WHERE THE SLOPES ARE

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Let N be a positive integer (the "level"), let $k \geq 2$ be an integer (the "weight"), and let $S_k(N, \mathbb{C})$ denote the finite-dimensional $\mathbb{C}$-vector space of cuspidal modular forms of weight k and trivial character on $\Gamma_0(N)$ defined over $\mathbb{C}$. Elements $f \in S_k(N, \mathbb{C})$ can be specified by giving their Fourier expansions

$$f = a_1 q + a_2 q^2 + \cdots = \sum_{n=0}^{\infty} a_n q^n,$$

where $q = e^{2\pi i z}$ and z is in the complex upper halfplane. This expansion is sometimes described as "the q-expansion at infinity" of the modular form f. There exists a natural basis of $S_k(N, \mathbb{C})$ consisting of forms all of whose Fourier coefficients are in fact rational. We denote the $\mathbb{Q}$-vector space spanned by this basis by $S_k(N, \mathbb{Q})$. Note that then we have

$$S_k(N, \mathbb{C}) = S_k(N, \mathbb{Q}) \otimes \mathbb{C}.$$

For each prime number p which does not divide N there is a linear operator T_p acting on $S_k(N, \mathbb{C})$, known as the p-th Hecke operator. (In fact, the T_p stabilize $S_k(N, \mathbb{Q})$.) A modular form which is an eigenvector for all of these linear operators simultaneously is called an *eigenform*; the space $S_k(N, \mathbb{C})$ has a basis made up of eigenforms, and the Fourier coefficients of these eigenforms can be normalized (by requiring $a_1 = 1$) to belong to a finite extension of $\mathbb{Q}$.

The eigenvalues of the T_p operator encode significant arithmetic information about the modular form and various other objects which can be attached to it (for example, a Galois representation). In our setting, the eigenvalue of T_p acting on an eigenform $f \in S_k(N, \mathbb{C})$ is a totally real algebraic number whose absolute value (with respect to any embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$) is between $-2p^{(k-1)/2}$ and $2p^{(k-1)/2}$. If we normalize the eigenvalues by dividing by $p^{(k-1)/2}$, the normalized eigenvalues are real numbers in the interval $[-2, 2]$, and we can ask about their distribution in that interval. The Sato-Tate Conjecture, still very

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much an open problem, predicts the properties of that distribution for a fixed modular form f and varying p. We can also, however, fix the prime p and consider the distribution as $k \to \infty$ of all the eigenvalues of T_p corresponding to eigenforms of weight k. This was done by Serre in [Ser97] and by Conrey, Duke, and Farmer in [CDF97].

The goal of this paper is to begin the study of an analogous question in the p-adic setting by presenting a wide range of numerical data. The unexpected regularities in the data suggest several interesting questions that deserve further investigation.

We fix a prime number p, then, and consider the situation in a p-adic setting. We choose an embedding of the algebraic closure of $\mathbb{Q}$ into the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$, and then we define

$$S_k(N, \mathbb{C}_p) = S_k(N, \mathbb{Q}) \otimes \mathbb{C}_p,$$

and similarly for $S_k(N, F)$ where F is any extension of $\mathbb{Q}_p$. In the p-adic context, it turns out that the right operator to consider is not T_p but rather the Atkin-Lehner U operator, which can be described by its action on q-expansions:

$$U\left(\sum a_n q^n\right) = \sum a_{np} q^n.$$

If p does not divide N, this operator does not stabilize the space $S_k(N, F)$, but it does stabilize the larger space $S_k(Np, F)$, and once again we can consider eigenforms and the corresponding eigenvalues of U.

Assume $p \nmid N$, and let $f \in S_k(Np, \mathbb{C}_p)$ be an eigenform for U, so that $U(f) = \lambda f$. The p-adic valuation of the eigenvalue λ turns out to play a crucial role in the p-adic theory. We shall call this valuation the *slope* of the eigenform f:

Definition. *Given an U-eigenform f of level Np, weight k and eigenvalue λ, we define the* slope *of f by*

$$\mathrm{slope}(f) = \mathrm{ord}_p(\lambda).$$

The name "slope" comes from the p-adic theory of Newton polygons: the slopes of the eigenforms in $S_k(Np, \mathbb{C}_p)$ are determined by the slopes of the Newton polygon of the characteristic polynomial of the U operator acting on this space.

We are interested in the distribution *of the slopes* of the U operator for fixed level and varying weight. (Thus, we are writing the eigenvalues as a p-adic unit times a power of p, and then we are ignoring the unit part.) All of our results are numerical, but we feel they are of sufficient interest and that they raise significant questions that need to be addressed on a theoretical level.

Though the questions we ask are supported by a substantial amount of numerical data, we are a little hesitant to label them as conjectures, basically because all our data is for small values of p and of k. On the other hand, we emphasize that the statements labeled as "Question" in what follows are indeed supported by a considerable amount of data.

