What *Do* the People Want

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What *Do* the People Want?

Mathematics and Voting

Fernando Q. Gouvêa
Response by Sandy Maisel

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What is the problem?

Consider an election with three or more candidates.

Alfred got 30% of the vote.
Beorn got 39% of the vote.
Canute got 31% of the vote.

Does Beorn win because 39% of the people like him?

What if 61% would have ranked him last of the three?

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A More Interesting Story

Should we hire Alice, Beth, Claire or Deanna?

Ranking	Number	Ranking	Number
$A > C > D > B$	3	$C > B > D > A$	2
$A > D > C > B$	6	$C > D > B > A$	5
$B > C > D > A$	3	$D > B > C > A$	2
$B > D > C > A$	5	$D > C > B > A$	4

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The vote comes out $A > B > C > D$.

Plurality result is $A > B > C > D$.
But then Deanna called to withdraw her name: she had a better offer!
Should they call Alice?

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Preferences without Deanna

Ranking	Number	Ranking	Number
$A > C > B$	3	$C > B > A$	2
$A > C > B$	6	$C > B > A$	5
$B > C > A$	3	$B > C > A$	2
$B > C > A$	5	$C > B > A$	4

Now Claire wins, and Alice is last!

You can check that it actually doesn't matter which candidate we delete; the ranking *always* reverses.

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Trying to do better

Marie Jean Antoine Nicolas de Caritat Condorcet (1743–1794)

In the 1780s, proposed a method for electing members of the Académie:

Have voters choose between each pair of candidates; to win, a candidate has to beat all others in head-to-head voting.

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Trying to do better

Jean Charles de Borda (1733–1799)

Condorcet's method:

- Takes too much time, and
- May fail to produce a winner at all.

Instead, have each voter *rank* the three candidates, assign two points for each first-place vote, one point for each second-place vote, and add up.

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Condorcet's Argument Against Borda

Condorcet discovered this profile of voter preferences:

Ranking	Number	Ranking	Number
$A > C > B$	1	$C > B > A$	1
$A > B > C$	30	$C > A > B$	10
$B > C > A$	10	$B > A > C$	29

Condorcet's question is: who should win?

- Pairwise matches: A wins.
- Simple plurality: B wins.
- Borda count: B wins.

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Condorcet's Paradox

Alas, Condorcet also discovered this!

Ranking	Number	Ranking	Number
$A > C > B$	1	$C > B > A$	4
$A > B > C$	5	$B > A > C$	1
$C > A > B$	2	$B > C > A$	2

In pairwise matches:

- A beats B.
- B beats C.
- C beats A.

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One More Example

Ranking	Number	Ranking	Number
$A > C > B$	2	$C > B > A$	4
$A > B > C$	3	$B > C > A$	2

Now we have:

- Vote for one: $A > C > B$ with $5 : 4 : 1$ votes.
- Vote for two: $B > C > A$ with $9 : 8 : 5$ votes.
- Borda count: $C > B > A$ with $12 : 11 : 10$ points.

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The Main Question

Does the election result reflect "what the people want" or does it just reflect the method we happened to choose?

Saari's challenge: tell me the preferences of everyone in your department, and tell me what result you'll like, and I'll design a voting procedure that will yield the outcome you prefer.

"Rationality"

"The majority prefers" is not transitive!

Borda won the debate, and his method was used until Napoleon forced the Académie to change it.

Notice that Condorcet cycles together with the problems with plurality vote make runoff systems problematic also.

The point is that "what the people want" is not a clearly defined notion, and that these attempts to define it precisely all produce paradoxical effects.

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A gap of many years…

Lots of methods were invented, but people had trouble formulating criteria for what would make a method good.

Some examples:

- Methods involving runoffs.
- Positional methods: assign $(w_1, w_2, w_3, \dots)$ points.
- Approval voting.

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An Arrow to the Heart

Kenneth Arrow (1921–) created a list of reasonable expectations for a voting system. Some of them are:

- The system produces a societal ranking of the candidates by examining the preferences of individual people.
- The result does not depend only on one person's preferences.
- If every voter prefers A to B, then A should be ranked higher than B.
- If the method ranks A higher then B and a third candidate C drops out, then B should not end up ranked higher than A.
- Voters' choices are not restricted.

IIA stated more technically: the relative ranking of A and B in the societal ranking should depend only on their relative rankings in the voters' lists.

Closely related is monotonicity: suppose A is ranked above B; if a voter changes his ranking in such a way as to rank A higher than before, in the overall ranking A should still rank above B.

Single Transferable Vote is not monotone:

Ranking	Number	Ranking	Number
$A > C > B$	7	$C > A > B$	5
$B > C > A$	4	$B > A > C$	1

As it stands, B and C get dropped after round 1, and so A wins.

If the unique $B > A > C$ voter switches to $A > B > C$, then B gets removed and C wins in the second count.

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The thing to do, then, is to look for a system that satisfies all of these conditions.

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The thing to do, then, is to create a system that satisfies all of these conditions.

HOWEVER:

Arrow was able to *prove* that no such system exists!

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Arrow's theorem seems to have killed the subject for quite a while.

Nevertheless, there were many unanswered questions:

- Can we figure out *why* Arrow's theorem is true?
- Can we describe the possible kinds of "paradoxical" behavior?
- Can we *quantify* the paradoxical cases? For example, are some voting methods less likely to produce paradoxes than others?
- Is there a way to compare methods?
- Is there a way to analyze whether methods offer opportunities for "strategic" voting?

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Recent Work

Steven Brams: uses game theory to analyze how voters will behave, and favors the use of approval voting.

Don Saari: focuses on the formal structure, using geometric methods to analyze voting methods; favors the use of the Borda count.

Neither can settle the non-mathematical question of which method is "better."

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A Sampler of Saari's Work

- For a three-candidate election, there exist preference profiles for which there are four different positional methods which produce four different rankings.
- For n candidates, this is true for k different rankings for any k between one and $n! - (n-1)!$.
- For three candidates and a large number of voters, about 69% of preference profiles will allow multiple outcomes depending on the positional method chosen.
- For most positional systems, the results are "maximally chaotic" when we drop candidates. ("Most" excludes the Borda count.)
- Arrow's theorem is true because of the existence of "Condorcet profiles."

(Not all that) Maximally Chaotic:
Suppose there are $N \geq 3$ candidates.

- Choose a ranking of the N candidates.
- There are N ways to drop one candidate, for each of those, choose a ranking of the $N-1$ remaining candidates.
- Repeat.

Then, for most positional systems, there exists a profile so that all of those rankings are realized.

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What Mathematics Can Tell Us

- Mathematicians are good at deriving formal consequences of precisely-stated assumptions.
- In this case, mathematics highlights that the notion of “what the people want” needs to be made precise.
- It also shows, however, that all precise methods have the potential to produce paradoxical results.
- There’s a chance, however, that by analyzing which problems can happen when, the results can guide us toward better choices.
- Now it’s up to the political scientists!

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Where to go to learn more

Steven J. Brams, *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures*. Princeton University Press, 2008.

Steven J. Brams and Peter C. Fishburn, *Approval Voting*. 2nd edition, Springer, 2007.

Jonathan K. Hodge and Richard E. Klima, *The Mathematics of Voting and Elections: A Hands-On Approach*. American Mathematical Society, 2005.

Donald G. Saari, *Chaotic Elections: A Mathematician Looks at Voting*. American Mathematical Society, 2001.

Alan D. Taylor, *Mathematics and Politics: Strategy, Voting, Power and Proof*. Springer, 1995.