Pathways from the Past

I: Using History to Teach
Numbers,
Numerals, &
Arithmetic

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Contents

First Thoughts for Teachers

What's in this packet?

This packet contains a set of history-based student activities for learning about

Writing Whole Numbers
Zero
Writing Fractions
Negative Numbers

They are intended to supplement the contents of your regular math textbook. The activities are primarily mathematical; the settings are historical. Each topic is independent of the others, to give you flexibility in fitting them into your lessons. Each section points out how its mathematical content fits into the math curriculum. It also indicates what connections might be made with topics in the history curriculum.

Why use history?

To learn mathematics well at any level, students need to understand the questions before you can expect the answers to make any sense to them. To teach mathematics well at any level, you need to help your students see the underlying questions and thought patterns that knit the details together. Understanding a question often depends on knowing the history of an idea:

Where did it come from?

Why is or was it important?

Who wanted the answer and what did they want it for?

Each step in the development of mathematics builds on what has come before. The things that students need to know now come from questions that needed to be answered in the past. **Yesterday's questions can help students understand, remember, and use today's mathematical tools to deal with tomorrow's problems.**

Most students are naturally curious about where things come from. That curiosity needs to be encouraged and nurtured. It is a powerful, but fragile, motivator for learning about anything at any level. With your help, it can lead your students to make sense of the mathematical processes they need to know.

How can I use history to do this?

There are several ways to use history in the classroom. The most common way to do it is simply to use *stories*. If chosen carefully, a story about a historical person or event can help students understand and remember a mathematical idea. The main drawback of using stories is that often they are only distantly connected to the mathematics.

This packet is primarily devoted to presenting student activities, so stories do not play a major role in the material that follows. Instead, we focus on the following ways to use history:

Overview — It is all too common for students to regard school mathematics as a random collection of unrelated bits of information. But that is not how mathematics actually gets created. People do things for a reason, and their work typically builds on previous work. Historical information helps students to see this "bigger picture." It also often explains why certain ideas were developed. Many crucial insights come from crossing boundaries and making connections between subjects. Part of the big picture is the fact that these links between different parts of mathematics exist, and paying attention to their history is a way of making students aware of them.

Context — Mathematics is a cultural product, created by people in a particular time and place and often affected by that context. Knowing more about this helps us understand how mathematics fits in with other human activities. For instance, the idea that numbers originally may have been developed to allow governments to keep track of data such as food production embeds arithmetic in a meaningful context right from the beginning. It also makes us think of the roles mathematics still plays in society. Collecting statistical data, for example, is something that governments still do!

Depth — Knowing the history of an idea usually leads to deeper understanding. For example, long after the basic ideas about negative numbers were discovered, mathematicians still found them difficult to deal with. They understood the formal rules for them, but they had trouble with the concept itself. Because the concept was troublesome, they did not see how to interpret those rules in a meaningful way. Learning about their difficulties helps us understand (and empathize with) the difficulties students might have. Knowing how such difficulties were resolved historically can also help students overcome these roadblocks for themselves.

Activities — History is a rich source of student activities. It can be as simple as asking students to research the life of a mathematician, or as elaborate as a project that seeks to lead students to reconstruct the historical path that led to a mathematical breakthrough. The activities in the worksheets of this packet ask students to do specific things that will deepen their understanding of particular mathematical ideas and help them practice their skills. The

Teacher Notes also occasionally suggest some broader questions and projects that might be appropriate from time to time, at least for some students.

What if I don't know much history myself?

We'll help. This booklet contains a summary of the historical background for each topic and detailed solutions for all of the activity sheets. For an easily readable, compact, inexpensive source of further background information, we unabashedly suggest our own book, *Math through the Ages: A Gentle History for Teachers and Others* (Oxton House Publishers, 2002). That book also contains a section describing "What to Read Next" and an extensive bibliography for anyone who wants to do serious historical research about a topic or a person.

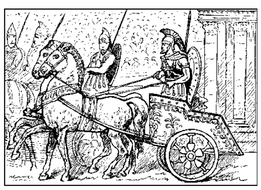
What do I really need to know first?

Not much—just a few small pieces of the Big Picture of the past several thousand years. Most of the mathematics we now learn in school comes from a tradition that began in the Ancient Near East, then developed and grew in Ancient Greece, India, and the medieval Islamic empire. Later this tradition migrated to late-Medieval and Renaissance Europe, and eventually became mathematics as it is now understood throughout the world. Some other cultures (Chinese, for example) developed their own independent mathematical traditions, but they have had relatively little influence on the substance of the mathematics that we now teach. That independence of cultures is one thing to keep in mind as you think about the history of numeration and arithmetic. Here are some other major historical reference points that your students might know from their history classes.

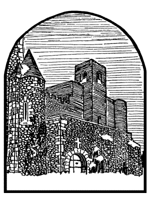
Ancient **Mesopotamia**, the region between the Tigris and Euphrates Rivers in what is now Iraq, is sometimes called the "cradle of civilization." If your students are studying ancient history, they will probably recognize the name of Hammurabi, a prominent ruler of the Babylonian Empire of the 17th century BCE. This was several thousand years after the beginnings of civilized society in that region, but it provides a convenient reference point. By Hammurabi's time, both writing and numeration were well developed tools for communication and commerce.

Ancient **Egypt** also had a well developed civilization by that time. It had developed around the Nile Valley in northern Africa. By about 3000 BCE, the country began to be unified under a single ruler (a Pharaoh). The pyramids date back to this millennium (3000–2000 BCE). Our main source of information about Egyptian mathematics comes from an artifact of a later time, the Ahmes Papyrus (or Rhind Papyrus) of about 1650 BCE.

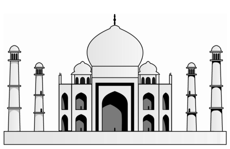
The Ancient **Greek civilization** dates from about the 6th century BCE to the Roman conquest, often identified with the Battle of Corinth in 146 BCE. In the 5th century BCE, Athens was the center of an intellectual and artistic culture. It was the era of dramatists Sophocles and Euripides, philosophers Socrates and Plato, and other prominent artists and intellectuals. When Alexander the Great's conquests spread Greek culture to the Near East and north Africa in the 4th century BCE, Alexandria (in Egypt) became another center of Greek learning. Most of the Greek mathematics we know about comes from the period after Alexander the Great, and a lot of it seems to be connected to the city of Alexandria. It is associated with names such as Euclid, who lived in Alexandria, and Archimedes, who lived in Syracuse (in Sicily).



When the **Roman Empire** absorbed Greece, it preserved Greek literature, art, and learning. As the Empire spread throughout Europe, so did Graeco-Roman culture, along with the language of Latin. The Roman Empire was the dominant political and cultural influence in Europe until the fall of Rome in 476 CE. The heavily geometric mathematics of the Greeks was gradually supplemented by a "subscientific" tradition of computational aids for building, commerce, and other practical pursuits.



The **fall of Rome** brought with it the political and social fragmentation of the **Middle Ages**. Between the 5th and 10th centuries CE, European scientific scholarship was essentially dormant, partially preserved but not materially enhanced by the dominance of the Christian Church and its monasteries. During that time, two other cultures that would influence Western mathematics were developing independently.

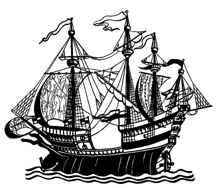


One was in **India**, where Hindu scholars were making great strides in astronomy, mathematics, and other intellectual pursuits. However, by the communication and travel standards of that time, India was very far from Europe, so almost none of this work found its way directly into the European tradition. That cultural link was provided by the Arabs, in a land geographically between India and Europe.

The rise of Islam began with the Hejira in 622 CE. A hundred years later, a loosely knit **Islamic Empire** stretched from India in the East to Spain in the West. It encompassed all of the African coast of the Mediterranean Sea, the entire Arabian Peninsula, and all of the Middle East, north to the Black and Caspian Seas. The caliphs (rulers) of the 9th century actively fostered the study of mathematics and science, drawing on both Greek and Indian sources. The common scholarly language throughout the Islamic Empire was Arabic.

The clash of European and Arabic powers in **the Crusades** opened the way to increased commerce between East and West. As goods and money flowed, so did ideas. By the 12th century, the scientific and mathematical advances of the Arabs were making their way into the European tradition as the Arabic manuscripts were gradually translated into Latin.

The exchange of ideas throughout Europe was aided by the use of **Latin** as the common language of scholarship. However, it was hampered by the fact that all copies of documents had to be done by hand. That changed in 1440 with the invention of **movable-type printing** by German inventor Johannes Gutenberg. The famous Gutenberg Bible of 1454 was the first book printed by this method. As printing spread, so did ideas. This invention was possibly the most influential technological breakthrough of the 15th century.



The **European Renaissance** of the 14th – 16th centuries awakened a renewed interest in science and mathematics, along with many extraordinary achievements in art, literature, and philosophy. Advances in shipbuilding and in craftsmanship of all kinds led to a wider range of commercial activity and exploration. This, in turn, called for more refined technological tools. It was the era of Christopher Columbus, Vasco da Gama, Ferdinand Magellan, John Cabot, and other seagoing adventurers. These activities required advances in such things as navigation, trigonometry, astronomy, and clock-making. Merchants learned arithmetic and developed double-entry bookkeeping.

The historical scope of this booklet and its activity sheets lies almost entirely within the boundaries of the foregoing milestones.

1

Writing Whole Numbers

Mathematical Focus	**Place Value**
Historical Connections	Ancient Egypt, c. 2000 BCE Ancient Mesopotamia, 2000–200 BCE Mayan Civilization, 1500 BCE The Roman Empire, 500 BCE India, 7th and 8th centuries CE

We write numbers using a *decimal place* system. The two italicized words refer to two different properties. *Decimal* says that our numeration system is based on ten; *place* tells us that the position of a symbol affects its value. Both properties are needed for the usual algorithms of arithmetic, but either can exist without the other.

This section explores historical numeration systems that did not have both of these properties together. By working with them, students will better understand the role of place value in our own system and how it relates to grouping by powers of ten. There are activities for four systems:

Egyptian Hieroglyphic — This system has no place value at all. It is based on ten, in the sense that each power-of-ten quantity has a different symbol, but the position in a symbol string does not affect a symbol's value.

Babylonian Cuneiform — This is a place value system based on 60, but it has no "zero" placeholder symbol, which results in confusing ambiguity.

Mayan — This is a place value system based mainly on 20, with one strange exception, and it does have zero placeholder symbol. Its symbol strings are written vertically.

Roman — The Roman system is based primarily on powers of ten, with a couple of extra symbols for five times some of them. It does not have place value in the usual sense, but a peculiar subtractive device depends on the relative positions of some symbols.