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1. Setting up the Problem

Let p be a prime number, $k \geq 2$ an even integer, and N a positive integer not divisible by p. Let ord_p be the p-adic valuation mapping, normalized by $\mathrm{ord}_p(p) = 1$. For any field F of characteristic zero, we write $S_k(N, F)$ to denote the F-vector space of cuspidal modular forms of weight k for $\Gamma_0(N)$ (with trivial character) whose Fourier coefficients all belong to F. We will essentially be concerned only with $F = \mathbb{Q}_p$, since the Newton polygon (and therefore the slopes) can be computed already in this context, though the eigenforms themselves may only be defined over some extension of $\mathbb{Q}_p$. Our computations will be restricted to the case $N = 1$ (and hence $k \geq 12$), but it seems reasonable to set up the problem for general level.

There are two natural inclusions of $S_k(N, F)$ into $S_k(Np, F)$; on q-expansions the first is the identity mapping and the second is the Atkin-Lehner V operator, which sends q to q^p. The subspace spanned by the images of both maps is called the space of *oldforms* in $S_k(Np, F)$; it has a natural complement called the space of *newforms*.

The Atkin-Lehner U operator maps $S_k(Np, F)$ to itself, acting on q-expansions by

$$U(\sum a_n q^n) = \sum a_{np} q^n.$$

This action stabilizes the space of newforms and also the space of oldforms. It follows from the Atkin-Lehner theory of change of level (see

[AL70]) that the action of U on newforms can be diagonalized (possibly after extending the base field), and that all the eigenvalues are equal to $\pm p^{(k-2)/2}$, and hence have slope equal to $(k-2)/2$. Thus, as far as the slopes are concerned, the interesting questions have to do with the action of U on the oldforms. This is best understood by relating it to the action of the Hecke operator T_p on forms of level N; this yields the theory of "twin eigenforms" discussed in [GM92], which we recall briefly.

The Hecke operator T_p can be diagonalized on $S_k(N, \mathbb{C}_p)$. Let $f \in S_k(N, \mathbb{C}_p)$ be a normalized cuspidal eigenform, and let a_p be the eigenvalue of T_p acting on f. Finally, let $f_1, f_2 \in S_k(Np, \mathbb{C}_p)$ be the two images of f under the maps described above. The U operator stabilizes the two-dimensional space generated by f_1 and f_2, and its characteristic polynomial is $x^2 - a_p x + p^{k-1}$. If this polynomial has two distinct roots, the action of U on this two-dimensional subspace can be diagonalized, and the eigenvalues will be precisely the two roots of the characteristic polynomial. Thus, the slopes of the two resulting eigenforms can be easily determined:

- If $\mathrm{ord}_p(a_p) < (k-1)/2$, the two eigenvalues have p-adic valuation equal to $\mathrm{ord}_p(a_p)$ and $k - 1 - \mathrm{ord}_p(a_p)$.
- If $\mathrm{ord}_p(a_p) \geq (k-1)/2$, then both eigenvalues have p-adic valuation $(k-1)/2$.

It has been conjectured by Ulmer that the polynomial $x^2 - a_p x + p^{k-1}$ always has two distinct roots. Specifically:

Conjecture (Ulmer)**.** *The action of U_p on $S_k(\Gamma_0(Np), \mathbb{Q}_p)$ is semisimple. In particular, the polynomial $x^2 - a_p x + p^{k-1}$ always has distinct roots.*

Coleman and Edixhoven have shown that this is true for $k = 2$ and that for general k it follows from the Tate Conjecture (see [CE98]).

It is easy to see that if the polynomial has a double root then we must have $a_p = \pm 2p^{(k-1)/2}$. For $N = 1$, it is possible to show that this cannot happen.

Theorem 1. *If $N = 1$, then $a_p \neq \pm 2p^{(k-1)/2}$, and therefore the polynomial $x^2 - a_p x + p^{k-1}$ always has distinct roots.*

Proof. **(Conrey and Farmer)**Let $f(x)$ be the characteristic polynomial of T_p acting on $S_k(1, \mathbb{Q})$. Suppose that one of the roots of $f(x)$ is equal to $\pm 2p^{(k-1)/2}$. Then, since $f(x)$ has rational coefficients and k is even, $\mp 2p^{(k-1)/2}$ must also be a root of $f(x)$. Hence, $f(x)$ must be divisible by $x^2 - 4p^{k-1}$.

Consider first the case $p \neq 3$. Then one knows from [CFW00], that

$$f(x) \equiv \begin{cases} (x-2)^d \pmod 3 \text{ if } p \equiv 1 \pmod 3 \\ x^d \pmod 3 \text{ if } p \equiv 2 \pmod 3 \end{cases}$$

It is easy to see that either factorization is inconsistent with divisibility by $x^2 - 4p^{k-1}$.

