Teaching Portfolio

George Melvin
Middlebury College

Table of Contents

A. Teaching Statement

As an undergraduate, the realisation that mathematics is a creative subject captivated my attention. By engaging with theorems and abstract definitions, I began to understand *what* it was I had been doing in my previous mathematics classes. As my friends and I began to develop our mathematical knowledge, we would excitedly discuss and explain our points of view to each other. The challenge of communicating my knowledge to my peers allowed me to engage with the material more deeply than studying on my own, and provided me with the confidence to express myself to my professors and peers outside the classroom.

Despite my own learning experiences, the idea of `*learning through discussion'* is not how I first approached teaching. When I began to teach mathematics, I relied heavily on lecturing and I tried to tell my students the story I had developed of a topic, expecting that the insights I had acquired would aid their learning experience. I discovered quickly that students needed to build their own understanding, and that being an effective educator required a broad and more interactive learning environment.

The predominant focus of my pedagogical development has been on providing more opportunities for students to interact with their studies through their peers. I have the strong belief that actively discussing mathematics engages student learning, promotes effective communication skills, and facilitates intellectual and personal inquiry. I have experimented with incorporating a more active approach to learning in several ways that I outline below.

At Middlebury College I have been investigating the role of discussion in the classroom in a Calculus II course I have been leading. At the beginning of every class I supply students with a worksheet-lecture note packet that I have created, containing both problem-solving and lecture note elements. Using these packets has allowed me to flip the role of problem-solving and note-taking in the classroom. In this way, the classroom experience is primarily guided by the students, who will investigate a new topic by working through a carefully chosen example in small discussion groups. Using their observations, students will be asked to `*Fill in the Blanks!*' to complete the statement of a theorem, or `*Get Creative!*' and provide their own conjectural mathematical proposition. As a class, we will then summarise the correct statement and invite student comments on comparison or clarification.

For example, in small groups students investigated a formula for the arc-length of a segment of a graph by deriving a formula for the arc-length of a piecewise linear function and then considering how the Mean Value Theorem could be applied to realise their formula as a Riemann sum. By initially engaging a new concept through problem-solving and discussion in this way, I believe that students find the foundation of abstract concepts and formulae more accessible and develop a deeper learning experience. Additionally, students gain confidence in expressing their knowledge by sharing their investigations with their peers.

An essential aspect of my commitment to classroom discussion is ensuring dialogue is approachable, tractable and a comfortable experience for all. I actively promote an interactive and open dialogue with, and among, students and look to engage them with the material in an informal manner. To this end, I have adopted several aspects of *small teaching* methods in my classroom. For example, I promote the use of a *learning diary* for students: students are required to bring a pocket notebook to class and at the end of each class I ask that they brainstorm with a neighbour and write down `*At least three new topics, ideas or examples that you have seen today. Of these topics, ideas or examples, which did you find more approachable; which did you find less approachable? Did you meet any examples that you think are important? If so, why do you think they are important?*' Allowing for immediate reflection at the end of class allows students to summarise with their peers that day's learning outcomes and promotes and initiates opportunities for student interaction outside the classroom.

In addition to providing activities allowing for active discussion in the classroom, I believe that it is important for students to be able to pursue their intellectual curiosity. I have endeavored to provide motivated students with research opportunities at all levels of the undergraduate mathematics curriculum, and I have incorporated elements of independent study and project-based group work into student assessment for a wide range of courses.

I have incorporated project-based group work as a part of assessment for entry-level and advanced courses I have taught at Middlebury College and Harvard University. In both settings, I placed a strong emphasis on presentation and exposition, asking for students to ensure that their work was accessible to non-experts. Prompting students to communicate their knowledge in this way requires students to engage with their understanding of a topic in a different manner: it is a distinct challenge to communicate to a non-mathematician the statement and solution of a problem. I believe this important transferable component of project-based assessment allows all participants to be engaged with the material, leads to a broader understanding of the topic at hand, and furnishes student growth outside the classroom. Students have been very receptive to these project-based activities and I am excited to investigate how to develop their role in future courses. In particular, in the multivariable calculus course I am currently teaching students will be required to submit a *Mathematica* project investigating the Frenet-Serret frame for space curves.

- At Middlebury, students in a Calculus II course were asked to work in small groups and create a poster outlining the development of a historically significant result. I adapted and created three projects for students to work through: *Newton's Development of the Binomial Series*; *Wallis and a Product Formula for Pi*; *Bernoulli, Euler and the Basel Problem*. Each project included opportunities for students to build on mathematics they had seen in class and investigate the history of a significant development in calculus. An essential requirement of the project was that groups had to create their poster *by hand*. In this way, students had to sit down together in discussion and think about how they could be visually and intellectually creative with the presentation of their material.
- At Harvard, students in an upper-division undergraduate *Smooth Manifolds* course were required to submit two papers during the semester based on a selection of further topics I had provided to them. I asked that students summarise technical arguments with the use of a conceptual diagram, or illuminating example, rather than a wrote exposition of a proof. It was important that students have the opportunity to submit several drafts to me on which I would provide feedback or point them towards lines of further enquiry based on their interests and strengths.

The teaching and mentoring roles I have undertaken have affirmed the great responsibility that college educators undertake to guide students toward their academic and personal goals. Furthermore, these experiences have demonstrated to me that an education where student discussion and intellectual inquiry are actively supported promotes an engaging and rewarding learning environment. As a consequence, I am committed to continuing my career at an institution where teaching is deeply valued and a rewarding student experience is a primary function. I am excited to continue my development as an educator of college-level mathematics students and I look forward to the challenge of initiating successful interactive learning environments in the future.

B. Teaching & Mentoring Responsibilities

I have extensive teaching experience from my time as an instructor of college-level mathematics, and I have been given a broad range of teaching and mentoring responsibilities. I have led entry-level calculus sections, taught upper-level undergraduate courses, devised research-level graduate courses, and organised research opportunities in mathematics for undergraduates.

B1. Teaching Responsibilities

As a course instructor I am solely responsible for course design and content, lecturing material, constructing and administering exams, creating homework, quizzes and project-work, leading discussion sections, and holding office hours.

I have been the instructor for the following courses at Middlebury College:

- *Multivariable Calculus (Math 223):* Spring 2018
  Lower-division undergraduate mathematics course.
- *Calculus II (Math 122): Fall 2017, Spring 2018*
  Lower-division undergraduate mathematics course.

I have been the instructor for the following courses at Harvard University:

- *Topology II: Smooth Manifolds (Math 132):* Spring 2017
  Upper-division undergraduate mathematics course.
- *Canonical Bases and Geometry (Math 284):* Spring 2017
  Graduate-level ‘Topics in Mathematics’ course

I have been the instructor for the following courses at the University of California, Berkeley:

- *Linear Algebra & Differential Equations (Math 54):* Summer 2010, 2011
  Lower-division undergraduate mathematics course.
- *Linear Algebra (Math 110):* Summer 2012
  Upper-division undergraduate mathematics course.
- *Abstract Algebra (Math 113):* Summer 2014
  Upper-division undergraduate mathematics course.

In addition, I have been a Teaching Fellow at Harvard University, a position that combines the role of instructor and discussion leader. I led sections in both instruction and discussion, being responsible for the delivery of course content, facilitating interactive discussion and problem-solving, constructing worksheets, and holding office hours. I have been a Teaching Fellow for the following courses at Harvard University:

- *Multivariable Calculus (Math 21A):* Fall 2016
  Lower-division undergraduate mathematics course.

I served as a Graduate Student Instructor (GSI) at the University of California, Berkeley, for a variety of lower- and upper-division mathematics courses:

- *Calculus II (Math 1B), Multivariable Calculus (Math 53), Linear Algebra & Differential Equations (Math 54), Linear Algebra (Math 110), Abstract Algebra (Math 113), Graduate Algebra (Math 250A).*

As a GSI I provide assistance and support to professor-led lectures. I am responsible for leading discussion sections 1-3 times per week, assessing student work, holding office hours and review sessions, and providing academic support to students. In this role I have encountered a broad range of student (~700 students over nine semesters) from a broad range of academic and socio-economic backgrounds.

B2. Mentoring Responsibilities

I have undertaken several different mentoring roles for students with a variety of mathematical backgrounds. In this section I will outline these roles in further detail.

As a graduate student at U. C. Berkeley, I was a graduate mentor in the Directed Reading Program (DRP) for several semesters. The DRP pairs motivated undergraduates at the University of California, Berkeley, with graduate mentors for a semester-long program of supervised individual learning. I supervised several undergraduate students in semester-long programs of independent study on topics in pure mathematics including:

- *representation theory of the symmetric group;*
- *Hilbert's Nullstellensatz and algebraic geometry;*
- *representation theory of finite-dimensional complex semisimple Lie algebras;*
- *representation theory of quivers, the Ringel-Hall algebra, and the canonical basis.*

I would arrange manageable (bi)weekly reading assignments supplemented with worksheets containing problem sets and discussions of proofs. At the end of the semester I prepare students for a 10-minute presentation of their work, offering advice on effective presentation skills.

In addition, in 2015 I organised a summer research experience at the University of California, Berkeley, for exceptional undergraduates from across the U.S., leading a group of six students on a graduate-level program of research. I provided appropriate preparation and mathematical background to students through daily lectures and problem-solving sessions. During the latter stage of the program I was responsible for providing academic support and encouragement in a research environment, supervising students in writing up their programs of research.