There is also a summary activity sheet comparing these four systems to each other and to our own Hindu-Arabic system. Besides learning about place value, students will get some valuable practice with powers and multiples of numbers as they work on these sheets. These unusual settings make the arithmetic useful and interesting, rather than routine drill done for its own sake.

Note: When the activity sheets refer to "numbers," they are written in our familiar Hindu-Arabic numeration system. There is a conceptual distinction between the numbers themselves and our way of writing them, but most students would find such an explanation more pedantic than helpful. We suggest that you ignore it unless a student raises the question.

Sheet 1-1: Egyptian Hieroglyphics

• Main Feature •
A base-ten system without place value

These activities illustrate grouping by tens without place value. In particular, they highlight the following features of the Egyptian hieroglyphic system, which was in use shortly after 2000 BCE, almost 4000 years ago.

- Multiple copies of a particular power of ten are denoted by repeating the symbol for that power.
- The order in which symbols are written does not affect their value. Descending order is useful for keeping track of values, but rearrangement for decorative purposes does not change the number represented.

- Historically, numbers were added using some sort of computational device: a counting board or an abacus. There was a separation between writing numbers down and doing arithmetic. Nevertheless, it is conceptually useful to consider how one would add numbers written in this system. Two numbers can be added by simply putting the symbols for each together. Ten symbols for a particular power of ten are then replaced by the symbol for the next power of ten. This is the point of the parenthetical "Why not?" question at the beginning of the sheet.

Students should notice the inconvenience of having to write so many symbols to represent even relatively small numbers. Question 2(b) is a good example of this. On the other hand, 1(b) and 2(c) show cases for which the Egyptians needed fewer symbols than we do.

Solutions

1. (a) 24 (b) 2,003,000 (c) 201,312

2. In each case, any two arrangements of the given symbols will work. Let your students simplify the lotus flower and astonished man symbols to make them easier to draw.
   (a) 𓍢𓍢𓍢𓍢𓍢∩∩∩|||||
   (b) 𓆐𓆐𓂭𓂭𓂭𓂭𓆼𓍢𓍢𓍢∩∩∩∩∩∩|||||||
   (c) 𓁨 |||

3. (a), (c), and (d) all stand for 11,122; (b) is 21,211; (e) is 111,202

4. (a) 𓂭𓆼𓍢𓍢𓍢𓍢|||| Check: 7200 + 4204 = 11,404
   (b) 𓆼∩||||||| Check: 563 + 454 = 1017
   (c) This example illustrates "borrowing" in the Egyptian system. One ∩ has to become ten copies of |, then one 𓍢 has to become ten copies of ∩, and finally one 𓆼 has to become ten copies of 𓍢 .
   𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢∩∩∩∩∩∩∩|||||||| Check: $1213 - 235 = 978$

5. This can be done by converting to our numeration system and multiplying. There are 34,050 soldiers, so there are $34{,}050 \times 3 = 102{,}150$ gold pieces. The hieroglyphic numeral is 𓆐𓆼𓆼𓍢∩∩∩∩∩. Calculating this *without* converting to our system is an instructive exercise in "carrying" in base-ten addition. Combine three copies of the numeral for the soldiers and exchange each group of ten like symbols for one symbol of the next higher power.

Sheet 1-2: Babylonian Numerals

• Main Feature •
A place system not based on ten

The Babylonian numeration system is sometimes called *cuneiform* ("kyoo-**nee**-uh-form" or "**kyoo**-nee-uh-form"), which means "wedge-shaped." It was used between 1900 and 1600 BCE in Mesopotamia, a region around the Tigris and Euphrates Rivers that is now part of Iraq. It is based on two wedge-shaped symbols, which looked something like 𒁹 and 𒌋. These symbols were quickly and easily pressed into soft clay tablets with a simple tool. Tablets that needed to be preserved were baked to form a hard, permanent record. Many of these tablets have survived.

(*Note:* We have simplified the shapes a bit to make them more clearly distinct from each other. Also, we write the symbols for a numeral on a single line for added clarity, rather than grouping or overlapping them, as the Babylonians did. The hand tablet graphic with question 8 shows how these symbols actually appeared.)

This system uses place value, but it is based on sixty, rather than on ten. That is, successive groups of symbols are multiplied by increasing powers of 60. The numbers 1 to 59 are made by adding combinations of the two basic symbols, with 𒁹 standing for *one* and 𒌋 for *ten.* By working with powers of 60, instead of powers of 10, students will see more clearly the essential features of any place value system.

Students may think it strange that the Babylonians thought about numbers in 60s much like we think in 10s, but we do it ourselves in very ordinary situations. For instance:

- We think in 60s when we tell time: 60 minutes in an hour, 60 seconds in a minute. That's 3600 seconds in an hour.

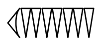
- We measure angles with a system based on 60: 360 ($= 6 \times 60$) degrees in a circle, 60 minutes in a degree of arc, 60 seconds in a minute of arc.

- Here's an example of measuring with 60s that is less well known. For ships and planes, distance is measured in nautical miles and speed is measured in knots. A nautical mile is the length of $\frac{1}{60}^{\text{th}}$ of a degree of the Equator. Thus, the full 360° length of the Equator is $60 \times 360 = 21,600$ nautical miles. A knot is one nautical mile per hour (60 minutes).

Why do we do it this way? Because of the Babylonians. These measurement customs have been handed down to us from the Babylonians of 4000 years ago!

This activity sheet introduces students to the Babylonian system in stages, starting with the numbers from 1 to 59 and progressing gradually to the powers-of-60 place values.

Solutions

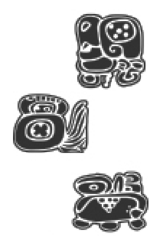
1. (a) 12 (b) 30 (c) 26
2. (a)  (b)  (c) 
3. (a) 201 (b) 674 (c) 1891
4. (a)  (b)  (c) 
5. This gets students to focus on the fact that $60^2 = 3600$ and to notice that $60^3 = 216{,}000$ is quite a large number for the fourth-place value.
6. $7883 = 2 \times 3600 + 11 \times 60 + 23$
7. Even with calculator assistance, working out these answers gives students some good arithmetic practice as they choose the appropriate multiples of the powers of 60.
   (a)  (b) 
8. The separate numerals represent 2 and 13. If they are 60s and 1s, respectively, then the number is 132, which makes sense for a flock of sheep, but not for a king's army. One possiblility is that the 1s place was skipped; if so, this represents $2 \cdot 60^2 + 13 \cdot 60 = 7980$. That's a large group of soldiers, but probably not a whole army. If both the 1s and 60s were skipped, then this represents $2 \cdot 60^3 + 13 \cdot 60^2 = 478{,}800$, a truly formidable force and one that may be too large for historical accuracy. Other possibilities are $2 \cdot 60^2 + 13 = 7213$, $2 \cdot 60^3 + 13 \cdot 60 = 432{,}780$, and $2 \cdot 60^3 + 13 = 432{,}013$, though this last number is less likely simply because the 13 probably would have been rounded off in tallying such a large number of soldiers.
   (*Historical Note:* This is the numeration system that was used during the First Babylonian Empire, which included the reign of King Hammurabi.)
9. This question highlights the importance of a zero placeholder. It is not easy. The figure is from a drawing of Ur Excavation Tablet 236 by Eleanor Robson. See Figure A.5.10 of *Mesopotamian Mathematics, 2100–1600 BC: Technical Constants in Bureaucracy and Education* by Eleanor Robson. Clarendon Press, Oxford,1999. Used by permission. The numerals of this question appear in the upper left quadrant of the tablet, which looks like the work of

somebody who was doing arithmetic exercises.
(*Note:* The symbols in the drawing are difficult to see clearly, partly because they are shaped less distinctly than the printed ones we use and partly because they overlap in groups on the tablet. For instance, the five 𒌋 marks on the upper left overlap so much that they look almost like a single symbol.)

Using the fact that the operation here is multiplication, the first part represents 50×30, *assuming* that these are numerals in the rightmost position (that is, counting 1s, not higher powers of 60). Now, $15 \times 30 = 1500$ and $1500 = 25 \times 60$, so the first answer, 𒎙𒐊, is in the 60s place; the 1s place has been skipped. Multiplying this answer by 𒐈 yields $1500 \times 3 = 4500$, which is $1 \times 3600 + 15 \times 60$, or 𒁹 𒌋𒐊, where again the 1s place has been skipped.

Sheet 1-3: Mayan Numerals

• Main Feature •
A non-decimal system with a placeholder

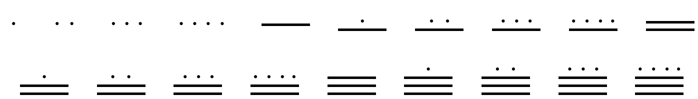


Halfway around the world, in a place that the Egyptians and Babylonians didn't know existed, lived a people called the Maya ("**mī** yuh"). The Maya lived in Central America, on and near the Yucatan Peninsula, now part of southern Mexico and the countries of Belize, Guatemala, and Honduras. During the centuries between 1500 BCE and 300 CE, the Mayan people had a very advanced civilization. As their population grew, they settled in larger and larger towns. They developed a written language, built large, colorfully decorated buildings, and created a detailed calendar for keeping track of the seasons, the phases of the moon, and similar things. These advances in language and science formed the basis for the rich Classic Period of Mayan culture, from about 300 to 900 CE.

The Maya had a numeration system similar to that of the Babylonians, but free from the spacing difficulty. There were two basic symbols, a dot, ·, for the number one, and a bar, ——, for the number five. These two symbols were used in combination to represent the numbers 1 through 19. These nineteen symbol groups were used to represent larger numbers by means of a place value system. The groups were arranged vertically, rather than horizontally, and were evaluated by adding the place value amounts for each group. The lowest group represented single units; the value of the second group was multiplied by 20, the value of the third by $18 \cdot 20$, the value of the fourth by $18 \cdot 20^2$, the value of the fifth by $18 \cdot 20^3$, and so on. Thus, the Mayan system was essentially based on twenty, except for the peculiar use of 18. The spacing difficulty of the Babylonian system was avoided by using a special "zero" symbol, , to show when a position was skipped. Questions 7–10 focus students on how place value works and how it depends on the base of the system.

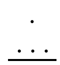
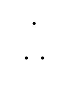
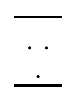
Solutions

1. This is an exercise in pattern recognition. These numerals for 1 through 19 are needed for the rest of this activity sheet.