Finally, if $p = 3$, we know, from [Hat79], that

$$f(x) \equiv (x-4)^d \pmod 8,$$

which again is inconsistent with divisibility by $x^2 - 4p^{k-1}$. Hence, no root of $f(x)$ can be equal to $\pm 2p^{(k-1)/2}$.

□

We will always have $N = 1$ in the computations below, and thus won't need to worry about double roots. In general, whenever the analogue of Theorem 1 holds, we can indeed read off the slopes of U acting on the oldforms in $S_k(Np, \mathbb{Q}_p)$ by determining the slopes of T_p acting on $S_k(N, \mathbb{Q}_p)$. Specifically, suppose $f \in S_k(N, \mathbb{Q}_p)$ is an eigenform for T_p with eigenvalue a_p, and suppose λ' and λ'' are the two roots of $x^2 - a_p x + p^{k-1}$, ordered so that $\mathrm{ord}_p(\lambda') \leq \mathrm{ord}_p(\lambda'')$. Then there are two U-eigenforms $f', f'' \in S_k(Np, \mathbb{C}_p)$ such that $U(f') = \lambda' f'$ and $U(f'') = \lambda'' f''$. Thus, for each slope obtained in level N one obtains a pair of slopes $\alpha' = \mathrm{ord}_p(\lambda') = \min(\mathrm{ord}_p(a_p), \frac{k-1}{2})$ and $\alpha'' = \mathrm{ord}_p(\lambda'')$ in level Np, satisfying

- $0 \leq \alpha' \leq \alpha'' \leq k-1$
- $\alpha' + \alpha'' = k-1$

with $\alpha' < \alpha''$ unless they are both equal to $(k-1)/2$. We define the *slope sequence for level N, weight k, and prime p* to be the ordered list of slopes

$$(\alpha_1, \alpha_2, \ldots, (k-1) - \alpha_2, (k-1) - \alpha_1)$$

for U acting on the oldforms in $S_k(Np, \mathbb{Q}_p)$, where we repeat slopes that occur with multiplicity. The number of elements in this sequence is equal to twice the dimension of $S_k(N, \mathbb{Q}_p)$. Since the slope sequence is symmetric under $\alpha \leftrightarrow (k-1) - \alpha$, we will usually specify it by giving only the first half of the slope sequence. Whenever $\mathrm{ord}_p(a_p) < (k-1)/2$, it follows from the discussion above that this first half is the same as the slope sequence for T_p acting on $S_k(N, \mathbb{Q}_p)$.

Since we know that the slopes are in the interval $[0, k-1]$ (and we want to vary k), it makes sense to normalize the slopes by dividing them by $k-1$.

Definition. *Suppose f is either a T_p-eigenform of level N or a U-eigenform f of level Np. Let k be the weight of f and let $a_p(f)$ be the eigenvalue (of T_p or of U). We define the* supersingularity *of f by*

$$\mathrm{ss}(f) = \frac{\mathrm{ord}_p(a_p(f))}{k-1}.$$

Let $f \in S_k(N, \mathbb{C}_p)$ be an eigenform for T_p, and (assuming Ulmer's Conjecture is true) let f', f'' be the two old U-eigenforms corresponding to it as above. Then, provided that $\mathrm{ss}(f) \leq 1/2$, we have

$$\mathrm{ss}(f') = \mathrm{ss}(f)$$

and

$$\mathrm{ss}(f'') = 1 - \mathrm{ss}(f),$$

and both numbers are in the interval $[0, 1]$. Thus, the sequence of supersingularities corresponding to old eigenforms of weight k and level Np is a normalized version of the slope sequence, and can be computed via the supersingularities of forms of level N, provided these are small enough.

(One can think of ss as a function on the *eigencurve* studied by Coleman and Mazur in [CM98]. It will be an continuous function on the eigencurve, except along the $k = 1$ locus. Notice, however, that classical eigenforms of weight 1 will always have slope zero; defining $\mathrm{ss}(f) = 0$ for such forms gives a continuous extension of ss to classical forms of weight 1. No such continuous extension is possible at points corresponding to non-ordinary forms of weight 1.)

We define the *supersingularity sequence in weight k*

$$(\eta_1, \eta_2, \ldots, 1 - \eta_2, 1 - \eta_1)$$

by

$$\eta_i = \mathrm{ss}(f_i) = \frac{\mathrm{slope}(f_i)}{k-1}$$

as f_i runs through the old eigenforms of weight k on $\Gamma_0(Np)$. The supersingularity sequence is contained in the interval $[0, 1]$ and is symmetric under $\eta \leftrightarrow 1 - \eta$.

