $j$,

$$\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a)) = \overline{w}_0^{-1}\pi^+\left(\left(\overline{w}_0 x_{-\mathbf{i}}(a)^T\right)^{-1}\right)\overline{w}_0 x_{-\mathbf{i}}(a)^T \pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right) t^{w_0}$$

so that

$$\pi^+\left(\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))\right) = \pi^+\left(x_{-\mathbf{i}}(a)^T \pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right) t^{w_0}\right).$$

By definition of the transpose map we obtain

$$\begin{aligned} x_{-\mathbf{i}}(a)^T &= x_{-i_m}(a_m)^T \cdots x_{-i_1}(a_1)^T \\ &= \alpha_{i_m}^\vee(a_m^{-1}) x_{i_m}(a_m) \cdots \alpha_{i_1}^\vee(a_1^{-1}) x_{i_1}(a_1) \\ &= \left(\prod_{j=1}^m \alpha_{i_j}^\vee(a_j^{-1})\right) x_{i_m}(b_m) \cdots x_{i_1}(b_1), \end{aligned}$$

where
$$b_j = a_j \prod_{l<j} a_l^{\langle \alpha_{i_j}, \alpha_{i_l}^\vee \rangle}, \quad j = 1, \dots, m.$$
Hence,
$$\pi^0 \left( x_{-\mathbf{i}}(a)^{-T} \right) = \prod_{j=1}^m \alpha_{i_j}^\vee (a_j).$$
Therefore, we obtain
$$x_{-\mathbf{i}}(a)^T \pi^0 \left( x_{-\mathbf{i}}(a)^{-T} \right) t^{w_0} = t^{w_0} x_{\mathbf{i}^{op}}(c^{op}),$$
where $c = (c_1, \dots, c_m) \in \mathbb{G}_m^m$ is defined by
$$c_j = (-w_0 \alpha_{i_j})(t) a_j^{-1} \prod_{l>j} a_l^{-\langle \alpha_{i_j}, \alpha_{i_l}^\vee \rangle}, \quad j = 1, \dots, m,$$
and
$$\pi^+ \left( x_{-\mathbf{i}}(a)^T \pi^0 \left( x_{-\mathbf{i}}(a)^{-T} \right) t^{w_0} \right) = x_{\mathbf{i}^{op}}(c^{op})$$
Putting this all together, and recalling the involution $i \mapsto i^*$ on $I$ (see Definition 4.2.6), we have
$$\begin{aligned} f_{B,\mathbf{i}}^{(2)}(t, a_1, \dots, a_m) &= \chi(u) \\ &= \chi(x_{\mathbf{i}^{op}}(c^{op})) \\ &= \sum_{j=1}^m \alpha_{i_j^*}(t) a_j^{-1} \prod_{l>j} a_l^{-\langle \alpha_{i_j}, \alpha_{i_l}^\vee \rangle}. \end{aligned}$$
By Proposition 4.3.18, we have, for any $(\lambda, a) \in X^\vee \times \mathbb{Z}^m$,
$$\mathrm{Trop}(f_{B,\mathbf{i}}^{(2)})(\lambda, a) = \min\{\langle \alpha_{i_j^*}, \lambda \rangle - a_j - \sum_{l=j+1}^m \langle \alpha_{i_j}, \alpha_{i_l}^\vee \rangle a_l \mid j = 1, \dots, m\}$$
Recalling that the root datum of the Langlands dual group ${}^L G$ is the dual root datum for $G$, we see that the locus $\mathrm{Trop}(f_{B,\mathbf{i}}^{(2)}) \geq 0$ is precisely the locus defined by the $\lambda$-inequalities for ${}^L G$ (Theorem 4.2.31).

Now we obtain the inequalities defining the string cone $C_{\mathbf{i}}$. Note that, if $b = zt\overline{w}_0 u \in M$ then
$$z = \pi^+ \left( \overline{w}_0^{-1} b^\iota \right)^\iota$$
Hence, we have
$$f_{B,\mathbf{i}}^{(1)}(t, a) = \chi \left( \pi^+ (\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))^\iota)^\iota \right).$$
Now,
$$\begin{aligned} &\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))^\iota \\ =&\overline{w}_0^{-1} t^{-w_0} \pi^0(x_{-\mathbf{i}}(a)^T) x_{-\mathbf{i}}(a)^{\iota T} \overline{w}_0 \pi^+((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})^\iota \end{aligned}$$

so that

$$\pi^+(\overline{w}_0^{-1}j(t, x_{-\mathbf{i}}(a))^\iota) = \pi^+((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})^\iota,$$

and we have

$$f^{(1)}_{B,\mathbf{i}}(t,a) = \chi\left(\pi^+((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})\right)$$

We can assume that $G$ is semisimple and simply-connected (see Remark 4.2.32)). In this situation, for any $g \in N_-TN_+$,

$$\chi_i(\pi^+(g)) = \frac{\Delta_{\omega_i, s_i\omega_i}(g)}{\Delta_{\omega_i,\omega_i}(g)}, \quad i \in I,$$

where $\Delta_{u\omega_i, v\omega_i}(g) = \omega_i(\pi^0(\overline{u}^{-1} g \overline{v}))$ is a generalised mino (see [36, Proposition 2.6]). Hence,

$$f^{(1)}_{B,\mathbf{i}}(t,a) = \sum_{i\in I} \frac{\Delta_{\omega_i, s_i\omega_i}((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})}{\Delta_{\omega_i,\omega_i}((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})}. \tag{4.4.13}$$