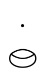
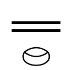
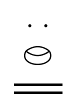
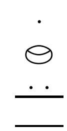


2. This example focuses attention on the details of the preceding paragraph.
   $2 \times (18 \times 20) + 11 \times 20 + 8 = 948$

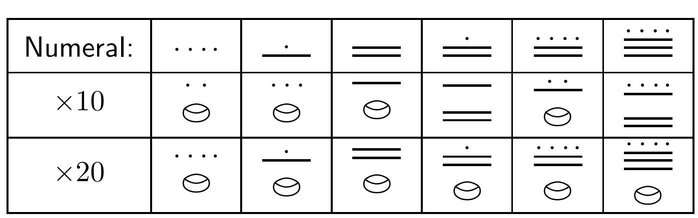
3. (a) $12 \times 20 + 11 = 251$
   (b) $5 \times (18 \times 20) + 8 \times 20 + 4 = 1964$
   (c) $10 \times (18 \times 20^2) + 3 \times (18 \times 20) + 7 \times 20 + 2 = 73{,}222$

4. (a)  (b)  (c) 

5. The point here is that, because the Mayan system is not based on powers of ten, the "zero" placeholders in that system do not match exactly with the zeros in our numerals.
   (a) $7 \times 20 = 140$ (b) $7 \times (10 \times 20) = 2520$ (c) $7 \times (18 \times 20^2) = 50{,}400$

6. (a)  (b)  (c)  (d) 

7. To multiply by 20, a Mayan would simply shift the digit over one place by appending a "zero" placeholder. Multiplying by 10 is not as easy; students might have to experiment a bit. The next question asks them to describe any patterns they see.



8. (a) Multiplication by 20 is easy: Put a "zero" placeholder under the numeral. Multiplying by 10 leads to two patterns, one easier than the other: If the given number is even, halve it and put a "zero" under it. If the number is odd, halve the even number before it and put two bars under it. (That is, multiply half the number before it by 20 and add 10.)

   (b) No; they do not work for two-place numerals. Putting a "zero" under a two-place numeral multiplies the lower place by 20, but the upper place by only 18. The simplest example is the dot-over-zero symbol for 20. Putting another "zero" under that makes it equal to 360, not 400. The same difficulty breaks the patterns for multiplying by 10.

9. Putting one more "zero" under this numeral multiplies its value by 20; putting two more "zeros" under it multiplies its value by 400. The original number is $8 \times (18 \times 20) = 2880$. $2880 \times 20 = 57{,}600 = 8 \times (18 \times 20^2)$. $2880 \times 400 = 1{,}152{,}000 = 8 \times (18 \times 20^3)$.

10. $19 \times 20 = 1 \times (18 \times 20) + 20 = 380$. The first of these would be correct in a pure base-20 system; the second uses the Mayan value of 360 for the third place in the numeral. There is some historical evidence to suggest that the Maya used both systems at one time or another.

Sheet 1-4: Roman Numerals

• Main Feature •
An additive system
(almost) without place value

The dominance of civilized Europe by the Roman Empire from about the first century BCE to the fifth century CE made Roman numeration the common European way of writing numbers for many centuries afterwards, even into the Renaissance. It is still used for decorative purposes. Like the Egyptian system, Roman numeration is additive and not positional (with one minor exception). Its basic numeral symbols are seven alphabet letters. These basic symbols and their corresponding values are listed in the table at right and on the activity sheet.

Symbol	Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Most of the time, the letters were given in order of descending value. To get the value of the entire numeral, the values of the basic symbols were added. For instance,

$$\text{CLXXII} = 100 + 50 + 10 + 10 + 1 + 1 = 172.$$

Larger numbers were written by putting a bar over a set of symbols to indicate multiplication by 1000. Thus, $\overline{\text{V}} = 5000$ and

$$\overline{\text{VII}}\text{CLXV} = 7000 + 100 + 50 + 10 + 5 = 7165.$$

Solutions

1. (a) 28 (b) 861 (c) 2137 (d) 6313 (e) 1,200,580
2. (a) XXXVII (b) CCLVI (c) MMXI (d) $\overline{\text{XX}}$CCCLXIII (e) $\overline{\text{MMI}}$

A peculiar feature of the Roman system is its subtractive device, which was a fairly late invention. If a basic symbol in a numeral had a smaller value than the one immediately to its right, then the smaller value was subtracted from the larger one to get the value of the pair. For instance,

$$\text{IV} = 5 - 1 = 4.$$

To avoid ambiguity, only symbols representing powers of ten could be subtracted, and they could be paired only with the next two larger values. For instance,

$$\text{MCMXCIV} = 1000 + 900 + 90 + 4 = 1994.$$

By this method, no more than three adjacent copies of the same basic symbol were needed in any numeral.

3. (a) 144 (b) 1999 (c) 3474
4. (a) CCCXXIV (b) CDLXXXIX (c) $\overline{\text{MMCCCXCVI}}$CMXLIV
5. (a) 1904 (b) MDCCCLIX You might ask students to look for a building in their town with a cornerstone marked in Roman numerals and bring in information about anything they find.
6. The information for this question is from the website *chart.copyrightdata.com*.
   (a) 1963 (b) 1935
   (c) The numeral reads 1944; the X should not be there.
   (d) The D is an error. Taken literally, MCMDXXVI is 1900 + 500 + 26 = 2426, which is an impossible date for this movie. Moreover, the Romans would never have written the number that way. Clever students might guess (correctly) that the D should have been an L, which makes sense. The movie was made in 1976.

The next three questions are more difficult than the others on this activity sheet. There are no obvious algorithms for doing these computations. Students will probably find different ways to organize their thinking and their work; allow for (and expect) some creative thought! These solutions are typical possibilities.

7. To add DCCCXLVIII and CDXXXIV, just put the symbols together, keeping the subtractive pairs together: (DCCCCD)(XLXXX)(VIIIIV). Now account for the subtractions by "cancelling" a subtracted quantity with a like added quantity, and rearrange the symbols in descending size order:

   (DDCC)(LXX)(VVII)

   Finally, convert any repeated symbols to the next larger size, as appropriate:

   MCCLXXXII

   Check: $848 + 434 = 1282$.

8. To subtract DCCCXLVII from MCCLXVI, break the problem into pieces that will allow for "borrowing," starting with units, then tens, and so on: (XVI minus VII)(L minus XL)(MCC minus DCCC), which is (IX)(X)(M minus DC, which is CD). Put the pieces together in descending order: CDXIX

   Check: $1266 - 847 = 419$.

9. This example avoids the complication of subtractive pairs in the multiplier; even so, it is not easy. To multiply CCLXXXIV by XVI, first use each multiplier digit separately: X times CCLXXXIV is MMDCCCXL; V times CCLXXXIV is DDCCLLLLXX; I times CCLXXXIV is CCLXXXIV. Now juxtapose the results and simplify, watching out for subtractive pairs:

   (MMDCCCXL)(DDCCLLLLXX)(CCLXXXIV)

   MMDDDCCCCCCCLLLLLXXXXIV

   $\overline{\text{IV}}$DXLIV

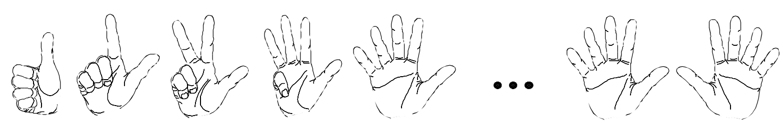
   Check: $284 \times 16 = 4544$.

Sheet 1-5: Hindu-Arabic Numerals

• Main Feature •
Comparing our system with the others

Our method for writing numbers is relatively new compared to the other systems described in this section. It was invented by the Hindus sometime before 600 CE and picked up by the Arabs during the Islamic expansion into India in the 7th and 8th centuries. That is why it is called the *Hindu-Arabic* system. The Europeans, in turn, learned it from the Arabs a few centuries later.

The Hindu-Arabic system uses place value and is based on powers of ten. Its basic symbols — 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 — are called *digits* and represent the numbers zero through nine. Nobody really knows why the number ten was originally chosen as the base for this system. The standard conjecture is that it was more biological than logical. Research indicates that this numeration system, like many others, emerged from finger counting, so it was natural that the base number should correspond to the number of fingers we human beings have. The very word we use for the basic numerals reflects this fact; *digitus* is the Latin word for *finger*.

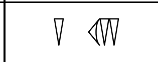
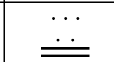
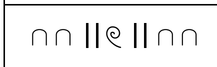
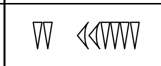
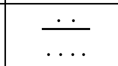
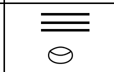
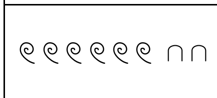
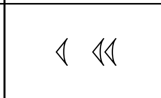
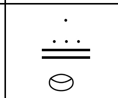
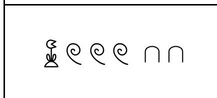
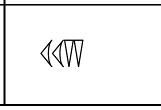
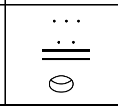


Despite its simplicity and efficiency, the Hindu-Arabic method of writing numbers did not displace the use of the Roman numeration system in Europe for several centuries. Old habits die hard. There were also practical reasons. For example, people worried about how easily a "2" can be changed into a "20" in the Hindu-Arabic system. Because of this, laws were passed saying that in legal documents numbers had to be written out in words. We still do this when we write checks.

One of the changes brought on by the Hindu-Arabic system is the fact that it is possible to compute directly with the numbers as written. The availability of cheap paper helped the new numbers to catch on. The advent of calculators has in some ways brought us back to having two systems: one for writing numbers and one (electronic) for actually doing computations.

Solutions

1. See Display 1.1. Two of the Babylonian entries illustrate the problem caused by the absence of a "zero" placeholder.

Egyptian	Babylonian	Mayan	Roman	Hindu-Arabic
			LXXII	72
			CXLIV	144
			CCC	300
			DCXX	620
			MCCCXX	1320

Display 1.1

2. This is an opinion question with a variety of legitimate answers. Look for some sense of reasonableness in student explanations.

3. This question links students' math work with their study of world history. In some of these cases, the dates span centuries and cannot be pinned down very precisely. However, these ten items have been chosen so that there is no ambiguous overlap of time periods. The proper order is:

   (b) by 3000 BCE
   (d) sometime in 2000 – 1000 BCE
   (a) 509 BCE
   (g) 331–323 BCE
   (c) 476 CE
   (j) 7th and 8th centuries
   (h) early 9th century
   (f) 13th century
   (i) 1452
   (e) 1492

2

Zero Is Not Nothing

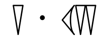
Mathematical Focus	**Properties of Zero**
Historical Connections	Ancient Mesopotamia, 2000–200 BCE India, 7th–11th centuries CE The Arab World, 9th century CE Europe, 12th–17th centuries CE

Most people think of zero as "nothing." The fact that it is *not* nothing lies at the root of at least two important advances in mathematics. These activities trace the development of zero from a place holder to a number and then from a number to an important algebraic tool. Along the way, students will see how to resolve potential confusions about the behavior of zero in arithmetic, such as the difference between division by zero and division into zero.