I have also focused on research opportunities for students with a background in lower-level mathematics. At Middlebury College, I have been working with a freshman investigating rearrangements of conditionally convergent series and preparing them to present their results at an undergraduate-focused conference. My role is to provide appropriate questions and avenues of inquiry, and to facilitate my student's exposure to mathematical concepts that are traditionally seen in a first course in real analysis. This has been a fun project and was motivated by questions the student asked during office hours.

Each of these experiences have been an exciting and unique challenge for me, requiring a careful consideration of student ability and background to develop appropriate programs of research. I look forward to investigating how similar opportunities may be provided to future mathematics students with a broad range of mathematical backgrounds.

C. Teaching Methods

To determine how best to proceed in a teaching role, it is vital to understand the academic background of your students. Prior to the first classroom interaction, I distribute an anonymous online survey focusing on student background, expectations for the course, and what activities students feel aid their learning experience. For example, in an upper-division Abstract Algebra summer course I asked students: *Why were they taking the course? What exposure to upper-division mathematics did they have? Had they written formal proofs before?* The responses allow me to determine how much time I should invest on certain preliminary topics, as well as create appropriate homework sets at the beginning of the course.

In all teaching situations, I outline my expectations and goals with a clear and thorough syllabus (see Appendix 1). I ensure that grade composition is set out and I indicate the policy for allowed student collaboration. I include thoughts on what I expect from students in the classroom and provide suggestions for their classroom etiquette. I also ask students to read and sign an ‘Expectations & Commitment Pledge’.

It is important that students are presented with a well-structured, concise narrative, placing new topics in context with earlier material. At the beginning of a course, I issue a proposed outline for the daily program of topics to be discussed. In each lecture I provide a complete set of typed lecture notes, highlighting aspects of a new definition or result with a variety of clear examples and accentuating key points (see Appendices 2, 3). Any notes that I provide are made available at a frequently updated course website. Student have responded positively to the notes that I provide, and lecture notes for a linear algebra course I taught have been used as a supplementary reference by Professor D. Lewis at UC Santa Cruz (see Appendix 9).

For introductory calculus courses, I structure lecture notes as a composition of guided problem-solving exercises, making use of prerequisite mathematics to guide students towards a new Theorem or Definition. In this way, I aim to have students discover new concepts for themselves, thereby providing motivation for a new result that may otherwise seem mysterious. For example, when introducing the directional derivative in a Multivariable Calculus course at Harvard University, students worked on a problem that guided them towards a formulation of the relationship between the numerical value of a directional derivative at a point and the direction in which a function is increasing most rapidly (see Appendix 5). In doing so, students discovered a connection between the directional derivative and the (familiar) single variable derivative. As a result, students commented that they now understood *why* this connection should exist.

D. Representative Course Materials

It is important to be well-prepared in any teaching or mentoring role. Designing a course is a considerable challenge and requires me to think about the overall course structure well in advance. The same preparation must be applied to all distributed material and it is essential that anything provided to students is clearly presented.

In the Appendices I provide representative examples of the following material:

- the course syllabus for a Spring 2018 Multivariable Calculus course at Middlebury College (Appendix 1);
- notes from a lecture on Lagrange's Theorem (Appendix 2);
- a worksheet from a Fall 2017 Calculus II class at Middlebury College (Appendix 3);
- a worksheet from an undergraduate individual reading program in Fall 2015 on the representation theory of the symmetric group (Appendix 4);
- a worksheet from a Fall 2016 Multivariable Calculus class at Harvard University (Appendix 5)
- a final exam from a Fall 2017 Calculus II course at the Middlebury College (Appendix 6).

E. Evaluation of Teaching Effectiveness

Interacting with students in a variety of teaching and mentoring roles has presented many challenges for me. It is important that I am able to determine if I am engaging the students effectively. Obtaining student feedback through anonymous evaluations provides a measure of my effectiveness as a teacher and focuses my efforts to improve my teaching.

In this section I provide a summary of student evaluations and comments related to my teaching effectiveness. In recognition of my dedication to being an effective teacher and mentor, I have been awarded a *Certificate of Excellence in Teaching* from Harvard University, and an *Outstanding Graduate Student Instructor* award from the University of California, Berkeley (see Appendices 7, 8).

Student Comments:

- *"George is an amazing teacher! He explains concepts with precision and clarity. The worksheets are extremely useful!"*

  Directed Reading Program Student Fall 2014, University of California Berkeley

- *"Always comes to class very, very prepared; puts in an incredible amount of effort. George is extremely helpful and is always willing to answer questions in office hours, via e-mail, through Piazza, and in class. One of the best teachers I have ever had."*

  Math 110 Student Summer 2012, University of California, Berkeley

- *“You have a real talent for teaching, especially when it comes to understanding and relating to difficulties your students are having. You are able to explain things in a way that is very approachable and I can see you've put a lot of work into doing your job well.”*
  Math 110 student Fall 2015, University of California, Berkeley

- *“I really appreciated how George changed his teaching methods when we gave feedback on the things we wanted to see changed, and I think it was more effective when he did.”*
  Math 21A student Fall 2016, Harvard University

- *“George was very engaging and dedicated lecturer, and I really enjoyed having the opportunity to take this course with him. It's clear that he's put a lot of thought into his pedagogical technique and lecturing style, the latter being among of the best I've seen so far as an undergrad in the math department. He was always very available to students throughout the course.”*
  Math 132 student Spring 2017, Harvard University

- *“The instructor was very effective in explaining new concepts. His manner of teaching helps me understand the topics better. I also liked the class environment.”*
  Math 122 student Fall 2018, Middlebury College

<u>*Student Evaluations*</u>:

Course Instructor:

At the University of California, Berkeley, course instructors are evaluated on two criteria: the effectiveness of their teaching, and the overall course.

Evaluations scored on a 7 point scale: 1 (not at all effective) - 7 (extremely effective)

	Semester	No. of responses	Effectiveness	Overall	Ave.
Math 54	Summer 2010	28	6.2	5.9	**6.05**
Math 54	Summer 2011	36	6.3	6.2	**6.25**
Math 110	Summer 2012	19	6.5	6.4	**6.45**
Math 113	Summer 2014	18	6.4	6.4	**6.4**

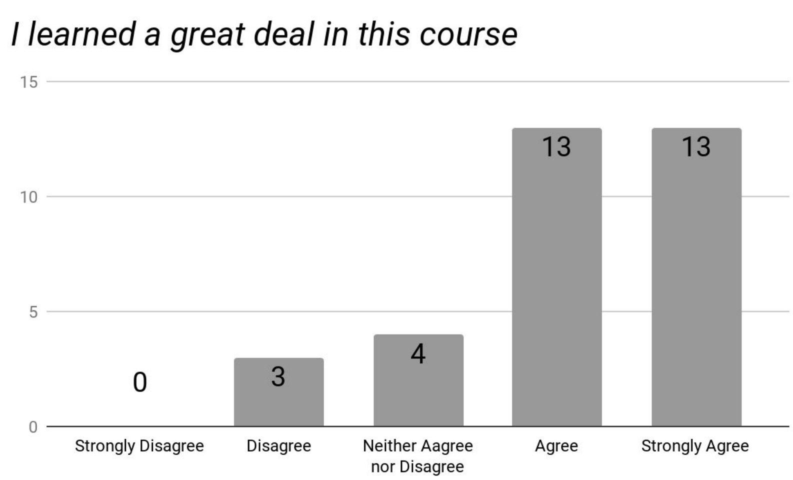
At Harvard University students are asked to respond to several prompts to evaluate course instructors. The following table shows the numerical evaluations I received from my students in my *Smooth Manifolds*, Spring 2017, course.

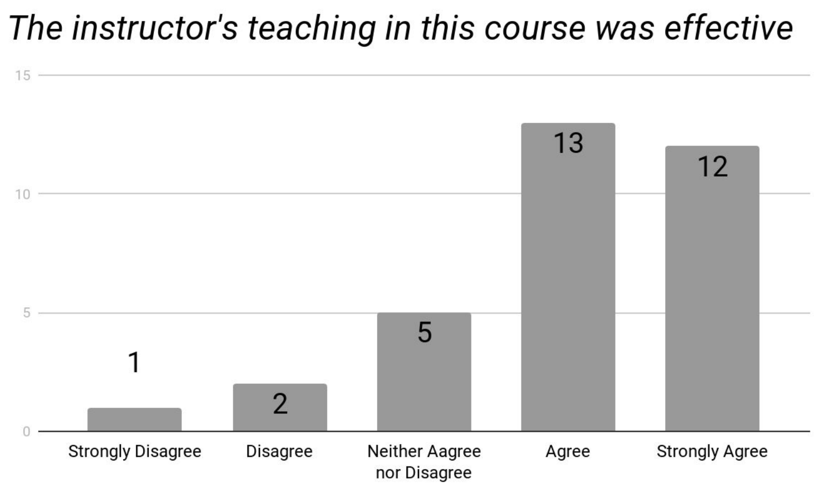
Evaluations scored on a 5 point scale:

1 = unsatisfactory, 2 = fair, 3 = good, 4 = very good, 5 = excellent.

	No. of Responses	Ave.
Evaluate your instructor overall	12	4.58
Gives effective lectures or presentations	12	4.42
Is accessible outside of class	12	4.92
Generates enthusiasm for the subject matter	12	4.75
Facilitates discussion and encourages participation	11	4.45

At Middlebury College students are asked to provide written responses to several questions related to course content and course instruction. Students are then asked to evaluate two Likert items. The following table shows responses I received from my students in my *Calculus II*, Fall 2017, course (34 responses).