Define the involutive antiautomorphism

$$\begin{array}{rccc} \tau_{w_0}: & G & \longrightarrow & G \\ & g & \longmapsto & \overline{w}_0 g^{-\iota T} \overline{w}_0^{-1}. \end{array}$$

Then,

$$(\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1}) = x_{-\mathbf{i}}(a)^{-T}\overline{w}_0^{-1} = \overline{w}_0^{-1}\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota)$$

and (4.4.13) becomes

$$\begin{aligned} &\sum_{i\in I} \frac{\Delta_{\omega_i, s_i\omega_i}((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})}{\Delta_{\omega_i,\omega_i}((\overline{w}_0 x_{-\mathbf{i}}(a)^T)^{-1})} \\ =&\sum_{i\in I} \frac{\Delta_{w_0\omega_i, s_i\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))}{\Delta_{w_0\omega_i,\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))}. \end{aligned}$$

We have the following formulae from [14, (4.6)],

$$\Delta_{u\omega_i, v\omega_i}(x) = \Delta_{-v\omega_i, -u\omega_i}(x^\iota) = \Delta_{w_0 v\omega_i, w_0 u\omega_i}(\tau_{w_0}(x)), \quad u, v \in W, i \in I.$$

Using these formulae we obtain

$$\sum_{i\in I} \frac{\Delta_{w_0\omega_i, s_i\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))}{\Delta_{w_0\omega_i,\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))} = \sum_{i\in I} \frac{\Delta_{-\omega_i, -w_0 s_i\omega_i}(x_{-\mathbf{i}}(a))}{\Delta_{w_0\omega_{i^*}, \omega_{i^*}}(x_{-\mathbf{i}}(a))}.$$

Observe that, for any $b = z\overline{w}_0 u \in B_-^{w_0}$, $\overline{w}_0^{-1} b \in N_- N_+$, so that

$$\Delta_{w_0\omega_i,\omega_i}(b) = 1, \quad i \in I.$$

Hence,

$$f_{B,\mathbf{i}}^{(1)}(t,a) = \sum_{i\in I} \Delta_{-\omega_i,-w_0 s_i\omega_i}(x_{-\mathbf{i}}(a)).$$

Finally, using [14, Corollary 5.9], we can compute the tropicalisation of a generalised minor: we have

$$\mathrm{Trop}(\Delta_{u\omega_i,v\omega_i}(x_{-\mathbf{i}}(a_1,\dots,a_m)) = \min\left\{\sum_{k=1}^{m} d_k^{(i)}(\pi)a_k \mid \pi \text{ is } \mathbf{i}\text{-trail from } -u\omega_i \text{ to } -v\omega_i \text{ in } V_{\omega_i}\right\}.$$

Thus,

$$\begin{aligned}\mathrm{Trop}(f_{B,\mathbf{i}}^{(1)})(\lambda,a) &= \min\{\mathrm{Trop}(\Delta_{-\omega_i,-w_0s_i\omega_i}(x_{-\mathbf{i}}(a))) \mid i\in I\}\\ &= \min\left\{\sum_{k=1}^{m} d_k^{(i)}(\pi)a_k \mid \pi \text{ is } \mathbf{i}\text{-trail from } \omega_i \text{ to } w_0s_i\omega_i \text{ in } V_{\omega_i},\ i\in I\right\}\end{aligned}$$

and the locus defined by $\mathrm{Trop}(f_{B,\mathbf{i}}^{(1)}) \geq 0$ is precisely the string cone $C_{\mathbf{i}}$ for ${}^LG$ (see Theorem 4.2.26).

We will now show that $q$ and $e$ are positive morphisms and

$$\mathrm{Trop}(q) = \mathrm{hw}, \quad \mathrm{Trop}(e) = \mathrm{wt}\,.$$

To show that $q$ is positive it suffices to show that the rational morphism $q\circ\underline{\theta}_{\mathbf{i}}$ is positive, for some $\mathbf{i}\in R(w_0)$. The result follows immediately since

$$q\circ\underline{\theta}_{\mathbf{i}}(t,a) = t \tag{4.4.14}$$

is a homomorphism. Then, $\mathrm{Trop}(q) = \mathrm{hw}$ follows from (4.4.14).

The argument for $e$ is bit more involved. Let $\mathbf{i}\in R(w_0)$. Using (4.4.6), we see that

$$\begin{aligned}\pi^0(x_{-\mathbf{i}}(a)) &= \pi^0\left(\pi^{\geq 0}(\overline{w}_0^{-1}\pi^-(j(t,x_{-\mathbf{i}}(a))))^T\right)\\ &= \pi^0(\overline{w}_0^{-1}\pi^-(j(t,x_{-\mathbf{i}}(a))))\\ &= \pi^0(\overline{w}_0^{-1}j(t,x_{-\mathbf{i}}(a))\pi^0(j(t,x_{-\mathbf{i}}(a)))^{-1})\\ &= t^{w_0}\pi^0(j(t,x_{-\mathbf{i}}(a)))^{-1}\end{aligned}$$

Hence, we find

$$e(j(t,x_{-\mathbf{i}}(a))) = \pi^0(j(t,x_{-\mathbf{i}}(a))) = t^{w_0}\pi^0(x_{-\mathbf{i}}(a))^{-1} = t^{w_0}\left(\prod_{j=1}^{m}\alpha_{i_j}^\vee(a_j)\right)$$