One of the main problems we want to consider is to understand the distribution of the supersingularities in the interval $[0, 1]$ when we fix the level N and let $k \to \infty$. This problem can be expressed in measure-theoretic terms, as in [Ser97]: considering N and p as fixed, for each k we define a probability measure μ_k on the interval $[0, 1]$ by putting a point mass at each supersingularity η_i: let $d_k = \dim S_k(N, \mathbb{Q}_p)$, and set

$$\mu_k = \frac{1}{2d_k} \sum_{i=1}^{d_k} \left(\delta_{\eta_i} + \delta_{1-\eta_i} \right),$$

where δ_x is the Dirac measure at x. The question then is whether the measures μ_k tend to a limit as $k \to \infty$, and if so to determine that limit measure. One can also consider several variants of this idea. For example, we might study the measure given by the first half of the slope sequence only (or, equivalently if we always have $\mathrm{ord}_p(a_p) < (k-1)/2$, by the slope sequence in level N).

2. Computations

For our computations, we restricted to the case $N = 1$, which then means that one only gets non-trivial results for even weights $k \geq 12$. For each prime number $p \leq 100$, we computed the Newton polygon of T_p acting on forms of weight k and level 1 for weights $k \leq 500$. Since in every case the slopes were less than $(k-1)/2$, the slopes we obtained are exactly the first half of the slope sequence for the U operator acting on oldforms of level p, as described above.

The method used for computation was straightforward: the space of cuspforms of weight k and level 1 has a basis consisting of forms $E_4^a E_6^b \Delta$, where E_4 and E_6 are the Eisenstein series of weight 4 and 6 respectively, Δ is the unique cuspform of weight 12, and $4a + 6b + 12 = k$. Using this explicit basis we determined the characteristic polynomial of T_p and computed its Newton slopes, then produced supersingularities by dividing by $k - 1$. The computation was done with the GP calculator [BBCO]; the basic GP functions we needed were based on a script originally written by Robert Coleman. The main constraint on the computation was the memory required for computing the characteristic polynomial: larger k meant working with a larger basis, and larger p meant that we needed to use more terms from the q-expansion of the modular forms. The full output of the computations can be found on the web at

`http://www.colby.edu/personal/fqgouvea/slopes/`

3. The slopes are smaller than expected

As already mentioned above, in every case we found that every slope in the Newton polygon of T_p acting on forms of level N was smaller than $(k-1)/2$. It is natural to ask whether this always happens.

Question. *Fix a prime number* p*. Let* $S_k(1, \mathbb{C}_p)$ *be the space of cusp-forms of weight* k *and level* 1*. Let* $f \in S_k(1, \mathbb{C}_p)$ *be an eigenform, and let* $a_p(f)$ *be the eigenvalue of* T_p *acting on* f*. Is it true that*

$$\mathrm{ord}_p(a_p(f)) < \frac{k-1}{2}$$

always?

Notice that a positive answer to this question implies that $a_p(f) \neq 0$, yielding a vast generalization of a famous conjecture of Lehmer about the case $k = 12$. This observation also shows that any generalization of this statement to higher level must be worded so as to exclude forms with complex multiplication and must also be stated for weight k sufficiently large.

4. The slopes are much smaller than expected

In fact, one sees much more. Even a cursory observation of the tables suggests that the slopes are much smaller than one might expect. In fact, we found that *in almost every case* the supersingularities for weight k and prime p are smaller than $1/(p+1)$. In other words, the inequality

$$\mathrm{ss}(f) \leq \frac{1}{p+1}$$

holds almost always for forms of level N. It follows that the sequence of supersingularities for weight k is almost always contained in $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$.

We need to be precise about what we mean by "almost always." What seems to emerge from the data is this: for almost all primes, the inequality is true for every eigenform. For a few exceptional primes, there is a thin set of weights for which we find supersingularities that are just a bit bigger than $1/(p+1)$.

Question. *Is it true that for almost all primes* p *the inequality*

$$\mathrm{ss}(f) \leq \frac{1}{p+1}$$

holds for every eigenform $f \in S_k(1, \mathbb{C}_p)$*? Equivalently, is it true that for almost all primes* p *we have*

$$\mathrm{ss}(f) \in \left[0, \frac{1}{p+1}\right] \cup \left[\frac{p}{p+1}, 1\right]$$

for every eigenform $f \in S_k(p, \mathbb{C}_p)$*?*

Prime p	Weights k
59	16, 46, 76, 106, 136, 166, 196, 226, 256
	286, 316, 346, 376, 406, 436, 466, 496
79	38, 44, 118, 124, 198, 204, 278, 284
	358, 364, 438, 444
2411	12
15271	16
187441	16
3371	20
64709	20
27310421	26

TABLE 1. Known exceptions to $\eta_i \leq 1/(p+1)$

To be more explicit about "almost all primes," in our computations exceptions to this inequality occurred only for $p = 59$ and $p = 79$; for each of these primes, the inequality fails to hold for the highest-slope form in certain weights. See Table 1 for the list of weights at which exceptional slopes appear; we discuss this list of weights further below. Other exceptions to the inequality, outside the range of this computation, can be read off from the results in [Gou97]; they correspond to forms of weights $k = 12, 16, 20$ that are non-ordinary with respect to large primes. The final entry in the table comes from a computation by Atkin. The full list of primes and weights for which we know of a slope that does not satisfy the inequality is given in Table 1. For each (p, k) pair, we found that exactly one slope in the first half of the slope sequence violates the inequality.