F. Undergraduate Research

It can be difficult for undergraduate students to appreciate the process that research mathematicians undertake to ask questions that lead to interesting theorems. This is especially true in areas of pure mathematics, where no obvious physical situation may be generating mathematical problems. Summer research opportunities allow exceptional undergraduates to experience what it is like to *be* a mathematician. However, it is a challenge to provide such students with accessible problems that generate interesting results. Student backgrounds must be taken into account when devising research projects so that all participants are engaged and can gain a sense of achievement. Furthermore,

In 2015 I organised an 8-week summer research opportunity at the University of California, Berkeley, for undergraduates from across the United States. The program's aim was to provide exceptional undergraduates with research experience in geometry and topology, preparing prospective graduate students for research.

In addition to organising practical matters (national advertisement, participant selection, student housing, program finances), I supervised a group of six students on a program of research relating to toric degenerations of flag varieties. I worked with my PhD advisor to design suitable research projects that would be accessible to the participants. Some students wrote software packages to test conjectures while others considered theoretical problems. Providing different types of projects ensured students could engage with a research problem that aligned with their academic background.

At Middlebury College I have been investigating approaches to providing research opportunities to undergraduates with backgrounds in lower-level mathematics. Currently, I am working with a freshman investigating rearrangements of conditionally convergent series and preparing them to present their results at an undergraduate-focused conference. This mentoring experience has been incredibly rewarding. My student has gained exposure to concepts typically seen in an upper-division real analysis course, and has been deeply engaged by the pursuit of asking and answering mathematical questions.

Having the opportunity to work closely with undergraduates on research projects is a hugely rewarding experience. Devising suitable problems that are accessible to either talented undergraduates or students with backgrounds in lower-level mathematics has been a new and exciting challenge, and I committed to providing similar opportunities in the future.

G. Technology

The advancement of technology in the learning environment, both hardware and software, provides a wealth of opportunities to today's educator. However, technology must be utilised in an appropriate and effective way, and not just used for '*technology's sake*'. I have experimented with technology in a number of ways in courses I have taught and I look forward to learning more about its benefits and successful implementation as I continue my career in education.

In this section I will briefly outline my experiences with learning management systems and experiments with the *flipped classroom* model.

At Harvard University and Middlebury College I have had the opportunity to implement the Canvas learning management system in my classes. I found Canvas to be a great tool to conduct class quizzes, provide immediate feedback on incorrect answers, and facilitate problem-solving discussion. Canvas has also provided a user-friendly platform to post video lectures and host student feedback and discussion. I hope to learn more about the possibilities that Canvas can provide when it comes to implementing a full *flipped classroom* approach to the learning environment.

In the last year, I have been excited and intrigued by approaching the mathematics classroom using the *flipped classroom* model. Providing students with appropriate and effective video lectures, and subsequent engaging classroom activities, requires careful consideration and foresight in a course. This semester I am experimenting with this approach in a multivariable calculus course I am teaching. At the moment I have allocated one third of lectures to be *flipped* lectures. For these lectures, students will be required to watch a short (15-20 minute) video lecture prior to attending class. Class time will then be devoted to working through problems designed to engage students with the concepts they have seen in the video lecture. I am excited to see how this experiment progresses and how receptive my students are to this approach to the classroom.

H. Professional Development

Engaging with students has become a major aspect of my passion for mathematics. It is important to me that I provide students with an outstanding education and that they are provided with ample opportunity to further themselves academically.

I have taken advantage of several workshops and courses in an effort to improve my teaching:

- In Spring 2015, I successfully completed the semester-long graduate course '*Mentoring in Higher Education'* at the University of California, Berkeley*.*
- I have attended the following workshops offered through the University of California, Berkeley, Teaching & Resource Center: '*How Students Learn*', '*Fostering Student Participation*'.
- At Harvard University I participated in the *Public Speaking for Teachers and Scholars* seminar course. My aim was to develop my own public speaking abilities and learn techniques that I could suggest to students in courses with a presentation component of their assessment.
- At Middlebury College I am an active participant in the *STEM Pedagogy Working Group*. This is a group of professors in STEM at Middlebury College who meet every month to share and discuss their pedagogical practices and experimentation with new approaches to their classroom environment.

These courses and workshops have been extremely valuable in improving my approach to effective teaching. For example, in project-based group work I provide students with suggestions to improve their presentation skills such as 'story-telling' or 'interactive slides'. I learned of these suggestions in the *Public Speaking for Teaching and Scholars* course and has enabled me to provide a more focused critique to students.

I. Conclusion

Teaching mathematics has been a tremendously enriching experience. My teaching and mentoring roles have given me a strong appreciation of the challenges facing students as they undertake programs of study in mathematics. Progressing from my position as a graduate student at the University of California, Berkeley, to a Faculty member at Harvard University and Middlebury College has affirmed the great responsibility that college educators undertake to guide students towards their academic goals. It has been important to me to be a personable and relatable teacher that students can trust with their difficulties and I look forward to sharing in future students' education. Having the opportunity to positively affect the student learning experience is an exciting prospect, and one that I am committed to realising in the years to come.

J. Appendices

1. Multivariable Calculus Spring 2018 Course Syllabus.

2. Lecture on Lagrange’s Theorem. Abstract Algebra, Summer 2014.
3. Calculus II Fall 2017 lecture notes.
4. Directed Reading Program Fall 2014 worksheet.
5. Multivariable Calculus Fall 2016 worksheet.
6. Calculus II Fall 2017 Final Exam.
7. UC Berkeley GSI Award.
8. Harvard University *Excellence in Teaching Award.*
9. Email from UCSC Professor

Middlebury College

Math 223A, Multivariable Calculus
Spring 2018
MWF 8-8.50am, 202 Warner Hall

The Basics

Instructor: George Melvin
Contact: gmelvin@middlebury.edu
Office: 312 Warner Hall
Office hours: M 2–3pm, W 4–5pm, Th 9–11am

Alternatively you are welcome to chat with me when my office door is open and I am available, or by appointment.

Course website: http://community.middlebury.edu/~gmelvin/current.html

Announcements and handouts can be found at the course website. The course website will be updated regularly; please check frequently.

Canvas & Piazza: This course has a Canvas site accessible via Course Hub. Please ensure you can access the piazza.com forum using the 'Piazza' tab on the course Canvas site.

Important dates:

3/8: Examination I
3/16: Last day to drop course
4/12: Examination II
5/?: Examination III

The Course

Course description: This course is an introduction to the analysis of multivariable functions and differential geometry. The course will comprise three modules:

1. Geometry of Euclidean space $\mathbb{E}^n$.
2. Differential calculus of several variable functions.
3. Generalisations of the Fundamental Theorem of Calculus to several variables.

Mathematical topics: The calculus of functions of more than one variable. Introductory vector analysis, analytic geometry of three dimensions, partial differentiation, multiple integration, line integrals, elementary vector field theory, and applications.

Prerequisites: Math 122, Math 200. If you are concerned about your preparation for this course then arrange to meet with me as soon as possible.

Textbook: *Vector Calculus*, by S. J. Colley (4th Ed.), Pearson. **It is important that you have the correct (4th) edition!**

Supplementary Resources: The following resources is supplementary to the lecture notes and is freely available for download:

The following recommended resources are supplementary to the lecture notes and will be placed on Reserve in the Davis Library.

- *Vector Calculus*, Marsden & Tromba
- *Multivariable Calculus*, Stewart

Many other calculus textbooks can be found in the QA303.xxx section of the Davis Library, in the Math Department and in my office (available for consultation whenever I'm around). **Caution:** some textbooks with *calculus* in the title are more suitable for a course in *real analysis* (Math 0323).

Additionally, you can consult the following excellent online resources:
- *Kahn Academy*, https://www.khanacademy.org/math/calculus-home
- *Paul's Online Math Notes*, http://tutorial.math.lamar.edu

In line with the College Honor Code you are expected to properly attribute any additional resource you consult. If you have a question regarding the suitability of a resource, or appropriate attribution practices, then let me know.

The Skinny

Grading: Your overall grade is given by the following prescription:

Examination I (two hours) - 15% (*March 8, 2018. 7-9pm. Location TBA*)
Examination II (two hours) - 25% (*April 12, 2018, 7-9pm. Location TBA*)
Examination III (three hour)- 35% (*May ??, 2018, ???. Location TBA*)
Mathematica Project - 10%
Homework - 15%

I reserve the right to modify this prescription until the end of the fifth week of the semester.

Assignment of Grades: The assignment of grades will be (roughly) as follows: let x be your final (weighted) score out of 100.

$x \geq 90$	**A**
$80 \leq x < 90$	**B**
$65 \leq x < 80$	**C**
$50 \leq x < 65$	**D**
$x < 50$	**F**

The appendage of a decoration ($\pm$) will be assigned at my discretion.

Grading policy: If you do better on Exam III than on Exam I (i.e. your score on Exam III, relative to the distribution of Exam III scores, improves from Exam I to Exam III) then your Exam I score will be disregarded and your Exam III score will constitute 50% of your overall grade.

Examination policy: You must take each Examination at the prescribed time. Known scheduling conflicts must be announced to me as soon as possible and additional arrangements will be made on a case-by-case basis. Failure to attend an Examination without excuse results in a failure for the Examination; there will be no make-up Examinations. Exceptions to this policy will only be granted in compelling circumstances.