Since homomorphisms are positive we see that $e$ is positive. Denote the conjugation map

$$\begin{aligned}c_{w_0}:\ T &\longrightarrow T\\ t &\longmapsto \overline{w}_0 t\overline{w}_0^{-1}\end{aligned}$$

Then, by Proposition 4.3.18, we have

$$\mathrm{Trop}(e) = \mathrm{Trop}(c_{w_0}) + \sum_{j=1}^{m} \mathrm{Trop}(\alpha_{i_j}^\vee)$$

and, for $(\lambda, a) \in X^\vee(T) \times X^\vee(\mathbb{G}_m^{\ell(w_0)})$,

$$\mathrm{Trop}(e)(\lambda, a) = \mathrm{Trop}(c_{w_0})(\lambda) + \sum_{j=1}^{\ell(w_0)} \mathrm{Trop}(\alpha_{i_j}^\vee)(a).$$

Making the canonical identifications $X^\vee(T) \cong X({}^LT)$, where ${}^LT \subseteq {}^LG$ is the torus dual to $T$, and $X^\vee(\mathbb{G}_m^{\ell(w_0)}) \cong \mathbb{Z}^{\ell(w_0)}$, we obtain $\mathrm{Trop}(c_{w_0})(\lambda) = w_0(\lambda)$, $\lambda \in X({}^LT)$, and

$$\mathrm{Trop}(e)(\lambda, a_1, \dots, a_m) = w_0(\lambda) + \sum_{j=1}^{m} a_j \alpha_{i_j}^\vee.$$

The result follows. □

**Remark 4.4.6.** A similar result to Theorem 4.4.5 is obtained by Berenstein-Kazhdan [13, Theorem 6.15]. Their result is a consequence of the fact that $(M_B, \Theta_0, f_B)$ can be given the structure of a *positive decorated geometric crystal*; we briefly outline their argument. By Theorem 4.3.30, the tropical locus $B_{f,\Theta_\mathbf{i}}$ is a Kashiwara crystal. The fibre of $\mathrm{Trop}(q)$ over $\lambda \in X^\vee(T)$ is shown to be a Kashiwara crystal isomorphic to $B(\lambda)$. They then use the following theorem of Joseph [114]: a family $\{\mathcal{C}_\lambda \mid \lambda \in X_+^\vee\}$ of highest weight crystals, so that $c_\lambda \in \mathcal{C}_\lambda$ is a unique highest weight element, is *closed* if, for any $\lambda, \mu \in X_+^\vee$, the correspondence $c_{\lambda+\mu} \mapsto (c_\lambda, c_\mu) \in \mathbb{C}_\lambda \otimes C_\mu$ extnds to an injective morphism of crystals $C_{\lambda+\mu} \to C_\lambda \otimes C_\mu$.

**Theorem 4.4.7** (Joseph, [114])**.** *If $\{\mathcal{C}_\lambda \mid \lambda \in X_+^\vee\}$ is a closed family of crystals then each $C_\lambda$ is isomorphic to $B(\lambda)$.*

Our contribution is the explicit identification of the fibre of $\mathrm{Trop}(q)$ with the set $B(\lambda)$ via the extended string cone and the identification of the $\lambda$-inequalities as the tropical locus of $f_{B,\mathbf{i}}^{(2)}$.

## 4.5 Future directions

In this final section we describe how the crystal structure appearing on the $B$-model side of mirror symmetry of partial flag varieties $X = {}^LG/{}^LP$ plays a conjectural organisational role with regards to certain integrable systems appearing on the $A$-model side of mirror symmetry for symplectic reductions of $X$. This will be the focus of future work. We focus on the case of polygon spaces $\mathcal{P}_{r,n}$ to be explicit. For background on completely integrable systems see [60].

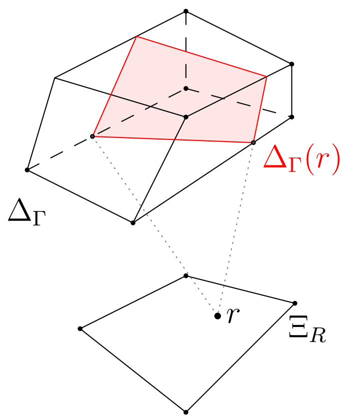


Figure 4.1: Relation between the moment polytopes $\Xi$, $\Delta_\Gamma$ and $\Delta_\Gamma(r)$.

Recall Examples 2.2.4, 2.4.6 and the construction of the polygon space $\mathcal{P}_{r,n}$ by symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2,n)$. Assume that $R \coloneqq |r| \in \mathbb{Z}_{>0}$, and suppose $\mathrm{Gr}_{\mathbb{C}}(2,n) = \mathrm{PGL}_n(\mathbb{C})/{}^L P$ admits Kahler form corresponding to $R\alpha_2^\vee$.

In [113, Section 3], Noharu-Ueda construct a family of completely integrable systems $\Psi_\Gamma : \mathrm{Gr}_{\mathbb{C}}(2,n) \to \mathbb{R}^{2(n-2)}$ parameterised by triangulations $\Gamma$ of some fixed $n$-gon $\Pi$. The functions in $\Psi_\Gamma$ are in bijection with $n-1$ consecutive edges of $\Pi$ and the $n-3$ diagonals defining $\Gamma$. Moreover, the integrable system $\Psi_\Gamma$ descends to an integrable system $\Phi_\Gamma : \mathcal{P}_{r,n} \to \mathbb{R}^{n-3}$ of the symplectic reduction $\mathcal{P}_{r,n}$ (these are the *bending systems* in [63]).