The story begins in Mesopotamia, the "Cradle of Civilization," sometime before 1600 BCE. By then, the Babylonians had a place value system for writing numbers based on grouping by sixty, much as we count 60 seconds in a minute and 60 minutes (3600 seconds) in an hour. They had two basic wedge-shaped symbols — 𒁹 for *one* and 𒌋 for *ten* — which were repeated in combination to stand for any counting number from 1 to 59. For instance, they wrote 72 as 𒁹 𒌋𒁹𒁹, with a small space separating the 60s place from the 1s place.

But there was a problem. The number 3612 was written 𒁹   𒌋𒁹𒁹 (one 3600 = 60^2 and twelve 1s) with a little extra space to show that the 60s place was empty. Since these marks were made quickly by pressing a wedge-shaped tool into soft clay tablets, the spacing wasn't always consistent. Knowing the actual value often depended on understanding the context of what was being described. Sometime between 700 and

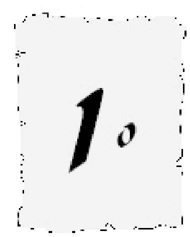
300 BCE, the Babylonians started using their end-of-sentence symbol (we'll use a dot) to show that a place was being skipped, so 72 and 3612 became, respectively,

 and 

When the Hindus developed the base-ten place value system (sometime before 600 CE), they used a small circle as the placeholder symbol. The Arabs learned this system in the 9th century, and their influence gradually spread it into Europe. This was done largely through the writings of the 9th-century Arab scholar Muḥammad Ibn Mūsa Al-Khwārizmī, whose book on arithmetic was translated into Latin in the 12th century and circulated throughout Europe.

In Al-Khwārizmī, zero is not thought of as a number; it is just a placeholder. In fact, he describes the Hindu-Arabic numeration system as using "nine symbols," meaning 1 through 9. He explained the role of zero like this:

> But when [ten] was put in the place of one and was made in the second place, and its form was the form of one, they needed a form for the tens because of the fact that it was similar to the form of one, so that they might know by means of it that it was [ten]. So they put one space in front of it and put in it a little circle like the letter o, so that by means of this they might know that the place of the units was empty and that no number was in it except the little circle...



(Translated from the Latin in "Thus spake Al-Khwārizmī" by John N. Crossley and Alan S. Henry, in *Historia Mathematica*, 17:103–131, 1990. The actual text has the Roman numeral "X" for "ten.")

The Indian word for this absence of quantity, *sunya*, became the Arabic *sifr*, then the Latin *zephirum* and *cifra*, and these words evolved into *zero* and *cipher*.

Sheet 2-1: Using a Placeholder

• MAIN FEATURE •
The importance of 0 as a symbol

These activities review the idea of place value and illustrate the importance of a place holder. They also highlight the distinction between a placeholder and a number, a significant issue in the history of the number system. The description of the Babylonian system here echoes and extends Activity 1-2 of Part 1.

Solutions

1. (a) 5 tens (b) 5 ones (c) 5 thousands

2. (a) 14 (b) 43 (c) 51

3. Students might notice that there are more than three correct answers to this, depending on whether two or more consecutive spaces are skipped. For instance, these numerals might be in the 60^3 and 60^2 places, or in the 60^5 and 60^3 places, etc. If they observe that, congratulate them on their understanding of the difficulty. The answers given here follow the pattern of the example just before this question.

   (a) $12 \cdot 60 + 21 = 741$ (b) $12 \cdot 60^2 + 21 = 43{,}221$ (c) $12 \cdot 60^2 + 21 \cdot 60 = 44{,}460$

4. These questions are like puzzle-book problems that many children (and adults, too) find fascinating. They are not easy. Students will probably need to use scrap paper for this part and the next. Besides emphasizing the role of a place holder, these questions provide arithmetic practice in a novel setting.

   (a) $203 + 107 = 310$ (b) $230 - 107 = 123$

   (c) $203 + 17 = 220$ (d) $550 + 505 = 1055$

   (e) $550 + 55 = 605$ (f) $550 - 505 = 45$ and $505 - 55 = 450$

5. These three questions, taken together, are actually easier than they first appear. Trying the various possible positions for 0 in the first numeral will lead to one or another of these results right away.

   (a) $2075 \times 4 = 8300$ (b) $2705 \times 4 = 10{,}820$ (c) $20{,}705 \times 4 = 82{,}820$

6. This question highlights the difference between place holder and number. The place holder 0 does *not* mean "multiply by zero"; it means "skip this place."

   (a) 7 tens (b) 4 hundreds

   (c) 5 hundreds and 6 ones (d) 2 thousands and 1 ten

Sheet 2-2: The Number Zero

• Main Feature •
Using 0 as a number

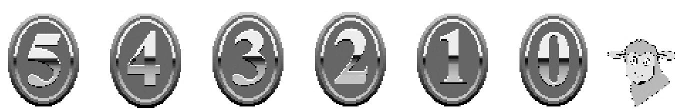
This activity sheet shows how the number zero behaves with respect to addition, subtraction, multiplication, and division. It pays special attention to distinguishing between division into zero and division by zero, a frequent point of confusion for students. It also looks at what it means to raise something to the 0 power.

As long as 0 was just a place holder, there was no need to worry about how it behaved in arithmetic. After all, it wasn't a "something" in any mathematical sense. It was just a punctuation mark — like a comma or a dash — to say which place to skip. But by the 9^{th} century CE, the Hindus had made a conceptual leap that ranks as one of the most important mathematical events of all time: They recognized this absence of quantity as a quantity in its own right. That is,

they began to treat zero as a number.

For instance, sometime around 850, the Indian mathematician Māhavīra wrote that a number multiplied by zero results in zero, and that zero subtracted from a number leaves the number unchanged. He also claimed that a number divided by zero remains unchanged. More than two centuries later, however, Bhāskara declared a number divided by zero to be an infinite quantity.

The main point here is not whether these Indian mathematicians got the right answers, but the fact that they asked such questions at all. To compute with zero, you first have to recognize it as *something*, an abstraction like *one*, *two*, *three*, etc. You must move from counting one goat or two cows or three sheep to thinking of *one*, *two*, *three* on their own, as things that can be manipulated without regard to what is being counted. Then you must take an extra step, to think of 1, 2, 3,... as ideas that exist *even if they aren't counting anything at all.* (Grammarians might say that it converts them from adjectives to nouns.) Then, and only then, does it make sense to treat zero as a number.



As the Al-Khowārīzmi quote above indicates, the Arabs did not recognize zero as a number. Neither had the Greeks, despite their impressive mathematical achievements many centuries before. European mathematics was heavily influenced by the Greeks and by the Arabs, so the idea that zero is a number took a long time to get established in Europe. Motivation for doing so came from the gradual acceptance of the Hindu-Arabic numeration system. As people started to compute in this system, it became necessary to explain how to add and multiply when one of the digits was 0. Nevertheless, even some of the most prominent mathematicians of the 16^{th} and 17^{th} centuries would not accept zero as a legitimate solution for an equation.

The mathematical issue is about making the most useful choices. We can *choose* to define the behavior of 0 in any way we please. The trick is to do it so that the essential “nice” properties of arithmetic are not damaged. When we do arithmetic with numbers that include the symbol 0, we need to know how to add, subtract, and multiply with that symbol.

People wanted arithmetic to work, so they quickly figured out how to add, subtract, and multiply with 0. Dividing by 0 was never needed. We could define it if we wanted to, but not without problems. For instance, we could choose to define $5 \div 0$ as 0 or 1 or 236, but that would conflict with the basic relationship between multiplication and division for nonzero numbers. We would have to keep accounting for this awkward exception. Several of these questions examine this issue.

Solutions

1. If you use this question and the next for some group discussion, that will help to sharpen students’ thinking and their ability to express mathematical ideas clearly. The simple questions of (b), (c), and (d) here set up a pattern of questions that will be asked for other arithmetic operations. Parts (b) and (c) together address commutativity: Does the operation work the same way in both orders? Part (d) asks about combining 0 with itself, a question that becomes problematic for division and exponentiation.

   (a) In common-sense terms of counting, if you have a particular number of things and add no more to them, then you should have the same number of things you started with.

   (b) $5 + 0 = 5$ (c) $0 + 107 = 107$ (d) $0 + 0 = 0$

2. (a) This invites another common-sense response. If you have some cats and take away none of them, you are left with the same number of cats.

   (b) $3 - 0 = 3$ (c) $1907 - 0 = 1907$ (d) $0 - 0 = 0$

   (e) You may have to help students here because the answer is largely historical. In Māhavīra’s time, negative numbers were not even considered. Numbers were thought of in terms of counting or measuring things. The idea that you could “take away” some quantity from no quantity would have been regarded as nonsense.

3. (a) Any number n multiplied by 0 should be the same as n copies of 0 added together. For instance, $5 \times 0 = 0 + 0 + 0 + 0 + 0$. Since $0 + 0 = 0$, the entire sum should be 0. Students might also consider $0 \times n$ as 0 copies of n “added together,” which would also result in 0.

   A different point of view is to say that we want multiplication by 10 to do the right thing. For that to happen with the usual method for multiplying numbers, we need to say $n \times 0 = 0$.

   (b) $7 \times 0 = 0$ (c) $0 \times 25 = 0$ (d) $0 \times 0 = 0$

4. This illustrates why division by 0 is undefined. There is no way to define it and preserve the fundamental property that $(a \div b) \times b = a$ for any number a.

   (a) Students might see this more easily if they write the example in fraction form: $5 \times (3 \div 5) = 5 \times \frac{3}{5} = 3$. In general, it should be true that $b \times \frac{a}{b} = a$ for any numbers a and b. So $0 \times (3 \div 0)$ should be 3, but, according to Māhavīra, $0 \times (3 \div 0) = 0 \times 3 = 0$.

   (b) If you divide a number by smaller and smaller numbers, the quotient gets bigger and bigger. So, as the divisor approaches 0, the quotient increases without limit; that is, it "approaches infinity."

   (c) If $7 \div 0 = B$, then $(7 \div 0) \times 0 = B \times 0$. But $(7 \div 0) \times 0$ should equal 7 (because multiplying by 0 should undo dividing by 0), whereas $B \times 0$ must equal 0 (because any number times 0 equals 0, as in part 3).

   (*Note:* Some students might suggest that a number divided by 0 ought to equal 0. The same kind of thing goes wrong in that case, too: $7 \div 0 = 0$ implies $(7 \div 0) \times 0 = 0 \times 0$, so again $7 = 0$.)