The structure of the table suggests that if there are any exceptional forms for a prime p, then there will be exceptional forms of relatively small weight, and the remaining exceptional forms will be in some sense related to these. This suggests the question

Question. *Let p be a prime. Is it true that if the inequality*

$$\mathrm{ss}(f) \leq \frac{1}{p+1}$$

holds when $f \in S_k(1, \mathbb{C}_p)$ and $k \leq p+1$, then it holds for all k and all $f \in S_k(1, \mathbb{C}_p)$?

We will postpone the discussion of the specific list of weights for which an exceptional form exists to a later section, and consider what these inequalities imply about the distribution of the supersingularities as k grows.

Assuming an affirmative answer to the questions above, we know that for almost all primes the measure μ_k is supported on $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$. This is not true for the other primes. However, if we focus on the exceptional slopes and compute the corresponding supersingularities, we see that in our examples $\mathrm{ss}(f)$ seems to get closer to $1/(p+1)$ as k grows. Let $p = 59$, for example; the sequence of supersingularities corresponding to the exceptional slopes in Table 1 is

$$0.066, 0.022, 0.026, 0.019, 0.022, 0.018, 0.020, 0.017, 0.019,$$
$$0.017, 0.019, 0.017, 0.018, 0.017, 0.018, 0.017, 0.018$$

Here, of course, $1/(p + 1) = 1/60 = 0.01666\ldots$, and the exceptional values of the supersingularity seem to be (slowly) approaching this value as the weight grows.

Encouraged by this, we can try to make "almost always" precise by using the measure-theoretic formulation:

Question. *Is it true that the sequence of measures $\{\mu_k\}$ converges, as $k \to \infty$, to the uniform measure on the set $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$?*

We have checked this by computing the first five moments of μ_k for our range of (p, k). The results strongly suggest an affirmative answer when there are no exceptional forms. When exceptional forms exist, the convergence is much slower, but it still seems reasonable that the limit measure will be as above.

It is natural, at this point, to ask what, if anything, has actually been proved in this direction. We know of no *general* results which suggest that "exceptional" slopes are rare. For specific primes, there are some hints. For $p = 3$, Lawren Smithline has shown in [Smi00] and [Smi] that for every form of level 1 and weight $k = 2 \cdot 3^a$ we have

$$\mathrm{slope}(f) < \frac{k}{4}.$$

For these values of k this is in fact equivalent to our inequality

$$\mathrm{slope}(f) \le \frac{k-1}{4}.$$

if we assume that all the slopes are integral (as we point out below, the slopes seem to be almost always integral).

For $p = 2$, Kevin Buzzard has formulated a conjectural description of *all* the slopes that implies, in particular, that the inequality

$$\mathrm{slope}(f) \le \frac{k-1}{3}$$

always holds for forms of level 1, i.e., that there are no exceptional forms for $p = 2$. (Buzzard's conjectural description also implies that all the 2-adic slopes for level 1 are integral.)

These hints suggest that something is going on. It seems to us that the fact that the inequality is so often true demands some explanation. In particular, one would like to know whether there is something special about the cases where it fails. It is natural to wonder whether one can identify a specific property of the modular forms or the Galois representations corresponding to exceptional pairs (p, k). See the discussion below on the possible connection to the Θ operator for more on this.

5. The slopes are almost always integers

Perhaps one of the more surprising outcomes of the computations is the fact that *almost all the slopes we obtained are integers*. Of all the observations we make, this is the one that is most likely to be merely an effect of the fact that we work only with small primes. The location of the exceptions, however, suggests that something else may be going on. Specifically, non-integral slopes occur in our computations only for $p = 59$ and $p = 79$, the same primes for which exceptional slopes occur. Furthermore, the fractional slopes we observe are connected to the exceptional slopes, in the following remarkable way:

1. A weight k for which there exists exceptional form with slope equal to 2 is preceded by a weight $k - 2$ for which the slope sequence contains two slopes equal to $1/2$. For $p = 59$, this happens for the pairs of weights $(74, 76)$ and $(104, 106)$; for $p = 79$, weights $(116, 118)$, $(122, 124)$.
2. A weight k for which there exists exceptional form with slope equal to 3 is preceded by a weight $k - 2$ for which the slope sequence contains two slopes equal to $3/2$, and that weight is preceded by a weight $k - 4$ whose slope sequence contains two slopes equal to $1/2$. This happens for $p = 59$ and weights $(132, 134, 136)$ and $(162, 164, 166)$; for $p = 79$, weights $(194, 196, 198)$ and $(200, 202, 204)$.
3. A weight k for which there exists exceptional form with slope equal to 4 is preceded by a weight $k - 2$ for which the slope sequence contains two slopes equal to $5/2$, that weight is preceded by a weight $k-4$ whose slope sequence contains two slopes equal to $3/2$, and that weight is preceded by a weight $k-6$ whose slope sequence contains two slopes equal to $1/2$. This happens for $p = 59$ and weights $(190, 192, 194, 196)$ and $(220, 222, 224, 226)$; for $p = 79$ and weights $(272, 274, 276, 278)$ and $(278, 280, 282, 284)$.