Let $\Delta_\Gamma \subseteq \mathbb{R}^{2(n-2)}$ be the moment polytope of the integrable system $\Psi_\Gamma$. Let $(u_1, \ldots, u_{n-1})$ be the coordinates on $\mathbb{R}^{2(n-2)}$ corresponding to $(n-1)$ consecutive edges, and $(v_1, \ldots, v_{n-3})$ the coordinates corresponding to the diagonals in $\Gamma$. Then, the moment polytope of $\Phi_\Gamma$ is shown to be the following subset of $\Delta_\Gamma$

$$\Delta_\Gamma(r) \coloneqq \{(u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-3}) \mid (u_1, \ldots, u_{n-1}, |r| - \sum_{i=1}^{n-1} u_i) = r\}$$

Recall the moment polytope $\Xi_R$ from Example 2.4.6. Thus, $r \in \Xi_R$. The above discussion is summarised in Figure 4.1.

For a particular triangulation $\Gamma_0$ of $\Pi$, Noharu-Ueda show ([113, Example 4.1]) that $\Psi_{\Gamma_0}$ is equivalent to the Gelfand-Tsetlin system [61] and that the moment polytope $\Delta_{\Gamma_0}$ is equivalent to a Gelfand-Tsetlin polytope $\mathrm{GT}_P^{(R)}$ consisting of all Gelfand-Tsetlin patterns

$$\begin{array}{cccccccc}
R & & \lambda_2^{(n-1)} & & & & & \\
 & \lambda_1^{(n-1)} & & \lambda_2^{(n-2)} & & & & \\
 & & \ddots & & \ddots & & & \\
 & & & \lambda_1^{(3)} & & \lambda_2^{(2)} & & 0 \\
 & & & & \lambda_1^{(2)} & & \lambda_2^{(1)} & \\
 & & & & & \lambda_1^{(1)} & &
\end{array} \tag{4.5.1}$$

Recall that each subtriangular array

$$\begin{array}{ccc}
\lambda_1^{(j)} & & \lambda_2^{(j-1)} \\
 & \lambda_1^{(j-1)} &
\end{array}$$

corresponds to the relation $\lambda_1^{(j)} \geq \lambda_1^{(j-1)} \geq \lambda_2^{(j-1)}$.

Recall the mirror family $(M_P, f_P)$ and explicit formula for the superpotential $f_P$ from Section 3.4. By Proposition 3.4.9, there is a monomial transformation of $(\mathbb{C}^\times)^{2(n-2)}$ such that the superpotential takes the form

$$f_P = \sum_{a \in \mathrm{GT}_P} \frac{z_{h(a)}}{z_{t(a)}}. \tag{4.5.2}$$

Here $\mathrm{GT}_P$ is the Gelfand-Tsetlin quiver of shape $P$. The tropical locus of $f_P$ with respect to the $z$-coordinates is now seen to be precisely the space of Gelfand-Tsetlin patterns of shape $P$. Namely, $\mathrm{Trop}(f_P)(\lambda, Z_i^{(j)}) \geq 0$, where $(\lambda, Z_i^{(j)}) \in X^\vee(Z(L_P)) \times \mathbb{Z}^{2(n-1)}$, if and only if

$$\begin{array}{cccccccc}
\lambda & & Z_2^{(n-1)} & & & & & \\
 & Z_1^{(n-1)} & & Z_2^{(n-2)} & & & & \\
 & & \ddots & & \ddots & & & \\
 & & & Z_1^{(3)} & & Z_2^{(2)} & & 0 \\
 & & & & Z_1^{(2)} & & Z_2^{(1)} & \\
 & & & & & Z_1^{(1)} & &
\end{array}$$

is a Gelfand-Tsetlin pattern. In particular, fixing $\lambda = R\alpha_2^\vee$ we obtain the Gelfand-Tsetlin patterns in (4.5.1).

We now describe a project for further research.

Recall the quantum structure map $q$ and the equivariant structure map $e$ for $(M_B, L_B)$ (Definition 3.1.3). Let $\lambda = R\alpha_2^\vee \in X^\vee(L_P) \cap X_+^\vee$. Then, by Theorem 4.4.5, $\mathrm{Trop}(q)$ is the highest weight map for the extended string cone, and $\mathrm{Trop}(e)$ is the weight map. By [1, Section 5], $\Delta_\lambda \coloneqq \mathrm{Trop}^{-1}(\lambda)$ is equivalent to the polytope $\mathrm{GT}_P^{(R)}$. Hence, $\Delta_\lambda$ can be identified with $\Delta_{\Gamma_0}$. Now, $\mathrm{Trop}(e)(\Delta_\lambda)$ is the convex hull of $W \cdot \lambda \subseteq \mathbb{R}X$. By Theorem 2.3.3, this is precisely the moment polytope $\Xi_R$. This situation (occuring on the $B$-model side) is similar to that described in Figure 4.1 (on the $A$-model side). This motivates the following conjecture.