Homework: Homework will be assigned daily and due for submission at the beginning of the following lecture. You are **strongly** encouraged to work with your peers on homework. You are warmly encouraged to discuss problems with me. However, you should write up solutions *on your own* and *in your own words.*

The goal of homework in this class is to provide you with an opportunity to practice the skills learned in class. If you required further practice then take the initiative and make use of additional resources.

Homework policy: For each homework submission, I will ask the grader to grade five problems. Each problem will be worth 2 points and graded as follows:

- **0 points:** Illegible work; no justification (even if answer is correct); > 3 minor errors or ≥ 1 major errors.
- **1 point:** Substantial progress made towards solution, but may have 2−3 minor errors or insufficient justification.
- **2 points:** Substantial progress made towards correct solution; ≤ 1 minor errors.

The lowest five homework scores will be dropped.
Mathematica project: Details will be announced no later than the end of Week 5 of the semester.
Attendance: Timely attendance of lectures is mandatory.. You must attend the section for which you are registered. If you expect regular class attendance to be a problem then let me know immediately and we can discuss your situation. In particular, this applies to extracurricular-explained absences. Changes to the attendance policy will only be granted in exceptional circumstances.
Late Days: You will be permitted *Late Days*™ that you can use throughout the semester. Late Days can be used to excuse late/absent attendance from class or missed weekly quizzes. Specific details can be found in the *Late Days* policy at the course website.
Suggested collaboration policy:

- You are strongly encouraged to collaborate with your peers. However, you should write up solutions *on your own* and *in your own words.*
- Any collaboration undertaken to obtain a solution on submitted work must be explicitly acknowledged; such acknowledgements will not be penalised. For example, write '*I worked with E. Noether, Archimedes, C. F. Gauss*' on your submission.
- You are strongly encouraged to be an active participant in the `piazza.com` forum acccessible via the course Canvas site.
- Other online forums (eg `math.stackexchange.com`) must not be utilised to complete or check your solutions to submitted work: consulting such forums is declared *cheating* in this class. You are reminded of your commitment to the College Honor Code.

The Important Stuff

Flipped learning: Throughout the semester we will frequently adopt a *flipped learning* approach to the classroom. Flipped learning requires you to engage with course material before coming to class by watching short video lectures. Time is then made available in class to focus on problem-solving and engaging with the course material at a deeper level. To benefit from this approach to learning requires a commitment on your part. You will be expected to have watched the required video lectures no later than the night before class, and to have worked through the introductory exercises.
Classroom etiquette: You are expected to be seated at your desk and ready to engage by the beginning of the class. You are expected to be courteous to your classmates and to help foster an inclusive, safe learning environment. Do not talk over each other; do not belittle someone's viewpoint; avoid the use of phrases like *'that's easy'*. Our learning environment will not be competitive. These remarks also apply to the `piazza.com` forum.

Mobile devices must be kept silent and non-vibrating throughout classtime. Use of your mobile device in-class is not permitted *unless otherwise specified*; if you are expecting a call then let me know. If you wish to use a laptop to take notes and/or document the class then you are free to do so. You must only use your computer as a note-taking device *unless otherwise specified*. Laptops should be silent throughout the class. Repeated violations of this policy will reflect negatively on your final score.
Learning etiquette: Be honest in your approach to learning. *Ask questions:* whenever you struggle with a topic, if an explanation is unclear, if notation is not defined; you will not be the only one with that question. *Ask questions even if you think they are stupid.*

I have found that one of the most difficult tasks to undertake when learning is asking a *useful* question. You can help yourself by adopting the following 3RA strategy when asking questions:

- Reflect: Ask yourself the question: does the question answer itself?

- Write: Write down your question down as a coherent sentence: does this help you see your way to a solution?

- Reformulate: Having constructed a coherent formulation of your question: can you ask a more pointed question?

- Ask: Ask your question!

Get to know your classmates and form study groups: discuss the material, work through problems, ask questions, help each other. If you understand something then challenge yourself by trying to explain your understanding to your peers. If someone struggles with your explanation then reformulate your argument: use examples, visual aids, simple analogies. The onus lies more greatly on the teacher to provide an adequate explanation than it does on the student to comprehend that explanation.

"What I cannot create, I do not understand"
Richard Feynman, 1918-1988
Renowned physicist, educator and Nobel Laureate.

Anti-discrimination commitment: I am firmly committed to diversity and equality in all areas of campus life. In this class we will promote an anti-discriminatory environment where everyone feels safe and welcome. I recognize that discrimination can be direct or indirect and take place at both institutional and personal levels. I believe that such discrimination is unacceptable and I am committed to providing equality of opportunity for all by eliminating any and all discrimination, harassment, bullying, or victimization. The success of this policy relies on the support and understanding of everyone in this class. We all have a responsibility not to be offensive to each other, or to participate in, or condone harassment or discrimination of any kind.

Accommodations: Students with documented disabilities who believe that they may need accommodations in this class are encouraged to contact me as early in the semester as possible to ensure that such accommodations are implemented in a timely fashion. Assistance is available to eligble students through Student Accessibility Services. Please contact one of the ADA Coordinators below for more information; all discussions will remain confidential.

- Jodi Litchfield (`litchfie@middlebury.edu`, or *802-443-5936*),
- Courtney Cioffredi (`ccioffredi@middlebury.edu`, or *802-443-2169*).

Finally, a parting thought: Please remember that MATHEMATICS IS DIFFICULT! But then again, getting admitted to Middlebury College is difficult and here you all are. We will strive towards a high level of rigor and it can be a struggle to wade through the mathematical marsh of complex concepts, technical tricks, and difficult definitions. However, if you are dedicated to your work, exercise your problem-solving abilities frequently, and talk about mathematics with your peers then I guarantee that you will be able to achieve your goals, and more, for this course.

Expectations & Commitment Pledge

Our course is a shared experience. As such, we *all* play an important role. To help ensure a successful experience, I expect all of us, myself included, to commit to the following practices:

- **Be here.** Attend lectures, arrive on time, and stay for the duration of the class period.
- **Be prepared.** Complete preview activities and readings prior to attending each class.
- **Be present.** Plan to participate in class by both asking and answering questions, as well as by taking part in discussions and group activities.
- **Be proactive** in your understanding. Complete assignments regularly. Ask questions. Attend office hours regularly.
- **Be honorable.** Follow the Honor Code for all graded work in this course.

- **Be respectful** of yourself, your classmates, your instructor, and the classroom experience. Value yours and others' efforts and contributions to this class. Be mindful that we each arrive with diffferent backgrounds and experiences. Your respectful presence and interactions with others are necessary to maintain a productive, safe, welcoming, and stimulating class environment. This respect should extend outside of our classroom to encompass our entire course experience, including assigned group work and homework collaboration.

I take these expectations very seriously. As such, I have signed the following commitment below and ask that all students in my courses sign and return the attached commitment as well.

I understand the above expectations and commit to uphold them to the best of my ability through our learning experience.

Math 223, Spring 2018. Expectations & Commitment Pledge

Our course is a shared experience. As such, we *all* play an important role. To help ensure a successful experience, I expect all of us, myself included, to commit to the following practices:

- **Be here.** Attend lectures, arrive on time, and stay for the duration of the class period.
- **Be prepared.** Complete preview activities and readings prior to attending each class.
- **Be present.** Plan to participate in class by both asking and answering questions, as well as by taking part in discussions and group activities.
- **Be proactive** in your understanding. Complete assignments regularly. Ask questions. Attend office hours regularly.
- **Be honorable.** Follow the Honor Code for all graded work in this course.
- **Be respectful** of yourself, your classmates, your instructor, and the classroom experience. Value yours and others' efforts and contributions to this class. Be mindful that we each arrive with diffferent backgrounds and experiences. Your respectful presence and interactions with others are necessary to maintain a productive, safe, welcoming, and stimulating class environment. This respect should extend outside of our classroom to encompass our entire course experience, including assigned group work and homework collaboration.

I take these expectations very seriously. As such, I have signed the following commitment below and ask that all students in my courses sign and return the attached commitment as well.

I understand the above expectations and commit to uphold them to the best of my ability through our learning experience.

____

Middlebury College

Math 223, Multivariable Calculus
Spring 2018
Prof. George Melvin

(Tentative) Timetable

	Monday	Tuesday	Wednesday	Thursday	Friday
2/12–2/16	introduction		*§1.1-1.3* linear algebra recap, displacement vectors		*§1.4* cross product
2/19 – 2/23	*§1.5* affine geometry: lines, planes etc.		*§1.7* coordinate systems I		**‼ NO CLASS ‼**
2/26 – 3/2	*§1.7* coordinate systems II		*§3.1* parameterised curves		*§3.3* vector fields I, flow lines
3/5 – 3/9	classification of quadric surfaces **NE**		**Case Study A:** Kepler's Laws **NE**	‼ Exam I ‼	*§2.1* functions of several variables I **PFDD**
3/12 – 3/16	*§2.1* functions of several variables II, level sets		*§2.2* limits, continuity		*§2.3* partial derivatives, the derivative I **DD**
3/19 – 3/23	*§2.3-2.4* the derivative II		*§2.5* chain rule		*§2.6* directional derivatives, gradient
4/2 – 4/6	*§3.3-3.4* vector fields II, grad, div, curl		*§4.2* extrema		*§4.3* Lagrange multipliers I
4/9 – 4/13	*§4.3-4.4* Lagrange multipliers II, applications		**Case Study B:** Lagrange multipliers with multiple constraints **NE**	‼ Exam II ‼	*§5.1-5.2* integration of functions of several variables I
4/16 – 4/20	*§5.2-5.4* integration of functions of several variables II		*§5.4* integration of functions of several variables III		Spring Symp. **‼ NO CLASS ‼**
4/23 – 4/27	*§5.5* change of variables		*§5.6* applications of integration		*§6.1* line integrals
4/30 – 5/4	*§6.2* Green's Theorem		*§6.3* Fudamental Theorem of Line Integrals		*§7.1* parameterised surfaces
5/7 – 5/11	*§7.2* surface integrals I		*§7.2* surface integrals II		*§7.3* Stokes' Theorem
5/14	*§7.3* Gauss' Theorem				

Notes:

1. Topics may not be covered precisely as timetabled.
2. This schedule is subject to change.
3. §$X.y$ refers to Chapter X, Section y of the course textbook *Vector Calculus*, by Colley (4th Edition).
4. PFDD = Pass/Fail/D Deadline, DD = Drop Deadline, NE = non-examinable
5. Lectures shaded red will follow the '*flipped classroom*' model. Further details can be found at the course website.