4. And so on. An exceptional form of slope n is and weight k is accompanied by a "trail" of pairs of forms of slope
$$\frac{2n-3}{2},\ \frac{2n-5}{2},\ \frac{2n-7}{2},\ \dots,\ \frac{1}{2}$$
and weight
$$k-2,\ k-4,\ k-6,\ \dots,\ k-2(n-1)$$
(one pair for each weight).

These patterns can overlap without interfering. For example, there are exceptional forms of slope 6 for $p = 79$ and weights 438 and 444. Each has its trail of weights for which fractional slopes exist. Because of the exceptional form at weight 438, there are forms of slope 1/2 at weight 428 and at each subsequent weight, up to forms of slope 9/2 at weight 436. Because of the exceptional form at weight 444, there are forms of slope 1/2 at weight 434 and at each subsequent weight, up to forms of slope 9/2 at weight 442. Hence, for example, at weight 436 we have *both* a pair of forms of slope 3/2 and a pair of forms of slope 9/2.

For the complete list of slope sequences which contain either exceptional or non-integral slopes, see
`http://www.colby.edu/personal/fqgouvea/slopes/exceptional.html`

This suggests the question:

Question. *How often are the slopes integral? Is it true that non-integral slopes occur only when p is a prime for which there exist exceptional slopes?*

All of the non-integral slopes we see are in fact half-integral, as the description above shows.

6. The shadow of the Θ operator

For this section, we limit ourselves to the two primes for which we have found exceptional and non-integral slopes, $p = 59$ and $p = 79$. As we mentioned above, the list of weights for which exceptional slopes occur seems to have some structure. We explore a possible connection between the list of exceptional weights and the Θ operator.

We recall the basic facts about Θ. As above, we restrict to the case of level $N = 1$. The Θ operator is the operator that acts on q-expansions as $q\frac{d}{dq}$, so that
$$\Theta\left(\sum a_n q^n\right) = \sum n a_n q^n.$$
(If we think of modular forms as functions on the complex upper half-plane, then $q = e^{2\pi i z}$ and Θ is just $\frac{d}{dz}$.) As is well known (see [Gou97]),

if f is a modular form then Θf is *not* a modular form, though it is "almost" modular in some sense (the p-adic story is a little different: see [Kat73, Gou88, CGJ95]). If we go ahead and formally compute the Hecke operators on Θf, we see that if f is an eigenform then so is Θf, and that $\mathrm{slope}(\Theta f) = 1 + \mathrm{slope}(f)$.

On the other hand, Θ does define an operator on modular forms modulo p, in which case it maps forms of weight k to forms of weight $k + p + 1$ (see [Joc82, Kat77]). The formula for the change in the slope no longer makes sense, of course. When one considers modular forms modulo p, one can only distinguish between forms of slope zero and forms of positive slope. What we can say, then, is that if f is a modular form modulo p then Θf will be a modular form modulo p and will always have positive slope.

Finally, recall that it is possible for forms whose weight differs by a multiple of $p-1$ to have identical q-expansions. (Basically, this is because the q-expansion of the Eisenstein series E_{p-1} is congruent to 1 modulo p.) Thus, if f is a modular form modulo p, one can ask what is the minimal weight k for which f lifts to a form of weight k in characteristic zero. Considering how the Θ operator affects this minimal weight leads to the theory of Θ-cycles, discussed in [Joc82]. We only need a small portion of the theory. Suppose f is a form modulo p of positive slope and weight k, $4 \le k \le p-1$. Consider the forms $f_i = \Theta^i f$. It is clear that $f_{p-1} = f$ (in the sense that they have the same q-expansion; remember that all of this is happening modulo p), so the f_i form a cycle. It is natural to ask whether there are any other values of i for which the minimal weight is small (i.e., less than $p-1$). It turns out that this happens only for $i = p-k+1$, in which case the form f_i is the reduction modulo p of a form of weight $p+3-k$.