**Conjecture 4.5.1.** Under the identification $\Delta_\lambda = \Delta_{\Gamma_0}$, the moment polytope $\Delta_{\Gamma_0}(r) \subseteq \Delta_{\Gamma_0}$ is equal to $\mathrm{Trop}^{-1}(\hat{r}) \cap \Delta_\lambda$

Conjecture 4.5.1 is similar to work of Rietsch-Williams [117].

If $\hat{r}$ is integral then we have the following heuristic interpretation of Conjecture 4.5.1: the lattice points of the moment polytope $\Delta_{\Gamma_0}(r)$ of the weight variety $\mathcal{P}_{r,n}$ is equal to the dimension of the $\hat{r}$-weight space in the irreducible representation $V(R\alpha_2^\vee)$ of $\mathrm{PGL}_n$. Similar results have been obtained via different methods in [62].

Additionally, it would be interesting to determine if this observation can be extended to more general weight varieties.

# Bibliography


[1] Valery Alexeev and Michel Brion. “Toric degenerations of spherical varieties”. In: *Sel. Math.* 10.4 (2004), pp. 453–478.

[2] Michael Francis Atiyah. “Convexity and commuting Hamiltonians”. In: *Bulletin of the London Math. Soc.* 14 (1982), pp. 1–15.

[3] Michèle Audin. *Torus actions on symplectic manifolds.* 2nd Ed. Birkhäuser, 2004.

[4] Denis Auroux. “Mirror symmetry and T-duality in the complement of an anticanonical divisor”. In: *J. of Gök. Geom. Top.,* 1 (2007), pp. 51–91.

[5] K. Baclawski. “A new rule for computing Clebsch-Gordan series”. In: *Adv. in Appl. Math.* 5 (1984), pp. 416–432.

[6] Victor Batyrev. “Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties”. In: *J. Algebraic Geometry* 3 (1994), pp. 493–535.

[7] Victor Batyrev. “Toric degenerations of Fano varieties and constructing mirror manifolds”. In: *The Fano Conference.* Univ. Torino, Turin. 2004, pp. 109–122.

[8] Victor Batyrev and Lev Borisov. “On Calabi-Yau complete intersections in toric varieties”. In: *Higher dimensional complex varieties.* de Gruyter, 1996, pp. 39–65.

[9] Victor Batyrev et al. “Conifold Transitions and Mirror symmetry for Calabi-Yau complete intersections in Grassmannians”. In: *Nuclear Phys. B* 514 (1998), pp. 640–666.

[10] Victor Batyrev et al. “Mirror symmetry and toric degnenerations of partial flag manifolds”. In: *Acta Math.* (2000), pp. 1–39.

[11] Kai Behrend. “Localization and Gromov-Witten invariants”. In: *Quantum Cohomology.* Vol. 1776. Lecture Notes in Math. Springer, 2002, pp. 3–38.

[12] Arkady Berenstein and David Kazhdan. “Geometric and unipotent crystals”. In: *Geom. Funct. Anal. Special Volume* Part I (2000), pp. 188–236.

[13] Arkady Berenstein and David Kazhdan. “Geometric and unipotent crystals II: From unipotent bicrystals to crystal bases”. In: (2007).

[14] Arkady Berenstein and Andrei Zelevinsky. “Tensor product multiplicities, canonical bases and totally positive varieties”. In: *Inventiones Mathematicae* 143 (2001), pp. 77–128.

[15] Arkady Berenstein and Andrei Zelevinsky. "Total positivity in Schubert varieties". In: *Comment. Math. Helv.* 72 (1997), pp. 128–166.

[16] Roman Bezrukavnikov, Michael Finkelberg, and Ivan Mirković. "Equivariant (K-)homology of affine Grassmannian and Toda lattice". In: *Compos. Math.* 141 (2005), pp. 746–768.

[17] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups.* Vol. 231. Graduate Texts in Mathematics. Springer, 2005.

[18] Nicolas Bourbaki. *Lie groups and Lie algebras. Ch. 4-6.* Elements of Mathematics. Springer-Verlag, 2002.

[19] A. Braverman and D. Gaitsgory. "Crystals via the affine Grassmannian". In: *Duke Math. J.* 107 (2001), pp. 561–575.

[20] A. Braverman and David Kazhdan. "$\gamma$-functions of representations and lifting, Visions in Mathematics". In: Modern Birkhäuser Classics. Birkhäuser, 2000, pp. 237–278.

[21] Michel Brion. "Lectures on the geometry of flag varieties". In: *Topics in Cohomological Studies of Algebraic Varieties: Impanga Lecture Notes.* Birkhäuser, 2005, pp. 33–85.

[22] Ben Brubaker, Daniel Bump, and Solomon Friedberg. "Weyl group multiple Dirichlet series, Eisenstein series and crystal bases". In: *Annals of Math.* 173 (2011), pp. 1081–1120.

[23] Daniel Bump and Anne Schilling. *Crystal bases: representations and combinatorics.* World Scientific, 2017.

[24] Philippe Caldero. "Toric degenerations of Schubert varities". In: *Trans. Groups* 7 (2002), pp. 51–60.

[25] P. Candelas, M. Lynker, and R. Schimmrigk. "Calabi-Yau manifolds in weighted $\mathbb{P}_4$". In: *Nuclear Phys. B* 341 (1990), pp. 383–402.

[26] Philip Candelas et al. "A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory". In: *Nuclear Phys. B* 359 (1991), pp. 21–74.

[27] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups.* Cambridge University Press, 1994.

[28] Reda Chhaibi. "Littelmann path model for geometric crystals". arXiv:1405.6437.