5. The parenthetical question should trigger a common-sense understanding here: Half of nothing is still nothing, etc. (a) 0 (b) 0 (c) 0

At this point, some students might ask about $0 \div 0$. This question is a little subtler than that of dividing a nonzero number by 0. Division is defined like this:

$$a \div b = q \text{ if and only if } b \times q = a.$$

If $b = 0$ and $a \neq 0$, then no number makes sense for q because $b \times q = 0 \times q = 0$. However, if both a and b are 0, then *any* number q will satisfy this definition! For instance, $0 \times 7 = 0$, so we could say $0 \div 0 = 7$. But if we do that, lots of basic arithmetic properties go wrong. For instance:

$$14 = 7 \times 2 = \frac{0}{0} \times 2 = \frac{0 \times 2}{0} = \frac{0}{0} = 7$$

Any nonzero choice for q will lead to a similar contradiction. Well, what about defining $0 \div 0 = 0$? In that case, we run into difficulties like this:

$$3 = 3 + 0 = \frac{3}{1} + \frac{0}{0} = \frac{3 \cdot 0 + 1 \cdot 0}{1 \cdot 0} = \frac{0}{0} = 0$$

Your sense of what your students know and need to know should determine whether or not you decide to deal with this issue here.

6. (a) It means that the number of 7s multiplied together is 5.

   (b) $7 \times 7 \times 7 \times 7 \times 7$ and $7 \times 7 \times 7$

   (c) $N = 8$ because the total number of 7s in the product is $5 + 3$.

(d) This is a difficult question for students who have not seen the idea already. The next question revisits this idea a bit more gently. $7^5 \times 7^0$ should be 7^5 because no more 7s are appended to the first product. Therefore, 7^0 must be 1, because it does not change 7^5 by multiplication.

7. This question repeats the idea of question 6, with a little more help.

   (a) $7^5 \times 7^0 = 7^5$ (b) $7^0 = 1$ (c) $2^0 = 1$ (d) $586^0 = 1$

   The implication is that any nonzero number to the 0 power must equal 1.

8. This reinforces the idea in Question 7. False. Counter-example: $2^0 = 3^0$ because both equal 1, but $2 \neq 3$.

Sheet 2-3: Zero in Equations

• Main Feature •
Using 0 as a tool in algebra

This activity sheet presumes that your students know some algebra and coordinate geometry. The terms *polynomial* and *quadratic equation* are used, but all we need is for students to know what those words mean. We assume that your students know what it means to "solve an equation" and to graph an expression in x and y.

All of the questions are based, directly or indirectly, on a fundamental property of our number system:

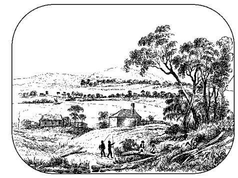
If the product of two numbers equals zero,
then at least one of them must be zero.

That is,

$$\text{if } ab = 0, \text{ then either } a = 0 \text{ or } b = 0 \text{ (or both).}$$

Mathematicians refer to this property by saying that our number system does not contain "zero divisors"; i.e., there are no pairs of nonzero numbers whose product is 0.

By the end of the 16th century, zero was widely accepted as a legitimate number. This was also a time in which the symbols for writing algebra were gradually becoming more standardized. Early in the next century, two mathematicians used zero in a way that transformed the theory of equations.



The first of these was Thomas Harriot (1560–1621), a man of many different talents, interests, and accomplishments. Among other things, in 1585 he was sent by Sir Walter Raleigh to help establish the ill-fated Virginia colony on Roanoke Island, now part of North Carolina. He was their surveyor, and he chronicled the settlers' activities and the natural resources of the area.[1] Harriot proposed a simple but powerful technique for solving algebraic equations:

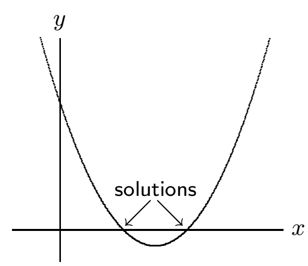
> Move all the terms of the equation to one side of the equal sign, so that the equation takes the form [some polynomial] $= 0$.

This is such a common part of elementary algebra today that we take it for granted, but it was a revolutionary step forward at the time. Here is a simple example of how it works.

> To solve $x^2 + 2 = 3x$, rewrite it as $x^2 - 3x + 2 = 0$. The left side can be factored into $(x-1)(x-2)$. Now, since this product equals 0, at least one of the factors must be 0. This allows us to find the two roots by solving the much simpler equations $x - 1 = 0$ and $x - 2 = 0$.

Harriot's Principle, as this is sometimes called, was popularized by Descartes in his writings on analytic geometry during the first half of the 17th century. When linked with Descartes' coordinate geometry, this principle becomes even more powerful. To see why, think about solving

$$\text{[some polynomial in } x\text{]} = 0.$$



If you graph

$$\text{[some polynomial in } x\text{]} = y$$

on an xy-coordinate plane, then the solutions (if any) will be where the graph crosses the x-axis. Even if the equation can't be solved exactly, a careful picture will give you a good approximation of its solutions.

The first three questions on this activity sheet are related. Taken together, they highlight a fundamental arithmetic property, cancellation. The fact that there are no zero divisors allows us to cancel a common nonzero factor from both sides of an equation. The formal description goes like this:[2]

[1]Harriot's life would make an excellent topic for student projects, particularly in conjunction with other subjects they are studying. It could be used to connect math with geography, history, and natural science.

[2]These arguments may be too formal for your students. They are here mainly for your information, to be used in whatever way you see fit.

If a, b, and c are numbers such that $ab = ac$ and $a \neq 0$, then $b = c$.

Justification: $ab = ac$ implies $ab - ac = 0$. But $ab - ac = a(b - c)$, so $a(b - c) = 0$. Now, *because there are no zero divisors*, $a \neq 0$ implies $b - c = 0$, so $b = c$.

By the way, it is quite proper to refer to this as *cancellation*, rather than as "dividing by a" or "multiplying by $\frac{1}{a}$." It is true even in the system of integers, where $\frac{1}{a}$ does not exist if $a \neq \pm 1$. Some people think of cancellation as a more obvious numerical fact and derive the no-zero-divisor property from that, as follows:

Suppose a and b are numbers such that $ab = 0$ and $a \neq 0$.
Then $ab = a \cdot 0$, so, by cancellation, $b = 0$.
That is, any product that equals 0 has at least one zero factor.

Solutions

1. This question makes students notice the fact that there are no zero divisors. The "explain why" part does not require a specific answer. Depending on the grade and ability level of your students, you might expect an informal response based on repeated addition: Nonzero numbers added together a nonzero number of times can never be zero, or something of that sort. More sophisticated students might justify it by cancellation or by "dividing out" one nonzero number to show that the other must be zero. Responses like these are good because they display an understanding of the basic idea.

2. This question presents examples of the cancellation law. Students should come away with an intuitive understanding that nonzero numbers can be cancelled, but 0 cannot.
   (a) $a = b$ (b) $a = b$ (c) nothing

3. This question presumes that students have some experience with equations and with being asked to justify or explain what they do.
   $3a = 3b$ implies $3a - 3b = 0$, so $3(a - b) = 0$. By #1, $3 \neq 0$ implies $a - b = 0$, so $a = b$.

4. North Carolina 

5. Besides reinforcing Harriot's Principle, these questions give students some practice in manipulating algebraic expressions and signed numbers.
   (a) $x^2 - 4x + 3 = 0$
   (b) $2x^3 - 4x^2 - 8x + 12 = 0$
   (c) $-4x^2 + 8x - 4 = 0$

6. The leading coefficient of each of these polynomials is easily factored out, so the polynomial can be written as that coefficient times a simpler polynomial. Since the coefficient is nonzero, the simpler polynomial must equal 0, so solving the simpler equation is equivalent to solving the original one.

   (5b) $x^3 - 2x^2 - 4x + 6 = 0$ (5c) $x^2 - 2x + 1 = 0$

Questions 7 and 8 do not presume that the students know how to factor quadratics or other polynomials. Therefore, the factored forms are given, so the answers come almost immediately from the fact that there are no zero divisors. If your students have had some experience with factoring polynomials, you could easily add more interesting questions by giving them equations that require factoring. Here are a few possibilities, along with their rearrangements and factored forms.

$$x^2 - 6 = 2x + 9 \;\Leftrightarrow\; x^2 + 2x - 15 = 0 \;\Leftrightarrow\; (x+5)(x-3) = 0$$
$$x^2 - 5x = 4(x-1) - x^2 \;\Leftrightarrow\; 2x^2 - 9x + 4 = 0 \;\Leftrightarrow\; (2x-1)(x-4) = 0$$
$$x^3 - 4 = 4x - x^2 \;\Leftrightarrow\; x^3 + x^2 - 4x - 4 = 0 \;\Leftrightarrow\; (x+1)(x+2)(x-2) = 0$$

7. The factored form is in the paragraph just before the question. This is the same as solving $(x-3)(x-1) = 0$. Since there are no zero divisors, this is true if and only if $x - 3 = 0$ or $x - 1 = 0$. Therefore the solutions are 3 and 1.

8. The idea here is the same as in #7; set each linear factor equal to 0 and solve.

   (a) $x = \frac{5}{2}$ or -3 (b) $x = -7$ or $\frac{2}{3}$

9. This question illustrates how Harriot's Principle and Descartes' geometry combine to make a powerful tool for approximating solutions to equations of all sorts. It also gives students practice with estimating and refining their first approximations, a valuable skill in all applications of mathematics. There is no single "best" answer. Here is a sequence of reasonable approximations, assuming that the tick marks on the x-axis are 1 unit apart. (*Note:* If you let students use calculators to do the routine computations, they will be better able to focus on the main idea.)

   $x = 2.5$; difference: 0.25 $(2.5^2 + 11 = 17.25;\ 7 \cdot 2.5 = 17.5)$
   $x = 4.5$; difference: 0.25 $(4.5^2 + 11 = 31.25;\ 7 \cdot 4.5 = 31.5)$

   $x = 2.4$; difference: $16.8 - 16.76 = 0.04$
   $x = 4.6$; difference: $32.2 - 32.16 = 0.04$

   $x = 2.38$; difference: $16.6644 - 16.66 = 0.0044$
   $x = 4.62$; difference: $32.3444 - 32.34 = 0.0044$

10. This question links back to an earlier point in this section — division by 0 leads to big trouble! The difficulty here is in the fourth step, "Divide by $(a - b)$." Since $a = b$, this is division by 0, which is not allowed.

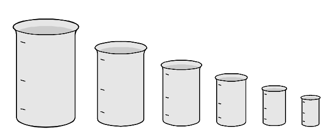
3

Broken Numbers

Mathematical Focus	**Fraction Arithmetic**
Historical Connections	Ancient Egypt, c. 2000 BCE Mesopotamia, 2000–200 BCE China, c. 100 BCE India, 7th–9th centuries CE Europe, 12th–18th centuries CE

Fractions have been part of mathematics for 4000 years or so, but the way we write and think about them developed much more recently. Some of our most common "rules of thumb" for fraction arithmetic depend more on the way we write fractions than on their values. For instance, "invert and multiply" makes no sense unless fractions are written as stacked pairs of numbers. This unit explores the history of fraction notation and its effect on how we compute. Along the way, it reinforces the common rules for working with fractions.