5 Lecture 5 - Lagrange's Theorem and applications. Cyclic groups.

Previous lecture - Next lecture

Keywords: Lagrange's Theorem. Groups of prime power are cyclic. Cyclic group, group generator.

In today's lecture we will prove one of the most fundamental results in group theory - **Lagrange's Theorem**. This result states that there are strong conditions on the existence of subgroups of a finite group. Moreover, we will classify all groups G such that $|G| = p$ is prime. We will also introduce a class of groups - the **cyclic groups** - and completely describe their structure; not bad for a day's work!

Remark. For the remainder of the course we will suppress the '$*$' when considering the law of composition in a group. As such, we will simply say 'Let G be a group', where the law of composition is implicitly understood to have been defined as part of the definition of G.[1]

5.1 Lagrange's Theorem

Theorem 5.1.1 (Lagrange's Theorem)**.** *Let G be a finite group, $H \subset G$ a subgroup. Then, the number of left cosets of H in G is $|G|/|H|$. That is,*

$$|G/H| = |G|/|H|.$$

Hence,

$$\boxed{\textbf{the order of } H \textbf{ divides the order of } G\textbf{.}}$$

Proof: Since G is finite then $H \subset G$ is finite and, by Corollary 4.2.3, every left coset of H in G has the same size, equal to $k = |H|$. Since a left coset of H in G is an equivalence class of the equivalence relation $\sim_H$ we know that there is a partition of G into equivalence classes. If there are r such equivalence classes (ie r left cosets of H), each of which has the same size k, then

$$|G| = \overset{r \text{ times}}{k + \cdots + k} = rk \quad \Longrightarrow \quad |G/H| = r = |G|/|H|.$$

□

Corollary 5.1.2. *Let G be a finite group, $g \in G$. Then, $o(g)$ divides $|G|$.*

Proof: Recall that $o(g) = |\langle g \rangle|$, and $\langle g \rangle \subset G$ is a subgroup of G. The result follows from Lagrange's Theorem. □

Hence,

$$\boxed{\textbf{the order of an element } g\textbf{, } o(g)\textbf{, divides the order of } G\textbf{.}}$$

Example 5.1.3.

1. Suppose that $H \subset D_8$ is a subgroup. Then, $|H|$ must be even. Indeed, Lagrange's Theorem implies that $|H|$ divides $|D_8| = 8$, we must have $|H| = 1, 2, 4, 8$. Note that Lagrange's Theorem **does not** imply that there must exist a subgroup of each of these orders. We will come back to this problem when we discuss Sylow's Theorems.

2. Let $f : \mathbb{Z}/5\mathbb{Z} \to S_4$ be a group homomorphism. Then, f must be the trivial homomorphism. Indeed, since $\ker f \subset \mathbb{Z}/5\mathbb{Z}$ is a subgroup then $|\ker f| = 1, 5$; if $|\ker f| = 1$ then f is injective. Hence, there must exist an element of S_5 of order 5, but 5 does not divide $|S_4| = 24$.

[1]Why do we make an issue of this? It could be possible to define two *different* laws of composition on a set G so that we obtain two *different* groups $(G, *)$ and $(G, \bullet)$ with the same underlying set.

Remark 5.1.4. Why have we repeatedly used the adjective 'left'? There is an analagous notion of a **right coset of** H **in** G: define an equivalence relation on G by

$$g \sim^H g' \quad \Leftrightarrow \quad g'g^{-1} \in H.$$

It can be shown that this defines an equivalence relation on G and the equivalence classes are of the form

$$[g] = \{hg \mid h \in H\} \stackrel{def}{=} Hg.$$

The resulting partition of G is called the **right** H**-partition of** G, and we denote the set of equivalence classes $H \setminus G$. There are analagous results to those obtained above for right cosets of H in G - in particular, there is an analogue of Lagrange's Theorem - **the number of right cosets of** H **in** G **equals** $|G|/|H|$ - so that, for a finite group G and a subgroup $H \subset G$

the number of right cosets equals the number of left cosets.

5.2 Groups of prime order

Let G be a group of prime order, so that $|G| = p$ is a prime. Let $g \in G$ be nontrivial. Thus, the subgroup $H = \langle g \rangle$ is a nontrivial subgroup of order $o(g)$ so that $H = G$, by Corollary 5.1.2. Hence, we have

$$G = \{e_G, g, g^2, \dots, g^{p-1}\}.$$

Moreover, $g^p = e_G$, for any nontrivial $g \in G$.

If G' is another group of order p then, for any nontrivial $h \in G'$, we find that

$$G' = \{e_{G'}, h, h^2, \dots, h^{p-1}\}.$$

It can then be shown that G and G' are isomorphic as groups.[2] In particular, any group G of prime order p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Hence, any group of prime order p is a **cyclic group** (to be defined in the next section) - a group generated by a single element.

5.3 Cyclic groups

Lagrange's Theorem is a simple consequence of the existence of a particular equivalence relation that can be defined on any finite group G, given a subgroup H, and has allowed us to classify[3] all finite groups of prime order. Combining Lagrange's Theorem with the basic arithmetic properties of $\mathbb{Z}$ from Lecture 2 allows us to to understand the structure of a larger class of groups - the **cyclic groups**.

Definition 5.3.1 (Cyclic group)**.** A group G is **cyclic** if there exists $x \in G$ such that

$$G = \langle x \rangle = \{\dots, x^{-1}, e_G, x, x^2, \dots\}.$$

We call such an x a **generator of** G, and say that G is **generated by** x.

Remark 5.3.2. Let G be a cyclic group with generator $x \in G$. Then, x^{-1} is also a generator of G. In general, there are many generators of a cyclic group.

Example 5.3.3. a) e_G is a generator of a cyclic group G if and only if G is the trivial group.

b) Let $G = \mathbb{Z}$. Then, G is cyclic and generated by 1. The set of all generators of G is $\{\pm 1\}$.

[2] What is the isomorphism between G and G'?

[3] This means that we have understood essentially all finite groups of prime order - they are isomorphic to $\mathbb{Z}/p\mathbb{Z}$, so that they they have the same structure as $\mathbb{Z}/p\mathbb{Z}$.

c) Let $n \in \mathbb{Z}_{>1}$ and $G = \mathbb{Z}/n\mathbb{Z}$. Then, G is cyclic and generated by 1. The set of generators of G is

$$\{\overline{x} \in \mathbb{Z}/n\mathbb{Z} \mid x \in \{1, \dots, n-1\},\ \gcd(x, n) = 1\}.$$

d) Let $G = \mu_6 = \{z \in \mathbb{C} \mid z^6 = 1\}$, considered as a subgroup of $(\mathbb{C}^\times, \cdot)$, the law of composition being multiplication of complex numbers - μ_6 is the group of sixth roots of unity. Then, μ_6 is cyclic with generator $w = \frac{1}{2}(1 + \sqrt{-3})$.[4]

e) The dihedral group D_8 is not a cyclic group as there does not exit any element of order 8. In general, D_{2n} is not cyclic.

f) S_n is cyclic if and only if $n = 2$.

g) $(\mathbb{Q}, +)$ is not cyclic.

In fact, the examples above describe **all possible cyclic groups**:

Theorem 5.3.4 (Structure Theorem of cyclic groups)**.** *Let G be a cyclic group generated by $x \in G$. Then,*

a) if G is infinite then G is isomorphic to $\mathbb{Z}$;

b) if G has order n then G is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Let $H \subset G$ be a nontrivial subgroup. Then,

a) (G infinite) H is isomorphic to $\mathbb{Z}$ and generated by x^i, where $i = \min\{r \in \mathbb{Z}_{>0} \mid x^r \in H\}$.

b) (G of finite order n) Suppose $|H| = k$ so that $n = km$, by Lagrange's Theorem. Then, H is cyclic and generated by x^m. Hence, H is isomorphic to $\mathbb{Z}/k\mathbb{Z}$.

Proof:

a) Suppose that G is infinite and $G = \langle x \rangle$. Then, by the definition of a cyclic group we have

$$G = \{\dots, x^{-1}, e_G, x, x^2, \dots, \}.$$

Define

$$f : \mathbb{Z} \to ;\ r \mapsto x^r.$$

Then, f is a group homomorphism. Moreover, f is injective - if $f(r) = e_G$ then $x^r = e_G = x^0$ so that $r = 0$, by Lemma 3.1.6 - and f is surjective, as x is a generator of G. Hence, f is an isomorphism and G is isomorphic to $\mathbb{Z}$.