[29] I. Ciocan-Fontanine. "The quantum cohomology ring of flag varieties". In: *Trans. Am. Math. Soc.* 351.7 (1999), pp. 2695–2729.

[30] David Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry.* Vol. 68. Math. Surv. & Mon. American Math. Society, 1999.

[31] Bruce Crauder and Rick Miranda. "Quantum cohomology of rational surfaces". In: *The moduli space of curves.* Progress in Mathematics. Birkhäuser, 1995, pp. 34–80.

[32] Bangming Deng et al. *Finite dimensional algebras and quantum groups.* Vol. 150. Math. Surv. & Mon. American Math. Society, 2008.

[33] Igor Dolgachev. *Lectures on invariant theory.* London Math. Soc. Lect. Notes. Cambridge University Press, 2003.

[34] Johannes Jisse Duistermaat and Johan A.C. Kolk. *Lie Groups.* Universitext. Springer, 2000.

[35] T. Eguchi, Kentaro Hori, and X.-S. Xiong. "Gravitational quantum cohomology". In: *Internat. J. Modern Phys. A* 12 (1997), pp. 1743–1782.

[36] Sergei Fomin and Andrei Zelevinsky. "Double Bruhat cells and total positivity". In: *J. Amer. Math. Soc.* 12 (1999), pp. 335–380.

[37] Philip Foth. "Moduli spaces of polygons and punctured Riemann spheres". In: *Canad. Math. Bull.* 43.2 (2000), pp. 162–173.

[38] Philip Foth and Yi Hu. "Toric degenerations of weight varieties and applications". Trav. Math., XVI. 2005.

[39] Kenji Fukaya et al. *Lagrangian Floer theory and mirror symmetry on compact toric manifolds.* Vol. 376. Astérisque. Soc. Math. de France, 2016.

[40] Kenji Fukaya et al. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I.* Studies in Adv. Math. American Math. Society, 2009.

[41] Kenji Fukaya et al. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II.* Studies in Adv. Math. American Math. Society, 2009.

[42] William Fulton and Rahul Pandharipande. "Notes on stable maps and quantum cohomology". In: *Algebraic Geometry, Santa Cruz 1996.* American Math. Society, 1997, pp. 45–96.

[43] I.M. Gelfand and Robert MacPherson. "Geometry in Grassmannians and a generalization of the dilogarithm". In: *Adv. in Math.* 44 (1982), pp. 279–312.

[44] I.M. Gelfand and Andrei Zelevinsky. "Multiplicites and regular basis for $\mathfrak{gl}_n$". In: *Group Theoretical Methods in Physics. Proc. Third Seminar.* Vol. 2. 1985, pp. 22–31.

[45] I.M. Gelfand and Andrei Zelevinsky. "Polytopes in the pattern space and canonical bases for irreducible representations of $\mathfrak{gl}_3$". In: *Group Theoretical Methods in Physics. Proc. Third Seminar.* Vol. 2. 1985, pp. 32–45.

[46] I.M. Gelfand, Andrei Zelevinsky, and M. Kapranov. "Hypergeometric functions and toric varieties". In: *Funct. Anal. Appl.* 23.2 (1989), pp. 94–106.

[47] Alexander Givental. "Equivariant Gromov-Witten invariants". In: *Int. Math. Res. Not.* 13 (1996), pp. 613–663.

[48] Alexander Givental. "Homological geometry and mirror symmetry". In: *Proceedings of the ICM Zurich 1994.* Birkhäuser, 1995.

[49] Alexander Givental. "Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture". In: *Topics in Singularity Theory.* Amer. Math. Soc. Trans. Ser. 2. American Math. Society, 1997.

[50] Alexander Givental and Bumsig Kim. “Quantum cohomology of flag manifolds and Toda lattices”. In: *Comm. Math. Phys* 168.3 (1995), pp. 609–641.

[51] Rebecca Goldin. “The cohomology ring of weight varieties and polygon spaces”. In: *Adv. in Math.* 160.2 (2001), pp. 175–204.

[52] Rebecca Goldin, Tara Holm, and Allen Knutson. “Orbifold cohomology of torus quotients”. In: *Duke Math. J.* 139.1 (2007), pp. 89–139.

[53] Rebecca Goldin and A.-L. Mare. “Cohomology of symplectic reductions of generic coadjoint orbits”. In: *Proc. Amer. Math. Soc.* 132.10 (2004), pp. 3069–3074.

[54] N. Gonciulea and V. Lakshmibai. “Degenerations of flag and Schubert varieties to toric varieties”. In: *Trans. Groups* 1.3 (1996), pp. 215–248.

[55] Brian Greene and M. Ronen Plesser. “Duality in Calabi-Yau moduli space”. In: *Nuclear Phys. B* 338.1 (1990), pp. 15–37.

[56] I Grojnowski and George Lusztig. “A comparison of bases of quantized enveloping algebras”. In: *Contemp. Math.* 153 (1993), pp. 11–19.

[57] Mark Gross. “Mirror symmetry and the Strominger-Yau-Zaslow conjecture”. In: *Current Developments in Math.* (2012), pp. 133–191.

[58] Victor Guilllemin, Eugene Lerman, and Shlomo Sternberg. *Symplectic Fibrations and Multiplicity Diagrams.* Cambridge University Press, 1996.