In earlier times, when people needed to account for portions of things, the things were broken down (sometimes literally) into smaller pieces and then the pieces were counted. The word *fraction*, which has the same root as *fracture* and *fragment*, reflects this idea. This evolved into primitive systems of weights and measures that made the basic measurement units smaller as more precision was needed. Some measurement systems still in use today are based on counting smaller units, rather than dealing with fractional parts. For instance, in the following list of liquid measures, each unit is half the size of its predecessor:

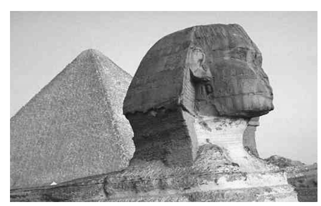


gallon, half-gallon, quart, pint, cup, gill.

The history of common fractions can be traced back to the Egyptian system of "parts" — what we today would call *unit fractions* — sometime in the second millennium BCE. At about the same time, Mesopotamian mathematics had a place value system that foreshadowed our system of decimal fractions. In neither case were fractions written as they are today. By 100 BCE, the Chinese were writing and working with fractions much as we do, but that was unknown to the Western world at the time. Hindu manuscripts as early as the 7th century CE show a similar approach, perhaps learned from the Chinese. These methods gradually became more widespread in Europe during the 12th to 16th centuries CE. Percent was used in European monetary transactions as early as the 15th century, but decimal fractions did not become commonplace for another 150 years. These activity sheets focus on those development stages separately, taking from each one something that helps students to see how fraction arithmetic works today.

Sheet 3-1: Unit Fractions

• MAIN FEATURE •
Using fractions with numerator 1



In the 17th century BCE, the basic operations of Egyptian arithmetic were addition, subtraction, and doubling. Multiplication were done by a system of proportions, using these three operations.[3] Their sense of fractions was very different from ours. They worked only with the idea of "the nth part," the unit fraction $\frac{1}{n}$ for any positive whole number n. To them, it made sense to consider "the fifth" (meaning $\frac{1}{5}$) or "the tenth" (meaning $\frac{1}{10}$), but not three fifths or seven tenths, etc. That is, there was only one basic "part" of each size. What we think of as other fractions, they would describe as sums of these basic parts, never using more than one of each size. For example, *three fifths* was thought of as "the half and the tenth."

This activity sheet asks students to work with such a system. At first we use a few of the Egyptian numerals, which are explained but will be familiar to students who did Activity Sheet 1-1. Then we shift to more familiar notation that captures the same idea. (For simplicity, we ignore the fact that the Egyptians had special symbols for the three common fractions $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.)

[3]See pp. 29–32 of Roger Cooke's *The History of Mathematics* (John Wiley & Sons, Inc., 1997) for an excellent description of this.

Solutions

1. This will not be easy for some students. They might need help seeing how the filled-in information for $\frac{5}{12}$ works. See Display 3.1 for the completed table.

$\overline{\cap\,\vert\vert}$	$\frac{1}{12}$
$\overline{\vert\vert\vert\vert\vert\vert}$	$\frac{2}{12} = \frac{1}{6}$
$\overline{\vert\vert\vert\vert}$	$\frac{3}{12} = \frac{1}{4}$
$\overline{\vert\vert\vert}$	$\frac{4}{12} = \frac{1}{3}$
$\overline{\vert\vert\vert}\quad\overline{\cap\,\vert\vert}$	$\frac{5}{12} = \frac{1}{3} + \frac{1}{12}$
$\overline{\vert\vert}$	$\frac{6}{12} = \frac{1}{2}$
$\overline{\vert\vert}\quad\overline{\cap\,\vert\vert}$	$\frac{7}{12} = \frac{1}{2} + \frac{1}{12}$
$\overline{\vert\vert}\quad\overline{\vert\vert\vert\vert\vert\vert}$	$\frac{8}{12} = \frac{1}{2} + \frac{1}{6}$
$\overline{\vert\vert}\quad\overline{\vert\vert\vert\vert}$	$\frac{9}{12} = \frac{1}{2} + \frac{1}{4}$
$\overline{\vert\vert}\quad\overline{\vert\vert\vert}$	$\frac{10}{12} = \frac{1}{2} + \frac{1}{3}$
$\overline{\vert\vert}\quad\overline{\vert\vert\vert}\quad\overline{\cap\,\vert\vert}$	$\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$
$\vert$	$\frac{12}{12} = 1$

Display 3.1

2. These questions provide some good practice in thinking about and manipulating fractions. No quick algorithm works easily for all such questions. One approach (called the "greedy method") is to find the largest unit fraction that is smaller than the number, subtract it, and repeat the process until the number is "used up."

   (a) $\dfrac{4}{7} = \dfrac{1}{2} + \dfrac{1}{14}$ ("the half and the fourteenth") (b) $\dfrac{11}{16} = \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{16}$

   (c) $\dfrac{13}{27} = \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27}$ (d) $\dfrac{23}{50} = \dfrac{1}{3} + \dfrac{1}{10} + \dfrac{1}{50} + \dfrac{1}{150}$

3. The top symbol represents $\frac{1}{10}$, as noted at the beginning of this sheet. The rest of the numerals are the successive multiples of $\frac{1}{10}$, up to $10 \cdot \frac{1}{10} = 1$. You might need to remind students that numerals next to each other are intended to be added. See Display 3.2 for the complete list of translated symbols.

4. This activity gives students practice in manipulating fractions, especially in subtraction. Some parts are not easy. The parts with even denominators are doubled simply by halving the denominator. But if the denominator is odd,

some ingenuity is needed. Modern students have the advantage of being able to use general fractions in order to find the appropriate unit fraction, a process unknown to the ancient Egyptians. See Display 3.3 for the completed table.

$$\begin{aligned} &\tfrac{1}{10} \\ \tfrac{1}{5} &= \tfrac{2}{10} \\ \tfrac{1}{4}+\tfrac{1}{20} &= \tfrac{3}{10} \\ \tfrac{1}{3}+\tfrac{1}{15} &= \tfrac{4}{10} \\ \tfrac{1}{2} &= \tfrac{5}{10} \\ \tfrac{1}{2}+\tfrac{1}{10} &= \tfrac{6}{10} \\ \tfrac{1}{2}+\tfrac{1}{5} &= \tfrac{7}{10} \\ \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{20} &= \tfrac{8}{10} \\ \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{15} &= \tfrac{9}{10} \\ 1 &= \tfrac{10}{10} \end{aligned}$$

Display 3.2

part	× 2	× 4	× 8
$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$\frac{1}{28}$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{1}{4}+\frac{1}{28}$
$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{5}+\frac{1}{45}$	$\frac{1}{3}+\frac{1}{9}$
$\frac{1}{13}$	$\frac{1}{7}+\frac{1}{91}$	$\frac{1}{4}+\frac{1}{18}+\frac{1}{468}$	$\frac{1}{2}+\frac{1}{9}+\frac{1}{234}$

Display 3.3

5. Students should see that half of any "part" is the "part" with the "size" (the denominator) doubled.

   (a) $\frac{1}{32}$ (b) $\frac{1}{30}$ (c) $\frac{1}{700}$

6. This generalizes the previous question. To divide a "part" by any whole number n, just multiply its "size" (its denominator) by n. For example, $\frac{1}{3} \div 7 = \frac{1}{21}$. (Of course, students may give a wide variety of examples.)

Sheet 3-2: Place Value Fractions

• MAIN FEATURE •
Forerunners of decimal fractions

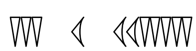
This sheet could be called "Decimal Fractions" because that is the modern idea it illustrates. However, the word *decimal* refers to base ten, and its historical forerunner was the Babylonian system based on *sixty*. This became the standard system for representing certain numbers in Europe as well. For example, it was used in trigonometric tables for a long time. And, of course, we still use it when we measure time.

These activities extend the ideas of Sheet 1-2, but do not depend on them. To make this sheet independent of that one, some basic information about the Babylonian numeration system is repeated.

The mathematics of Mesopotamia from 2500 BCE to 300 BCE or so is usually called "Babylonian." This oversimplifies the history of that region. During that very long period of time, a variety of civilizations occupied that area. The Sumerians invented the method of cuneiform writing on clay tablets sometime during the third millennium BCE, and successive civilizations adopted it. The Babylonians are perhaps best known for the reign of Hammurabi in the 18th century BCE. The history of this area would be a good topic for student projects linking mathematics with social studies.[4] We conform to custom by calling all of the mathematics of this era and region "Babylonian."

Solutions

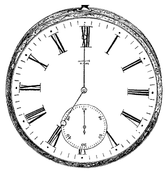
Questions 1–4 revisit the Babylonian place value system for whole numbers, a prerequisite for understanding their system of fractions.

1. (a) 24 (b) 16 (c) 41
2. (a)  (b)  (c) 
3. Students without calculators may need some scrap paper for this one.
   (a) $1 \cdot 60 + 13 = 73$ (b) $10 \cdot 60^2 + 12 \cdot 60 + 23 = 36{,}743$
4. The algorithm for this is the same as it is in our base-ten system: Find the largest power of the base that is smaller than the number; see how many copies of it you can subtract; then repeat the process with what's left until you use up the number. Our long standard long division algorithm does exactly this, in a very compact "shorthand" notation.
   (a) $13 \cdot 60 + 32$ 
   (b) $3 \cdot 60^2 + 10 \cdot 60 + 25$ 

The convenient base-sixty notation used here is the one used by historians of ancient mathematics. You may have to help students see how it is analogous to our own decimal notation. The semicolon, introduced in the examples just before #7, corresponds to our decimal point. It separates the whole-number part from the fractional part. The commas just separate the "digits" from each other; they are

[4] See pages 43–44 of *The History of Mathematics* by Roger Cooke (John Wiley & Sons, 1997) for a summary listing of the major civilizations of that area during this time period.

needed because we sometimes need two of our digits to describe a single place value in their base-sixty system.



You might point out to students that we measure time with a (partial) base-sixty system: hours, minutes, seconds. In fact, many microwave ovens "count down" the time with a display a lot like the notation used here. For instance, it will count down the seconds for 2 minutes of cooking time like this: 2:00, 1:59, 1:58, 1:57....