If $H \subset G$ is a nontrivial subgroup and $i = \min\{r \in \mathbb{Z}_{>0} \mid x^r \in H\}$ then we claim that x^i generates H: we need only show that any nontrivial $h \in H$ is of the form $h = x^{ia}$, for some $a \in Z$. So, let $h \in H$ be nontrivial. Then, we must have $h = x^r$, for some $r \in \mathbb{Z}$. By the division algorithm we can find $q, b \in \mathbb{Z}$ with $0 \leq b < i$ such that $r = qi + b$. Hence, we see that

$$x^b = x^{r-qi} = x^r(x^i)^{-q} \in H.$$

Since $0 \leq b < i$ and i is the minimal positive integer such that $x^i \in H$, we must have that $b = 0$ so that $r = qi$. Hence, $h = (x^i)^q$ and

$$H = \langle x^i \rangle = \{\dots, x^{-i}, e_G, x^i, x^{2i}, \dots, \}.$$

[4] This is a particular example of a more general result: any finite subgroup of $(\mathbb{C}^\times, \cdot)$ is cyclic.

b) If G is finite of order n and cyclic, then

$$G = \langle x \rangle = \{e_G, x, \dots, x^{n-1}\},$$

and $x^n = e_G$ by Lemma 3.1.6. Define

$$f : \mathbb{Z}/n\mathbb{Z} \to G \; ; \; \overline{r} \mapsto x^r.$$

This function is well-defined: if $\overline{r} = \overline{s}$, so that $r - s \in n\mathbb{Z}$, then

$$f(\overline{r}) = x^r = x^{s+nk} = x^s(x^n)^k = x^s(e_G)^k = x^s = f(\overline{s}).$$

Moreover, f is an isomorphism of groups. In a similar way as proved in a), it can be shown that any nontrivial subgroup H is of the stated form.

$\square$

Example 5.3.5. Let $n = 10 = 2.5$. Then, the subgroups of $\mathbb{Z}/10\mathbb{Z}$ are

$$\{\overline{0}\},\ \{\overline{0}, \overline{5}\} \cong \mathbb{Z}/2\mathbb{Z},\ \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\} \cong \mathbb{Z}/5\mathbb{Z},\ \mathbb{Z}/10\mathbb{Z}.$$

Contact: `gmelvin@middlebury.edu`

October 6 Lecture

An exp-traordinary function

In today's lecture we will define a very interesting function. Investigating this function will lead us to the notion of an *inverse function*, and will take our mathematical journey back to the familiar calculus realm of differentiation and integration.

1 Defining a function via a series Let x be any real number and consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Mathematical workout - flex those muscles

Use the ratio test to show that the above series is (absolutely) convergent, for every real number x.

By assigning to every real number x the limit of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have provided the definition for a *well-defined* function

$$(\text{input}) \qquad x \quad \mapsto \quad \exp(x) \overset{def}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (\text{output})$$

We will call the function $\exp(x)$ just defined **the exponential function**. Let's investigate some of its basic properties.

Check your understanding

Using the definition of $\exp(x)$, show that $\exp(0) = 1$, $\exp(x) > 1$, for any $x > 0$, and $\exp(x) > 1 + x$, for any $x > 0$.

2 A remarkable property We are going to investigate a remarkable property of the exponential function. Let x be a real number. For each $m = 0, 1, 2, \ldots$, denote the m^{th} partial sum of the series $\exp(x)$ by $s_m(x)$, so

$$s_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

Check your understanding

Let x be a real number.

1. Write down the expressions for $s_0(x)$, $s_1(x)$, $s_2(x)$, $s_3(x)$.

2. Show that $s_1(x)s_1(y) = s_1(x+y) +$ *higher order terms.*

3. Show that $s_2(x)s_2(y) = s_2(x+y) +$ *higher order terms.*

4. **Guess the pattern!** Complete the following statement

$$s_3(x)s_3(y) = \underline{\hspace{4cm}} + \textit{ higher order terms}$$

5. **Guess the general pattern!** Complete the following statement: for every $k = 0, 1, 2, \ldots$

$$s_k(x)s_k(y) = \underline{\hspace{4cm}} + \textit{ higher order terms}$$

Recall that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} s_n(x).$$

Complete the following statement:

$$\boxed{\exp(x) \cdot \exp(y) = \underline{\hspace{5cm}} \qquad (*)}$$

Property $(*)$ has lots of remarkable consequences. For example, suppose that x is any positive real number. Then,

$$\begin{aligned} 1 &= \exp(0) \\ &= \exp(x + (-x)) \\ &= \exp(x) \cdot \exp(-x) \end{aligned}$$

In particular,

- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) \neq 0$, for any real number x.

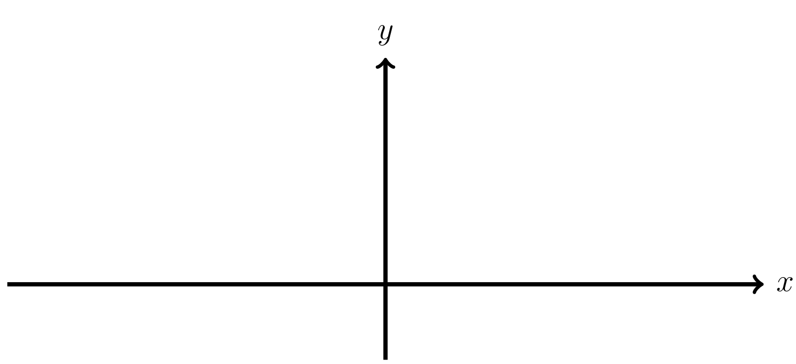
Check your understanding

1. Let x be a real number and write $x = 2y = y + y$. Use $(*)$ to show that $\exp(x) = \exp(2y) \geq 0$. Deduce that $\exp(x) > 0$, for every real number x. (*Hint: use* $\exp(2y) = \exp(y + y)$.)

2. Let $x < y$ and write $y = x + h$, where $h > 0$. Use $(*)$ to show that $\exp(y) > \exp(x)$. (*Hint: recall that* $\exp(h) > 1$ *whenever* $h > 0$)

   Hence, the exponential function is **strictly increasing**.

3. Based on your investigations, draw the graph of the function $\exp(x)$.



Summary

- $\exp(x+y) = \exp(x)\cdot\exp(y)$, for any real numbers x, y.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) > 0$, for any real number x.
- $\exp(x)$ **is a strictly increasing function**. $(**)$

We will discuss the consequences of Property $(**)$ in the next Lecture. In particular, over the next week or so we will show the following

- $\exp(x)$ has a *functional inverse*: there is a function $L(y)$ satisfying
$$\exp(L(y)) = y, \qquad L(\exp(x)) = x.$$
- $\exp(x)$ is a differentiable function; hence, $\exp(x)$ is a continuous function.
- $\exp(x)$ is the solution of a *differential equation*:
$$\frac{d}{dx}f(x) = f(x).$$

Remark 2.1. 1. Observe that
$$\exp(1) = \sum_{n=0}^{\infty}\frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$$
This series is a series with positive terms, which implies that its sequence of partial sums (s_m) is strictly increasing. In particular, for any $m = 0, 1, 2, \ldots$,
$$s_m < \exp(1) \quad \text{and} \quad \lim_{m\to\infty} s_m = \exp(1).$$
Notice that $s_2 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$ and
$$\sum_{n=2}^{\infty}\frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{2}.\frac{1}{1-\frac{1}{2}} = 1$$
Hence,
$$2.5 = \frac{5}{2} = s_3 < \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < 1 + 1 + 1 = 3$$
so that
$$2.5 < \exp(1) < 3.$$

2. In Problem Set 4 you will have the opportunity to show that
$$\exp(1) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$
and, more generally,
$$\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

October 20 2014. DRP. Symmetric Group

Recall the **symmetric group on n letters S_n**:

$$S_n = \mathrm{Perm}\{1, \dots, n\} = \{f : \{1, \dots, n\} \to \{1, \dots, n\} \mid f \text{ bijective}\}.$$

Some facts about the symmetric group:

1. Any $\sigma \in S_n$ admit a decomposition into disjoint cycles $\sigma = C_1 \dots C_k$, where $C_i = (i_1 \dots i_{r_i})$, and $C_i \cap C_j = \varnothing$. The **cycle type** of σ is the (nonincreasing) list $(r_1, \dots, r_k)$, where r_i is the length of the cycle C_i, $r_1 \geq r_2 \geq \dots r_k \geq 1$ and $\sum r_i = n$.
2. $\sigma, \tau \in S_n$ are conjugate if and only if they have the same cycle type.
3. The number of conjugacy classes is equal to $P(n)$, the number of integer partitions of n. Hence, the number of irreducible complex characters of S_n is equal to $P(n)$.

Let $\lambda = (\lambda_k, \dots, \lambda_1)$ be a partition of n (we assume the sequence is always nonincreasing). The **Young subgroup** $Y_\lambda \subset S_n$ is the subgroup of S_n consisting of those elements $\sigma \in S_n$ such that

$$\sigma(i) \in \{\lambda_j, \dots, \lambda_{j+1} - 1\}, \quad \text{for every } i \in \{\lambda_j, \dots, \lambda_{j+1} - 1\}.$$

Exercise. Let $\lambda = (3, 2)$ be a partition of 5. Show that Y_λ is isomorphic to $S_3 \times S_2$, and deduce that $|Y_\lambda| = 3!2!$. More generally, show that, for any λ, a partition of n, Y_λ is isomorphic to $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$; hence $|Y_\lambda| = \lambda_1! \cdots \lambda_k!$.