[59] Victor Guilllemin and Shlomo Sternberg. “Convexity properties of the moment map”. In: *Inventiones Mathematicae* 67 (1982), pp. 515–38.

[60] Victor Guilllemin and Shlomo Sternberg. *Symplectic Techniques in Physics.* Cambridge University Press, 1984.

[61] Victor Guilllemin and Shlomo Sternberg. “The Gel’fand-Cetlin system and quantization of the complex flag manifolds”. In: *Journal of Funct. Anal.* 52.1 (1983), pp. 106–128.

[62] Megumi Harada and Kiumars Kaveh. “Toric degenerations, integrable systems and Newton-Okounkov bodies”. In: *Inventiones Mathematicae* 202.3 (2015), pp. 927–985.

[63] Jean-Claude Hausmann and Allen Knutson. “Polygon spaces and Grassmannians”. In: *Ensign. Math.* 43 (1997), pp. 173–198.

[64] Jean-Claude Hausmann and Allen Knutson. “The cohomology ring of polygon spaces”. In: *Ann. de l’Ins. Four.* 48.1 (1998), pp. 281–321.

[65] Gert Heckman. “Projections of Orbits and Asymptotic Behaviour of Multiplicities for Compact Lie Groups”. PhD thesis. University of Leiden, 1980.

[66] Jin Hong and Seok-Jin Kang. *Introduction to Quantum Groups and Crystal Bases.* American Math. Society, 2002.

[67] Kentaro Hori et al. *Mirror symmetry.* Vol. 1. CMIM. American Math. Society, 2003.

[68] James E. Humphreys. *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics 9. Springer, 1972.

[69] Jens Carsten Jantzen. *Lectures on quantum groups.* Vol. 6. Graduate Studies in Mathematics. American Math. Society, 1996.

[70] Jens Carsten Jantzen. *Representations of Algebraic Groups.* 2nd Ed. Vol. 107. Math. Surv. & Mon. American Math. Society, 2003.

[71] Anthony Joseph. "Consequences of the Littelmann path theory for the structure of the Kashiwara $B(\infty)$ crystal". In: *Highlights in Lie algebraic methods.* Ed. by Anthony Joseph, A. Melnikov, and I. Penkov. Vol. 295. Progress in Mathematics. Birkhäuser, 2012, pp. 25–64.

[72] Joel Kamnitzer. "The crystal structure on the set of Mirkovic-Vilonen polytopes". In: *Adv. in Math.* 215 (2007), pp. 66–93.

[73] Anton Kapustin and Lev Rozansky. "Three-dimensional topological field theory and symplectic algebraic geometry II". In: *Commun. Number Theory Phys.* 4 (2010), pp. 463–549.

[74] Anton Kapustin, Lev Rozansky, and Natalia Saulina. "Three-dimensional topological field theory and symplectic algebraic geometry". In: *Nuclear Phys. B* 816 (2009), pp. 295–355.

[75] Masaki Kashiwara. "On crystal bases". In: *Representations of Groups, Proc. Canadian Math. Soc.* American Math. Society, 1995, pp. 155–197.

[76] Masaki Kashiwara. "On crystal bases of the Q-analogue of universal enveloping algebras". In: *Duke Math. J.* 63.2 (1991), pp. 465–516.

[77] Masaki Kashiwara. "The crystal base and Littlemann's refined Demazure character formula". In: *Duke Math. J.* 71.3 (1993), pp. 839–858.

[78] Masaki Kashiwara and T. Nakashima. "Crystal graphs for representations of the $q$-analogue of classical Lie algebras". In: *J. of Algebra* 165 (1994), pp. 295–345.

[79] Masaki Kashiwara and Yoshihisa Saito. "Geometric construction of crystal bases". In: *Duke Math. J.* 89.1 (1997), pp. 9–36.

[80] L. Katzarkov, Maxim Kontsevich, and Tony Pantev. "Bogomolov-Tian-Todorov theorems for Landau-Ginzburg models". arXiv:1409.5996.

[81] Sean Keel and Jenia Tevelev. "Geometry of Chow quotients and Grassmannians". In: *Duke Math. J.* 134.2 (2006), pp. 259–311.

[82] Bumsig Kim. "On equivariant quantum cohomology". In: *Int. Math. Res. Not.* 17 (1996), pp. 841–851.

[83] Bumsig Kim. "Quantum cohomology of flag manifolds $G/B$ and quantum Toda lattices". In: *Annals of Math.* 149.1 (1999), pp. 129–148.

[84] Frances Kirwan. *Cohomology of quotients in symplectic and algebraic geometry.* Vol. 31. Math. Notes. Princeton University Press, 1984.

[85] Alexander A. Klyachko. "Spatial polygons and stable configurations of points in the projective line". In: *Algebraic Geometry and Its Applications.* Ed. by Alexander Tikhomirov and Andrej Tyurin. Vol. 25. Aspects of Math. Springer, 1994, pp. 67–84.

[86] Allen Knutson. "Weight Varieties". PhD thesis. M.I.T., 1996.

[87] Allen Knutson, Thomas Lam, and David Speyer. "Projections of Richardson varieties". In: *J. Reine Angew. Math.* 687 (2014), pp. 133–157.

[88] Maxim Kontsevich. "Homological algebra of mirror symmetry". In: *Proceedings of the ICM Zurich 1994.* Birkhäuser, 1995, pp. 120–139.

[89] Bertram Kostant. "A formula for the multiplicity of a weight". In: *Trans. Am. Math. Soc.* 93 (1959), pp. 53–73.