The upshot, for us, is simply this: given a form of positive slope and weight $k < p-1$, there must also exist a form of positive slope and weight $p+3-k$. This explains part of the data above: for $p = 59$, the existence of a form of weight 16 and slope 1 forces the existence of a form of weight 46 and positive slope; for $p = 79$ the existence of a form of weight 38 and slope 1 forces the existence of a form of weight 44. The theory does not predict that these forms should have slope 1, but it turns out that they do. (This is not really surprising, since the slopes tend to be integers and they also tend to be "as small as possible.")

Notice that this whole theory refers *only* to modular forms modulo p. If f is an eigenform, then all of the $\Theta^i f$ are eigenforms modulo p; by the Deligne-Serre Lemma (see [DS74, Lemma 6.11] or [AS86, Prop. 1.2.2]), they lift to eigenforms in characteristic zero (but recall that the $\Theta^i f$ are not themselves modular forms, so the lifts will only be congruent to them). We know these eigenforms will have positive slope (which, as we noted, is "visible" modulo p as the fact that $U(f) \equiv 0 \pmod{p}$), but there seems to no reason to predict anything further about their slope.

Nevertheless, it seems that the Θ operator has a "shadow" in characteristic zero. To see this, consider the exceptional slopes for $p = 59$. For the case of weight 16, the occurrence of a large slope seems to be "accidental," but its "propagation" to higher weights seems to be linked to the Θ operator. Let f_0 be the (unique) form of weight 16 and level 1; its 59-adic slope is 1, because its 59-th Fourier coefficient is divisible (once) by 59. If we reduce f_0 modulo 59, then its image under Θ can be lifted to an eigenform of weight 76. This is exactly the exceptional form of weight 76, and it has slope 2. The same pattern continues as we iterate Θ, both starting with the form of weight 16 and starting with the form of weight 46.

Thus, it seems that, at least in the case of forms with exceptional slope, if we start from a form f of slope a and weight k, the Θ operator does somehow "produce" a form of weight $k + p + 1$ and slope $a + 1$ (though of course it will only be congruent to Θf modulo p, so that the congruence is not strong enough to explain the relation between the slopes). If this is correct, it means that for $p = 59$ there will be two infinite sequences of forms of exceptional slope, one in weights $16 + 60i$ and the other in weights $46 + 60i$. In each sequence, the slope is equal to $i + 1$ and each exceptional form is congruent modulo 59 to the image under Θ of the previous form in the sequence. For small enough weights, we have checked that this is indeed what happens.

The same analysis explains the two sequences of exceptional forms modulo 79, of weights $38 + 80i$ and $44 + 80i$. Again, the forms in each sequence seem to represent a shadow in characteristic zero of the Θ operator on forms modulo p.

Question. *Let f be an eigenform of level* 1*, weight k, and exceptional slope a. Is it true that for every integer $i > 0$ there exists an eigenform f_i of level* 1*, weight $k + i(p + 1)$, and slope $a + i$, such that $f_i \equiv \Theta^i f \pmod{p}$?*

One can easily compute what happens to the supersingularity along such a family. Under the same assumptions as in the question above, suppose

$$\mathrm{ss}(f) = \frac{a}{k-1} = \frac{1}{p+1} + e,$$

where, since we are assuming f is exceptional, we have $e > 0$. Assuming the existence of the forms f_i, set

$$\mathrm{ss}(f_i) = \frac{1}{p+1} + e_i.$$

Then one easily computes that

$$e_i = \frac{e}{1 + i\frac{p+1}{k-1}}.$$

In particular, we have $e_i > 0$ for all i, so that all of the f_i are exceptional. Note also that $e_i \to 0$ as $i \to \infty$, as suggested in question.

It might be useful to point out that the data show that if we start with a non-exceptional f it need not be true that a "shadow of Θf" will exist. One can see this by simply looking at the slope sequences at weight k and weight $k + p + 1$ and noting that the presence of a in the first sequence does not necessarily imply that $a + 1$ appears in the second. So we could formulate a broader question:

Question. *Let $f \in S_k(1, \mathbb{C}_p)$ be an eigenform of slope a. Under what conditions does there exist an eigenform $f_1 \in S_{k+p+1}(1, \mathbb{C}_p)$ which is of slope $a + 1$ and is congruent to Θf?*

(Note that there always does exist an eigenform which is congruent to Θf, so the crucial question here concerns the behavior of the slopes.)

At the level of the Galois representations attached to modular forms modulo p, the Θ operator corresponds to a Tate twist. Thus, it follows from the discussion above that for $p = 59$ or $p = 79$ all the forms with exceptionally large slope are attached to Galois representations modulo p which seem to be connected by Tate twists.

In fact, the situation is even stranger. Let us take $p = 59$ start once again with the unique cuspform f of weight 16; it has slope 1. As pointed out above, the reduction modulo 59 of Θf is an eigenform of weight 76; its lift to characteristic zero is an eigenform of slope 2, which is therefore exceptional. We could also consider, however, the form $E_{58} f$, whose reduction modulo 59 is an eigenform of weight $16 + 58 = 74$ whose q-expansion modulo 59 is identical to that of our initial form. It too must lift to an eigenform in characteristic zero. In fact, there are *two* such lifts, and both of them have slope $1/2$. The two lifts are defined over a ramified quadratic extension of $\mathbb{Q}_{59}$, and they are Galois-conjugate and congruent to each other.