We are going to define representations of S_n using Young subgroups: this will give us $P(n)$ representations and we will (eventually) show that we can use these representations to recover all irreducible representations of S_n!

Fix $\lambda = (\lambda_k, \dots, \lambda_k)$ a partition of n. Consider the set of left cosets S_n/Y_λ, and denote a set of coset representatives $\{x_1, \dots, x_r\}$; thus

$$S_n/Y_\lambda = \{x_1 Y_\lambda, \dots, x_r Y_\lambda\}.$$

Exercise. Show that $r = n!/\lambda_1! \cdots \lambda_k!$. For $n = 4$, $\lambda = (2, 2)$, find explicit representatives $x_1, \dots, x_6 \in S_4$

Let $M_\lambda = \mathbb{C}^r$ and define the following representation of S_n on M_λ: first, fix a bijection between

$$\{x_1, \dots, x_r\} \leftrightarrow \{e_1, \dots, e_r\},$$

where e_i are the standard basis vectors in $\mathbb{C}^r$. For $\sigma \in S_n$, define

$$\sigma \cdot e_i = e_j, \text{ where } x_j Y_\lambda = (\sigma x_i) Y_\lambda$$

and extend this linearly (ie, if $v = \sum a_i e_i$ then $\sigma \cdot v = \sum a_i \sigma \cdot e_i$).

For example, when $n = 3$, $\lambda = (2, 1)$, we have

$$Y_\lambda = \{\sigma \mid \sigma(1) = 1, \sigma(\{2, 3\}) = \{2, 3\}\} = \langle (23) \rangle.$$

Then, we can choose representatives $x_1 = e, x_2 = (12), x_3 = (13)$. Then, we would have

$$(123) \cdot e_1 = e_2, \text{ since } (123)eY_\lambda = (12)Y_\lambda,$$

$$(123) \cdot e_2 = e_3, \text{ since } (123)(12)Y_\lambda = (13)Y_\lambda,$$

$$(123) \cdot e_3 = e_1, \text{ since } (123)(13)Y_\lambda = eY_\lambda.$$

Exercise. Determine the character $\chi_{(2,1)}$ of the representation $M_{(2,1)}$ (recall that characters are constant on conjugacy classes!). If you have time, try to determine the character $\chi_{(2,2)}$ of the representation $M_{(2,2)}$ above.

The representation we've just constructed is called the **permutation representation on S_n/Y_λ**.

Let's now try and understand the character χ_λ of the representation M_λ just constructed:

Exercise. Let $\sigma \in S_n$, M_λ be the permutation representation constructed above (so you've fixed some representatives $\{x_1, \dots, x_r\}$ etc). Suppose that $\sigma \in C_\mu$, where C_μ is a conjugacy class in S_n.

1. Explain why
$$\chi_\lambda(\sigma) = |\{x_j \mid \sigma x_j Y_\lambda = x_j Y_\lambda\}|.$$
(Hint: the character is defined as the trace of any matrix representation of σ). Use this to deduce that
$$\chi_\lambda(\sigma) = \frac{|\{\tau \in S_n \mid \tau^{-1}\sigma\tau \in Y_\lambda\}|}{|Y_\lambda|}.$$
(Hint: what happens if you choose different coset representatives?)

2. Suppose that $\tau \in S_n$ is such that $\tau^{-1}\sigma\tau \in Y_\lambda$. Let $\mathrm{Cent}_{S_n}(\sigma)$ be the centraliser of σ. Show that $w = z\tau$ also satisfies $w^{-1}\sigma w \in H$, for any $z \in \mathrm{Cent}_{S_n}(\sigma)$.

3. By considering the intersection $C_\mu \cap Y_\lambda$ and the previous exercise, show that
$$|\{\tau \in S_n \mid \tau^{-1}\sigma\tau \in Y_\lambda\}| = |C_\mu \cap Y_\lambda||\mathrm{Cent}_{S_n}(\sigma)|.$$
(Hint: can you find a bijection between the set on the LHS and $(C_\mu \cap Y_\lambda) \times \mathrm{Cent}_{S_n}(\sigma)$)

4. Deduce that
$$\chi_\lambda(\sigma) = \frac{n!}{\lambda_1! \cdots \lambda_k!}\frac{|C_\mu \cap Y_\lambda|}{|C_\mu|}.$$

5. * Can you see how to count $|C_\mu \cap Y_\lambda|$? (This is quite difficult...! A formula is given on p.55 of Fulton-Harris; it's the big ugly looking sum and product that appears just before '...where the sum is over all collections...') it might help to look at some actual examples (ie, fix n, μ, λ).

Math 21A, Harvard University. Fall 2016. Worksheet 10/20
Instructor: George Melvin

These problems are supplementary; they are not required but they may help you learn.
** problems are related to material that we will see later in the course.*

Learning objectives

1. *To learn the method of Lagrange multipliers.*
2. *To apply the method of Lagrange multipliers to constraint problems.*
3. *To learn sufficient conditions that ensure a solution to global extrema problems.*
4. *To apply the method of Lagrange multipliers to global extrema problems.*

Definitions, formulae, results

- **The Method of Lagrange Multipliers** Let $f(x,y)$ be a nice function, $g(x,y)=0$ be a curve. The **Lagrange equations** are the following equations in three unknowns λ, x, y:

$$\boxed{\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= 0 \end{aligned}}$$

  If $f(x,y)$ attains a local maximum/minimum on the curve $g(x,y)=0$ at a point (x_0,y_0) then, either

  (a) (λ, x_0, y_0) is a solution to the Lagrange equations (for some λ), or
  (b) $\nabla g(x_0,y_0) = 0$.

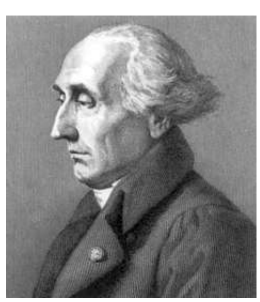
  There is an analogous statement for $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$.

- Solving the Lagrange equations can be hard!

- **Bolzano Theorem** Let $D \subset \mathbb{R}^2$ be a bounded region that includes all of it's boundary points, and $f(x,y)$ be a continuous function defined everywhere on D. Then, $f(x,y)$ attains a global maximum $f(x_1,y_1)$ and a global minimum $f(x_2,y_2)$, at points (x_1,y_1) and (x_2,y_2) in D.

- **To find the global maximum/minimum of $f(x,y)$ on a region D:**

  (a) determine the local maxima/minima of $f(x,y)$ on the interior of D,
  (b) determine the maxima/minima of $f(x,y)$ on the boundary of D (using Method of Lagrange, for example),
  (c) compare the local extrema found above to determine the global extrema.



Joseph-Louis Lagrange, *1736-1813* Bernard Bolzano, *1781-1848*

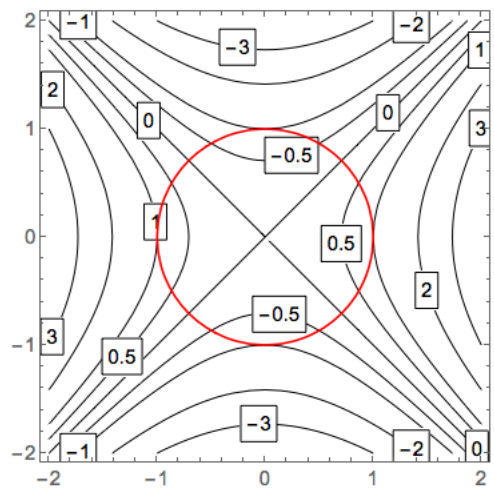
Extremizing functions on curves

1. (a) Let $f(x, y) = y$. Does $f(x, y)$ admit any local maxima/minima in the plane?

   (b) Restrict the inputs of $f(x, y)$ to those (x, y) such that $x^2 + y^2 = 1$. Without performing any calculations, where does $f(x, y)$ attain its maximum? its minimum?

   (c) Draw the circle $x^2 + y^2 = 1$ and the level curves $f(x, y) = 1, -1$. What is the relationship you observe between the level curves, the circle $x^2 + y^2 = 1$, and the points you found in (a).

   (d) Now, let $h(x, y) = x^2 - y^2$. Some level curves of $h(x, y)$ are drawn below, along with the circle $x^2 + y^2 = 1$. Use the plot to determine the points (x, y) on the circle where $h(x, y)$ is maximised/minimised.



   Can you describe a relationship between the points you've found, the level curves of $f(x, y)$ and the curve $x^2 + y^2 = 1$?

   (e) Let $f(x, y)$ be a function, $g(x, y) = 0$ a curve in the plane. Based on your investigations above, which of the following statements sounds reasonable? (*Several may sound reasonable, or none at all*)

      i. $f(x, y)$ is maximised/minimised at the points (a, b) on the curve $g(x, y) = 0$, such that the tangent line to $g(x, y) = 0$ at (a, b) is parallel to the tangent line (at (a, b)) of some level curve of $f(x, y)$.

      ii. $f(x, y)$ is maximised/minimised at the points (a, b) such that $\nabla f(a, b)$ is orthogonal to $\nabla g(a, b)$.

      iii. $f(x, y)$ is maximised/minimised at the points (a, b) such that $\nabla f(a, b) = 0$ and $\nabla g(a, b) = 0$.

      iv. $f(x, y)$ is maximised/minmised at the points (a, b) such that $\nabla f(a, b)$ and $\nabla g(a, b)$ are parallel.