[90] Bertram Kostant. "On convexity, the Weyl group and the Iwasawa decomposition". In: *Ann. Sci. Ec. Norm. Sup.* 6 (1973), pp. 413–55.

[91] Bertram Kostant. "Quantization and representation theory I: prequantization". In: *Lecture in modern analysis and applications III.* Vol. 170. Lecture Notes in Math. Springer, 1970.

[92] V. Lakshmibai and K. Raghavan. *Standard monomial theory: an invariant theoretic approach.* Enc. Math. Sci. Springer-Verlag, 2008.

[93] Thomas Lam. "Whittaker functions, geometric crystals and quantum Schubert calculus". arXiv:1308.5451.

[94] Thomas Lam and Nicolas Templier. "The mirror conjecture for miniscule flag varieties". arXiv:1705.00758.

[95] Peter Littelmann. "Cones, crystals, and patterns". In: *Trans. Groups* 3.2 (1998), pp. 145–179.

[96] Peter Littelmann. "Paths and root operators in representation theory". In: *Annals of Math.* 142 (1995), pp. 499–525.

[97] George Lusztig. "Canonical bases arising from quantized enveloping algebras". In: *J. Amer. Math. Soc.* 3.2 (1990), pp. 447–498.

[98] George Lusztig. "Canonical bases arising from quantized enveloping algebras II". In: *Prog. Theor. Phys. Supp.* 102 (1990), pp. 175–201.

[99] George Lusztig. *Introduction to quantum groups.* Vol. 110. Progress in Mathematics. Birkhäuser, 1993.

[100] George Lusztig. "Quantum deformations of certain simple modules over enveloping algebras". In: *Adv. in Math.* 70 (1988), pp. 237–249.

[101] George Lusztig. "Quantum groups at roots of 1". In: *Geom. Ded.* 35 (1990), pp. 89–113.

[102] George Lusztig. "Quivers, perverse sheaves, and quantized enveloping algebras". In: *J. Amer. Math. Soc.* 4.2 (1991), pp. 365–421.

[103] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry.* Vol. 161. Graduate Studies in Mathematics. American Math. Society, 2015.

[104] Jerrold Marsden and Alan Weinstein. "Reduction of symplectic manifolds with symmetry". In: *Reports on Mathematical Physics* 5 (1974), pp. 121–30.

[105] Robert Marsh and Konstanze Rietsch. "The $B$-model connection and mirror symmetry for Grassmannians". arXiv:1307.1085.

[106] Olivier Mathieu. "Good bases for $G$-modules". In: *Geom. Ded.* 36 (1990), pp. 51–66.

[107] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* 2nd Ed. Oxford University Press, 2005.

[108] P. McMullen. "The polytope algebra". In: *Adv. in Math.* 78 (1989), pp. 76–130.

[109] Kenneth Meyer. "Symmetries and integrals in mathematics". In: *Dynamical Systems.* Academic Press, New York, 1973, pp. 259–272.

[110] David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory.* 3rd Ed. Vol. 34. Ergebnisse der Math. und ihrer Grenzgebiete. 2. Folge 2. Springer-Verlag, 1994.

[111] T. Nakashima. "Geometric crystals on Schubert varieties". In: *J. Geom. Phys.* 53 (2005), pp. 197–225.

[112] Takeo Nishinou, Yuichi Nohara, and Kazushi Ueda. "Toric degenerations of Gelfand-Cetlin systems and potential functions". In: *Adv. in Math.* 224 (2010), pp. 648–706.

[113] Yuichi Nohara and Kazushi Ueda. "Toric degenerations of integrable systems on Grassmannians and polygon spaces". In: *Nagoya Math. J.* 214 (2014), pp. 125–168.

[114] *Quantum groups and their primitive ideals.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1995.

[115] Konstanze Rietsch. "A mirror symmetric construction of $qH^*_T(G/P)_{(q)}$". In: *Adv. in Math.* 217 (2008), pp. 2401–2442.

[116] Konstanze Rietsch. "A mirror symmetric solution to the quantum Toda lattice". In: *Comm. Math. Phys* 309.1 (2012), pp. 23–49.

[117] Konstanze Rietsch and Lauren Williams. "Cluster duality and mirror symmetry for Grassmannians". arXiv:1507.07817.

[118] Marc Rosso. "Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra". In: *Comm. Math. Phys* 117 (1988), pp. 581–593.

[119] Yoshihisa Saito. "PBW basis of quantized universal enveloping algebras". In: *Publ. Res. Inst. Math. Sci.* 30.2 (1994), pp. 209–232.

[120] Jean-Marie Souriau. *Structure des Systèmes Dynamiques*. Dunod Paris, 1970.

[121] David Speyer and Bernd Sturmfels. "The tropical Grassmannian". In: *Adv. Geom.* 4 (2004), pp. 389–411.

[122] Tonny Albert Springer. *Linear algebra groups*. 2nd Ed. Birkhäuser, 1998.

[123] A. Strominger, Sing-Tung Yau, and Eric Zaslow. "Mirror symmetry is T-duality". In: *Nuclear Phys. B* 479 (1996), pp. 243–259.

[124] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*. Univ. Lect. Notes. American Math. Society, 1996.

[125] Constantin Teleman. *Gauge theory and mirror symmetry*. 2014. URL: arxiv.org/abs/1404.6305.