One more step will make the overall pattern clear. From Θf in weight 76 we can go the $\Theta^2 f$ in weight 136; its lift to characteristic zero has slope 3 and is therefore exceptional. We can also consider $E_{58}\Theta f$, which is an eigenform modulo 58 of weight 134. It has two lifts to characteristic zero, both of slope $3/2$. Or we could look at $E_{58}^2 f$, which is an eigenform modulo p of weight 132. It has two lifts to characteristic zero, both of slope $1/2$.

Once again, we have only been able to check this pattern for small weights. What it suggests, however, is that every form on our list whose slope is either exceptional or non-integral corresponds, modulo p, to a Galois representation which is a Tate twist of the representation corresponding to the "initial" form of weight $k < p + 1$. This reinforces the feeling that there is a connection between forms whose slopes are unusually large and forms whose slopes are non-integral, and that all these forms correspond to Galois representations with unusual[1] properties. Why this should be the case seems completely mysterious.

Question. *Is there a representation-theoretic characterization of eigenforms that are of exceptional or non-integral slope?*

7. The slopes are too constant

Finally, we would like to observe that our computations strongly support Kevin Buzzard's observation (arising from his computations for $p = 2$) that the slopes of oldforms seem to be far more constant as the weight varies than one would expect. In this regard, recall that in [GM92] we conjectured that if we looked at two (sufficiently large) weights k_1 and k_2 such that $k_1 \equiv k_2 \pmod{p^n(p-1)}$, then the slope sequences for these two weights should be identical up to slope n. For $n = 0$, this is a theorem of Hida (see [Hid86a, Hid86b]). In general, Coleman [Col97] and Wan [Wan98] have shown that it is true if we strengthen the hypothesis to $k_1 \equiv k_2 \pmod{p^{M(n)}(p-1)}$, where $M(n)$ is a quadratic function of n.

What one actually sees in the data, however, is much stronger. Consider, for example, the case where $p = 5$, $n = 2$, $k_i = 112 + 100i$. Our conjecture would predict that the portion of the slope sequences that has slopes less than or equal to 2 would be the same for two such weights. Table 2 gives the (lower halves) of the slope sequences for $i = 0, 1, 2, 3$. What we see is that the *entire* (lower half of the) slope sequence for weight k_i reappears in weight k_{i+1}. This suggests that something immensely stronger than the conjectures in [GM92] should

[1] The 59-adic representation attached to the unique form of weight 16 is known to have unusual properties. Specifically, the image of its reduction modulo 59 in $\mathrm{PGL}_2(\mathbb{F}_{59})$ is isomorphic to the symmetric group S_4; see [SD73], [SD75], [SD77], and [Hab83]. On the other hand, we are unaware of anything unusual about the 79-adic representation attached to the form of weight 38.

Weight k	Slope Sequence (lower half)
112	(1, 5, 5, 5, 10, 11, 14, 15, 16)
212	(1, 5, 5, 5, 10, 11, 14, 15, 16, 20, 21, 24, 25, 27, 30, 31, 34)
312	(1, 5, 5, 5, 10, 11, 14, 15, 16, 20, 21, 24, 25, 27, 30, 31, 34, 36, 37, 40, 41, 45, 46, 47, 50, 51)
412	(1, 5, 5, 5, 10, 11, 14, 15, 16, 20, 21, 24, 25, 27, 30, 31, 34, 36, 37, 40, 41, 45, 46, 47, 50, 51, 55, 55, 55, 59, 60, 63, 64, 65)

TABLE 2. Slope sequences for $p = 5$

be true, at least for the slopes of oldforms. Further examples of this behavior can easily be extracted from the data.

8. Conclusions

Our computational results suggest several surprising regularities in the behavior of the slopes of p-oldforms for fixed p and varying k. It is quite possible that there are still more observations to make. For example, if it is true that the slopes for weight k are integers between 0 and $(k-1)/(p+1)$, with what multiplicities do these integers occur? The data for small primes suggests that here too the behavior is quite regular. Can one come up with a precise conjecture? Such a conjecture would be closely related to the above conjectures about the distribution of the supersingularities in the interval $[0,1]$. One could also consider the behavior of the slope for fixed k and varying p; in this case, the appropriate normalization seems to be to multiply the p-supersingularity by $p+1$, so that the normalized supersingularities will be in $[0,1]$. Finally, one could ask whether one can use the data to obtain predictions of what the *non-classical* slopes (i.e., the slopes corresponding to overconvergent p-adic modular forms of weight k) should be.

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