Lagrange multipliers

2. We are going to solve a Lagrange multiplier problem for $f(x, y) = 2x - 3y$ on the ellipse $4x^2 + y^2 = 10$.

   (a) Write down the Lagrange equations.

   (b) Find maxima/minima of $f(x, y) = 2x - 3y$ on the curve $4x^2 + y^2 = 10$.

3. The earth revolves around the sun on an elliptical trajectory, with the center of the sun as a focus of the ellipse (*You don't need to know what this means*). In the celestial plane containing the centres of the earth and the sun, with coordinates (x, y), the ellipse is given by the equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$. The centre of the sun is at $(-4, 0)$ in these coordinates. Thus, the distance (squared) of the center of the earth from the center of the sun is $d(x, y) = (x + 4)^2 + y^2$. At which points (a, b) is the center of the earth closest to the center of the sun? (*Hint: minimising the distance is the same as minimising* $d(x, y)$)

Global extrema

4. For which regions D is it true that a continuous function $f(x, y)$ must attain a maximum and minimum value on D?

   (a) D is the set of points (x, y) such that $|x| \leq 1$ and $|y| \leq 3$.

   (b) D is the set of points (x, y) such that $-1 \leq x^2 - y^2 \leq 1$. (*The plot in Problem 1 may help*)

   (c) D is the set of points such that $4x^2 + y^2 \leq 10$.

   (d) D is the set of points such that $|2x - y| \leq 2$.

5. Find the global maximum and minimum of the function $f(x, y) = 3xy - y + 5$ on the disc $x^2 + y^2 \leq 1$.

Middlebury College

Calculus II: Fall 2017

FRIDAY DECEMBER 15

EXAMINATION III

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

DO NOT OPEN THIS PACKET UNTIL INSTRUCTED

Instructions:

- Sign the Honor Code Pledge below.
- Write your name on this exam and any extra sheets you hand in.
- You will have 180 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt **at least five** of Problems 2, 3, 4, 5, 6, 7, 8.
- Your final score will be the sum of your score for Problem 1 and the highest possible score obtained from five of the seven remaining problems.
- There are **6 blank pages** attached for scratchwork.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

QUESTION 1:	/20
QUESTION 2:	/20
QUESTION 3:	/20
QUESTION 4:	/20
QUESTION 5:	/20
QUESTION 6:	/20
QUESTION 7:	/20
QUESTION 8:	/20
TOTAL:	/100

NAME: ______________________________

"I have neither given nor received unauthorized aid on this assignment"

1. (20 points) True/False. **You do not need to justify your solution.**

(a)

$$\int_1^{27} \frac{dt}{t} = 3\int_1^9 \frac{dt}{t}$$

(b) Every bounded sequence (a_n) is convergent.

(c) Let $A = [-2, 3]$. Define the function $f(x) = -3x - x^2$ with domain A. Then, the function $f(x)$ admits an inverse function.

(d) The following series is divergent

$$\sum_{n=1}^{\infty} \frac{\cos(2n)}{\sqrt{n^3+5}}$$

(e) Using the inverse trigonometric substitution $x = 3\tan(t)$, it is shown that

$$\int f(x)dx = t + \sec(t) + \sin(2t) + C$$

Then,

$$f(x) = \arctan\left(\frac{x}{3}\right) + \frac{\sqrt{x^2+9}}{3} + \frac{6x}{9+x^2}$$

(f) Let $f(x)$ be an infinitely differentiable function with domain $(-\infty, \infty)$. Assume $|f^{(n)}(x)| \leq (n-1)!$, for any x and for any n. Then, there exists x such that $f(x)$ is not equal to its Taylor series centred at $c = 0$.

(g) For $|x| < 1$, the function $f(x) = \frac{x}{4-x^2}$ admits the power series representation (centred at $c = 0$)

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{4^{k+1}} = \frac{x}{4} + \frac{x^3}{4^2} + \frac{x^5}{4^3} + \ldots$$

(h) $f(x) = \sqrt{1-x^2}$ is a solution of the differential equation $f'(x) = x\,(f(x))^3$.

(i) The Taylor series centred at c associated to an infinitely differentiable function $f(x)$ has interval of convergence $(-c, c)$.

(j) The following improper integral is divergent.

$$\int_0^3 \frac{6-x}{\sqrt{6x-x^2}}dx$$

Solution: Write T(rue) or F(alse) in the corresponding box below

a)	b)	c)	d)	e)
f)	g)	h)	i)	j)

2. (a) (10 points) Determine a solution to the following antiderivative problem.

$$\int 2x \arctan(x) dx$$

(b) (10 points) Determine whether the following improper integral converges. If convergent, determine its limit.

$$\int_1^2 \frac{1}{\sqrt{x^2-1}}\, dx$$

3. Consider the sequence $(a_n)_{n\geq 1}$ defined as follows:

$$a_1 = 5, \qquad a_{n+1} = \frac{3a_n + 2}{4}, \qquad n = 1, 2, 3, \ldots$$

(a) (5 points) Using induction, show that $(a_n)_{n\geq 1}$ is strictly decreasing.

(b) (5 points) Using induction, show that $2 < a_n \leq 5$, for $n = 1, 2, 3, \ldots$.

(c) (5 points) Explain carefully why the sequence $(a_n)_{n\geq 1}$ is convergent.

(d) (5 points) Determine the limit $L = \lim_{n\to\infty} a_n$.

4. Consider the sequence

$$G_n = \int_0^\infty x^{n-1} \exp(-x) dx, \qquad n = 1, 2, 3, \ldots$$

You may assume without proof that G_n is a convergent Type I improper integral, for every n.

(a) (5 points) Show that $G_1 = 1$.

(b) (10 points) Using integration by parts, show that $G_{n+1} = nG_n$.

(c) (5 points) Using induction, show that $G_n = (n-1)!$, for $n = 1, 2, 3, \ldots$. (Recall that we define $0! = 1$.)

5. Let $f(x)$ be the function whose Taylor Series centred at $c = 10$ is

$$\sum_{n=1}^{\infty} \frac{(x-10)^n}{7^n n^2}$$

(a) (10 points) Determine the interval of convergence of the power series.

(b) (5 points) What is $f^{(10)}(10)$?

(c) (5 points) Let $F(x)$ be the antiderivative of $f(x)$ satisfying $F(10) = 1$. Write down $T_3(x)$, the 3^{rd}-degree Taylor polynomial of $F(x)$ centred at $c = 10$.

6. (a) (10 points) Using induction, show that

$$\frac{1}{3 \cdot 6 \cdot 9 \cdot \cdots \cdot 3n} = \frac{1}{3^n} \cdot \frac{1}{n!}, \qquad n = 1, 2, 3, \ldots$$

(b) (10 points) Show that the following series is convergent and determine its limit.

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 6 \cdot 9 \cdot \cdots \cdot 3n} = \frac{1}{3} + \frac{1}{18} + \frac{1}{162} + \ldots$$

7. (a) (10 points) Determine the first five terms of the Taylor series centred at $c = 2$ of the function $f(x) = \log(x)$.

(b) (10 points) Using part (a), or otherwise, determine the limit of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n n} = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \dots$$

You may assume, without proof, that (i) the above series is convergent, and (ii) the Taylor series of $f(x) = \log(x)$ centred at $c = 2$ has interval of convergence $(0, 4]$.

8. (a) (10 points) Find a solution to the initial-value problem

$$\cos(x)y' = y\sin(x), \qquad y(0) = 1$$

(b) (10 points) Determine the general solution to the differential equation

$$xy' - y = \frac{x^2}{1+x^2}$$

University of California
Berkeley

2013

the Graduate Divison
and
the Graduate Council
Advisory Committee for GSI Affairs
hereby recognize

George Melvin

as an
Outstanding Graduate Student Instructor

Andrew J. Szeri
Dean
Graduate Division

Laura Stoker
Chair
Advisory Committee for GSI Affairs

Rosemary A. Joyce
Associate Dean
Graduate Division

Linda von Hoene
Director
GSI Teaching and Resource Center

Graduate Student Instructor
Teaching & Resource Center

HARVARD UNIVERSITY

CERTIFICATE OF TEACHING EXCELLENCE

George Melvin

has been recognized for excellence in teaching
during the Spring semester of 2017.

This certificate acknowledges a special contribution to the teaching
of undergraduates in Harvard College.

Robert A. Lue
Richard L. Menschel Faculty Director
Derek Bok Center for Teaching & Learning

Jay M. Harris
Dean of Undergraduate Education

DEREK BOK CENTER FOR TEACHING AND LEARNING

advanced linear algebra notes

Debra Lewis <lewis@ucsc.edu>
To: gmelvin@berkeley.edu

12 February 2014 at 06:19

George,

I'm teaching the advanced linear algebra course at UCSC this quarter, and one of the students, Sergey Kojoian, highly recommended your lecture notes. I'm on UCSC's Committee on Computing and Telecommunications, and on Monday we were discussing the propagation across the web of course materials that weren't intended for mass distribution, so... would you be willing to share your notes with my class, and if so, how? I use Piazza, and could post the pdf there, or put a link to your website. If you'd prefer not to have them distributed to Slugs, quite alright.

Thanks.

Deb