5. (a) 2, 24, 31 $\quad 2 \cdot 60^2 + 24 \cdot 60 + 31$
   (b) 30, 1, 17 $\quad 30 \cdot 60^2 + 1 \cdot 60 + 17$
   (c) 1, 11, 50, 12 $\quad 1 \cdot 60^3 + 11 \cdot 60^2 + 50 \cdot 60 + 12$

The following questions focus on how the Babylonian system of fractions is similar to our own decimal system. Working through these activities will give students a deeper understanding of how our system of decimal fractions works. It will also provide some practice in working with common fractions.

6. The factors of 60, besides 1 and 60, are 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. (This means that a common fraction with any of these denominators can be expressed as a "one-place" fraction in the Babylonian system.)

7. $7; 15 = 7 + \frac{15}{60} = 7\frac{1}{4}$
   $5; 22, 30 = 5 + \frac{22}{60} + \frac{30}{3600} = 5 + \frac{1350}{3600} = 5 + \frac{3 \cdot 450}{8 \cdot 450} = 5\frac{3}{8}$

8. Some scrap paper and a calculator would be useful here.
   (a) $1; 20 = 1 + \frac{20}{60} = 1\frac{1}{3}$
   (b) $2; 30, 30 = 2 + \frac{1}{2} + \frac{1}{120} = 2\frac{61}{120}$
   (c) $3; 24; 36 = 3 + \frac{24}{60} + \frac{36}{3600} = 3 + \frac{2}{5} + \frac{1}{100} = 3\frac{41}{100}$
   (d) $4; 1, 1, 1 = 4 + \frac{1}{60} + \frac{1}{3600} + \frac{1}{216{,}000} = 4 + \frac{3600+60+1}{216{,}000} = 4\frac{3661}{216{,}000}$ (*Note:* This fraction is in lowest terms; 3661 is not divisible by 2, 3, or 5, which are the only prime factors of 60^3.)

Questions 9 and 10 illustrate the power of a place value system. You can easily expand on these sample question by making up similar ones of your own. You can emphasize the efficiency of the system by having your students convert the questions into to common fractions and add, but that is a *very tedious* exercise for these examples. Simpler cases are not hard to construct.

One important note is in order here. We have avoided without comment these two significant difficulties with the Babylonian system of fractions:

- There was no "zero" placeholder. (This was explained in Unit 2.)

- There was no mark (like the semicolon we used) to separate the whole-number part from the fractional part. So they would write $2 \times 30 = 1$, and not $2 \times 0;30 = 1$. Of course, exactly the same equation could mean $2 \times 30 = 1,0$!

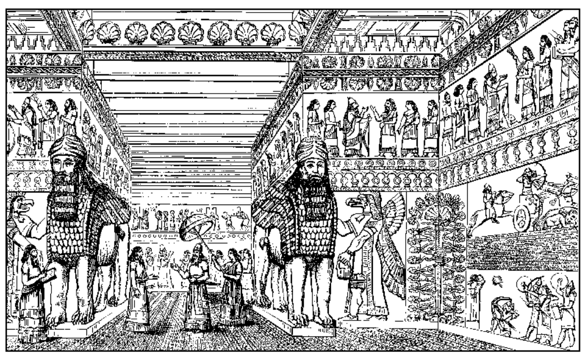
Context was the only way to resolve the ambiguities caused by the absence of such symbols. If you construct your own examples and questions, you might want to avoid cases in which a zero value occurs.

9. (a) $1\frac{3}{4} + 1\frac{1}{3} = 2 + \frac{9+4}{12} = 3\frac{1}{12}$

   (b) Sum: 6; 30 Check: $2\frac{2}{3} + 3\frac{5}{6} = 5 + \frac{4+5}{6} = 6\frac{1}{2}$

10. Parts (a) and (b) involve "carrying"; part (d) involves "borrowing."

    (a) 6; 5, 31, 32 (b) 11; 2, 39, 5 (c) 7; 5, 17, 8 (d) ; 59, 53, 30



Sheet 3-3: Name and Count

• Main Feature •
Fractions and size

This activity sheet focuses on numerators and denominators. To understand them, we should think of fractions primarily in terms of counting, rather than division. This is a subtle matter of emphasis. Think of a fraction as counting copies of a single, small enough part. Instead of measuring out a pint and a cup of milk for a recipe, it's easier to measure three cups. Instead of representing a fractional amount by identifying the largest single part within it and then exhausting the rest by successively smaller parts, simply look for a small part that can be counted up enough times to produce exactly the amount you want. Two numbers then specify the total amount: the size of the unit part, and the number of times it is "counted."

That was how the early Chinese mathematicians thought about fractions. Their *Nine Chapters on the Mathematical Art*, from about 100 BCE, contains a notation for fractions much like ours. (The one difference is that they avoided "improper fractions" such as $\frac{7}{3}$; they would write $2\frac{1}{3}$ instead.) A similar notation appears in Hindu manuscripts as early as the 7th century CE. They wrote the two numbers one over the other, with the size of the part below the number of times it was to be counted. No line separated one number from the other. For instance (using modern numerals), the fifth part of the basic unit taken three times would be written as $\begin{matrix}3\\5\end{matrix}$.

This Hindu custom of writing fractions as one number over another spread to Europe around the time of the Crusades. Latin writers of the Middle Ages used the terms *numerator* ("counter" – how many) and *denominator* ("namer" – of what size) as a convenient way to distinguish the top number of a fraction from the bottom one. If we still spoke Latin, these terms would make much more sense to students!

The horizontal bar between the top and bottom numbers was inserted by the Arabs by sometime in the 12th century. It appeared in most Latin manuscripts from then on, except for the early days of printing (the late 15th and early 16th centuries), when it probably was omitted because of typesetting problems. It gradually came back into use in the 16th and 17th centuries. Curiously, although 3/4 is easier to typeset than $\frac{3}{4}$, this "slash" notation did not appear until about 1850.

The questions this worksheet help to develop a sequence of important ideas:

- denominators name the size of the pieces being counted;
- numerators count the same-size pieces;
- larger denominators result in smaller fractions;
- larger numerators result in larger fractions;
- changing the denominator affects the numerator;
- common denominators make it easy to compare fractions.

Solutions

1. Here are some suggestions. *Denominator* is related to "denomination" (a particular value of money or a specific branch of a religion) and "nominate" (to name a candidate for a job or election). *Numerator* is related to "enumerate," (to count something) and "numeral" (a counting symbol). Students may come up with other suitable words.

2. The size of the unit part is $\frac{1}{8}$; you have 3 of them. Doublng the denominator gives you $\frac{3}{16}$, which is less pizza because the unit part is now smaller. (You have only half as much pizza.)

3. This question focuses students' attention on what the denominator of a fraction tells us. In particular, it emphasizes the fact that larger denominators represent smaller sizes.

$$\frac{1}{256} < \frac{1}{64} < \frac{1}{12} < \frac{1}{9} < \frac{1}{8} < \frac{1}{6} < \frac{1}{3} < \frac{1}{2}$$

4. This question is a "steppingstone" to bring out the usefulness of common denominators. Its simplicity emphasizes the fact that conversion to a common denominator reduces a fraction situation to a whole number situation.

$$\frac{1}{9} < \frac{2}{9} < \frac{4}{9} < \frac{5}{9} < \frac{7}{9} < \frac{8}{9}$$

5. This exercise illustrates the contrast between numerators and denominators in determining the size of a fraction. Increasing the numerator increases the size of the fraction; increasing the denominator decreases the size of the fraction. The larger fraction in each pair is as follows.

   (a) $\frac{5}{7}$ (b) $\frac{4}{7}$ (c) $\frac{5}{8}$ (d) $\frac{4}{5}$ (e) $\frac{2}{5}$ (f) $\frac{23}{35}$ (g) $\frac{53}{91}$ (h) $\frac{15}{37}$ (i) $\frac{45}{83}$

6. These questions provide a real-world instance of expressing a fraction using different denominators. In order to use common denominators, students must understand that multiplying both numerator and denominator by the same number does not change the value of the fraction.

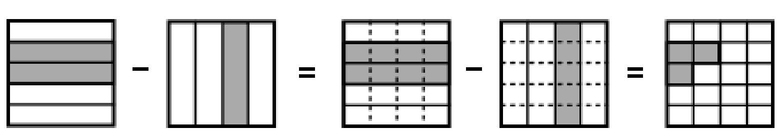
   (a) $\frac{1}{4}$ (b) 2 (c) 3 (d) 4 (e) $\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16}$

7. (a) $\frac{1}{5} = \frac{4}{20}$ (b) $\frac{4}{7} = \frac{12}{21}$ (c) $\frac{2}{3} = \frac{10}{15}$ (d) $\frac{9}{12} = \frac{3}{4} = \frac{15}{20}$

8. These questions should be done without a calculator. The denominators in this exercise are deliberately chosen so that the least common denominator is just the product of the two given ones, and the computations are essentially single-digit facts that should be automatic for students by now.

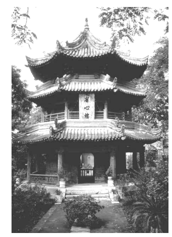
   (a) $\frac{4}{7} = \frac{36}{63}$ and $\frac{5}{9} = \frac{35}{63}$, so $\frac{4}{7}$ is $\frac{1}{63}$ larger than $\frac{5}{9}$.

   (b) $\frac{5}{6} = \frac{55}{66}$ and $\frac{9}{11} = \frac{54}{66}$, so $\frac{5}{6}$ is $\frac{1}{66}$ larger than $\frac{9}{11}$.



Sheet 3-4: Working with Parts

• MAIN FEATURE •
Fraction arithmetic



The Chinese book called *Nine Chapters on the Mathematical Art*, shows that the Chinese of 100 BCE thought about and wrote fractions much as we do today. Many of the usual rules for operating with fractions appear in the *Nine Chapters*: how to reduce a fraction that is not in lowest terms, how to add fractions, and how to multiply them. For instance, the rule for addition (translated into our terminology) looks like this:

> Each numerator is multiplied by the denominators of the other fractions. Add them as the dividend, multiply the denominators as the divisor. Divide; if there is a remainder, let it be the numerator and the divisor be the denominator.[5]

This is pretty much what we still do!

For multiplying and dividing, the method in the *Nine Chapters* also involved finding a common denominator. This made the process of division natural and obvious. For example, to divide $\frac{2}{3}$ by $\frac{4}{5}$, they would first multiply both the numerator and denominator of each fraction by the denominator of the other, so that

$$\frac{2}{3} \div \frac{4}{5} \quad \text{becomes} \quad \frac{2\cdot 5}{3\cdot 5} \div \frac{3\cdot 4}{3\cdot 5}; \quad \text{that is,} \quad \frac{10}{15} \div \frac{12}{15}.$$

Now that both fractions are written in the same measurement unit (denominator), the question becomes a whole-number division problem: dividing the numerator of the first fraction by the numerator of the second. In this case,

$$\frac{2}{3} \div \frac{4}{5} = \frac{2\cdot 5}{3\cdot 5} \div \frac{3\cdot 4}{3\cdot 5} = \frac{10}{15} \div \frac{12}{15} = 10 \div 12 = \frac{5}{6}.$$

$$\boxed{\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}}$$

The way in which fractions were written affected the development of their arithmetic. For instance, the "invert and multiply" rule for dividing fractions was used by the Hindu mathematician Mahāvīra

[5]Kangshen Shen, John N. Crossley, and Anthony W.-C. Lun. *The Nine Chapters on the Mathematical Art: Companion and Commentary.* Oxford Univ. Press, Oxford & New York, 1999; p. 70.

around 850 CE. However, it was not part of European arithmetic until the 16th century, probably because it made no sense unless fractions, including fractions larger than 1, were written as one number over another.

The questions of this activity sheet build on the ideas of Activity Sheet 3-3. They focus on the basic arithmetic operations for fractions, drawing on the rules used by the Chinese, the Hindus, and the Arabs for historical motivation. All of these rules are based on one fundamental idea:

> **Find a small enough measurement unit for all the fractions to be integral multiples of that unit. Then the computation becomes whole-number arithmetic.**

In other words, the key to all fraction arithmetic is *finding a common denominator*.

Solutions

1. The purpose of this first, very simple question is to make sure that students understand the basic idea of counting parts.

   (a) 5; 3 (b) $\frac{3}{10}$; less; it is only half as much.

2. This is an easy exercise in finding common denominators.

   (a) $\frac{2}{8}+\frac{1}{8}=\frac{3}{8}$ (b) $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$ (c) $\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$ (d) $\frac{5}{20}=\frac{4}{20}=\frac{9}{20}$

   (e) Yes. You can multiply the numerator and denominator given here by any nonzero whole number to get another correct answer.

3. The *Nine Chapters* rule for adding fractions differs from our usual method in two ways. Watch for these two differences in students' answers. (1) The denominator of the sum is always the product of the denominators of the summands, even if there is a smaller common denominator. Presumably the Chinese would have reduced the sum to a simpler fraction after adding, but not during the addition process, as we sometimes do. (2) The sum is never written as an improper fraction; any sum greater than 1 should be expressed as a "mixed number." (*Note:* Using a calculator for part (c) will allow students to focus on the main idea more clearly.)

   (a) $$\frac{2}{3}+\frac{3}{5}=\frac{5\cdot 2+3\cdot 3}{3\cdot 5}=\frac{19}{15}=1\frac{4}{15}$$

   (b) $$\frac{1}{2}+\frac{4}{5}+\frac{7}{8}=\frac{5\cdot 8\cdot 1+2\cdot 8\cdot 4+2\cdot 5\cdot 7}{2\cdot 5\cdot 8}=\frac{40+64+70}{80}=\frac{174}{80}=2\frac{14}{80}$$

   (c) $$\frac{1}{3}+\frac{3}{4}+\frac{5}{8}+\frac{1}{6}=\frac{4\cdot 8\cdot 6\cdot 1+3\cdot 8\cdot 6\cdot 3+3\cdot 4\cdot 6\cdot 5+3\cdot 4\cdot 8\cdot 1}{3\cdot 4\cdot 8\cdot 6}$$
   $$=\frac{192+432+360+96}{576}=\frac{1080}{576}=1\frac{504}{576}$$

This example can be used to illustrate the convenience of reducing to a simpler denominator. The sum reduces to $1\frac{7}{8}$.

4. As mentioned in the notes for #3, there are two main differences: (1) the denominator of the sum is always the product of the denominators of the summands, and (2) the sum is never an improper fraction. (*Note:* What is important for operations with fractions is to have *a* common denominator. It doesn't need to be the *least* common denominator. The Chinese method avoids the problem of finding a least common denominator by always using the most obvious choice of common denominator.)

5. As in the rule for addition, the denominator of the difference is *always* the product of the denominators of the original fractions, even if there is a smaller common denominator. The Chinese might have reduced the difference to a simpler fraction later, but not during the subtraction process. In all of these examples, the first fraction is larger than the second. This reflects the fact that, at the time of the *Nine Chapters*, the idea of negatives as numbers was still many centuries in the future. You might also want to observe that the "Divide" step is not necessary because the result will always be less than the first fraction.
   (a) $\frac{2}{3} - \frac{3}{5} = \frac{2 \cdot 5 - 3 \cdot 3}{3 \cdot 5} = \frac{10-9}{15} = \frac{1}{15}$
   (b) $\frac{7}{8} - \frac{1}{6} = \frac{7 \cdot 6 - 1 \cdot 8}{8 \cdot 6} = \frac{42-8}{48} = \frac{34}{48}$
   (c) $\frac{3}{4} - \frac{5}{8} = \frac{3 \cdot 8 - 5 \cdot 4}{4 \cdot 8} = \frac{24-20}{32} = \frac{4}{32}$

6. (a) Cancel any factors that are in both the numerator and the denominator.
   (b) (3a): $1\frac{4}{15}$ is already in lowest terms. (3b): $2\frac{14}{80} = 1\frac{7}{40}$ (3c): $1\frac{504}{576} = 1\frac{7}{8}$
   (c) $\frac{54}{90} = \frac{2 \cdot 3 \cdot 9}{2 \cdot 5 \cdot 9} = \frac{3}{5}$ $\quad \frac{462}{495} = \frac{2 \cdot 3 \cdot 7 \cdot 11}{3 \cdot 3 \cdot 5 \cdot 11} = \frac{2 \cdot 7}{3 \cdot 5} = \frac{14}{15}$

7. Do your students already know that the Chinese never used "improper" fractions — that is, fractions in which the numerator is larger than the denominator? It would be good to tell or remind them about this here. It explains why there is no division step at the end of their rule. Since the Chinese only used fractions that were less than 1, the product would always be less than 1. The fact that multiplying a quantity by a number less than 1 makes the result smaller is an important, common-sense idea that should be reinforced.
   (a) $\frac{6}{7} \times \frac{5}{8} = \frac{6 \cdot 5}{7 \cdot 8} = \frac{30}{56}$
   (b) $\frac{11}{12} \times \frac{7}{9} = \frac{11 \cdot 7}{12 \cdot 9} = \frac{77}{108}$

(c) $\frac{5}{6} \times \frac{1}{6} = \frac{5 \cdot 1}{6 \cdot 6} = \frac{5}{36}$

(d) $\frac{12}{100} \times \frac{3}{100} = \frac{12 \cdot 3}{100 \cdot 100} = \frac{36}{10,000}$

8. Our common "invert and multiply" rule for dividing one fraction by another is efficient, but it hides the common-sense idea of division. The Chinese method makes the process a much more natural reflection of the concept. These questions invite a class discussion about why both methods always give the same results.

(a) $\frac{5}{8} \div \frac{3}{7} = \frac{35}{56} \div \frac{24}{56} = 1\frac{11}{24}$ $\qquad$ $\frac{3}{10} \div \frac{4}{9} = \frac{27}{90} \div \frac{40}{90} = \frac{27}{40}$

(b) $\frac{5}{8} \div \frac{3}{7} = \frac{5}{8} \times \frac{7}{3} = \frac{35}{24} = 1\frac{11}{24}$ $\qquad$ $\frac{3}{10} \div \frac{4}{9} = \frac{3}{10} \times \frac{9}{4} = \frac{27}{40}$

Sheet 3-5: Decimals

• MAIN FEATURE •
Fractions in our base-ten system

Decimal fractions appeared in Chinese and Arabic mathematics, but these ideas did not migrate to the West. In Europe, the first use of decimals for fractions occurred in the 16th century. They were made popular by Simon Stevin's 1585 book, *The Tenth*. Stevin, a Flemish mathematician and engineer, showed how writing fractions as decimals allows operations on fractions to be carried out using the much simpler algorithms of whole-number arithmetic. Within a generation, the use of decimal fractions by scientists such as Johannes Kepler and John Napier paved the way for their general acceptance.

However, the use of a period as the *decimal point* was not uniformly adopted. For quite a while, many different symbols — including an apostrophe, a small wedge, a left parenthesis, a comma, and a raised dot — were used to separate the whole and fractional parts of a number. In 1729, the first arithmetic book printed in America used a comma for this purpose, but later books tended to favor the use of a period. Usage in other parts of the world continues to be varied, with the comma and the raised dot still being used in many countries. Most English-speaking countries use the period, but many other nations prefer the comma. International agencies and publications often accept both comma and period. Modern computer systems allow the user to choose, as a language setting, whether the decimal separator should be written as a comma or a period.

Stevin's innovation and its application to science and practical computation had an important effect on how people understood numbers. Up to Stevin's time, things like $\sqrt{2}$ or π were not quite considered numbers. They were ratios that corresponded to certain geometric objects, but when it came to thinking of them as numbers, people felt queasy. The invention of decimals allowed people to think of $\sqrt{2}$ as 1.414 and of π as 3.1416 (more or less). It's no coincidence that it was Stevin who first thought of the real numbers as points on a number line and who declared that all real numbers should have equal status.

With the advent of calculators in the middle of the 20th century, it seemed as if decimals had won the day permanently. But the old system of numerators and denominators still has many advantages, both computational and theoretical, and it has proved to be remarkably resilient. We now have calculators and computer programs that can work with common fractions. Percentages are used in commerce, common fractions and mixed numbers appear in recipes, and decimals occur in scientific measurements. These multiple representations are a matter of convenience and also a reminder of the rich history behind ideas we use every day.

Interdisciplinary Projects: For a worthwhile project that combines science, math, and composition, ask students to find out some things that Stevin, Kepler, and Napier were famous for and write up their findings. You can also make a connection to geography, modern history, and literature by way of Stevin's nationality. He was Flemish; that is, he was born in Flanders. Where was/is Flanders? (Roughly, it is the northern part of Belgium now, but more can be said about that.) *In Flanders Fields* is a famous poem; what is it about? (Written by Major John McCrae, a Canadian military doctor, it refers to one of the major battlefields of World War I. This, too, can be expanded into a longer assignment, if you wish.)

Solutions

This activity sheet focuses on representing fractions as decimals and computing with decimals. It begins very simply, to check on student familiarity with the place value structure of our system.

1. From left to right: units, tenths, hundredths, thousandths.

2. (a) $0.7 = \frac{7}{10}$ It is in lowest terms.

   (b) $0.75 = \frac{75}{100} = \frac{3}{4}$

   (c) $0.008 = \frac{8}{1000} = \frac{1}{125}$

   (d) $0.33 = \frac{33}{100}$ It is in lowest terms. (It is *not* $\frac{1}{3}$. A student who makes this error will give you a "teachable moment" — use it to expand a bit on the difference between these two numbers.)

   (e) $2.12 = \frac{212}{100} = \frac{53}{25}$

   (f) $1.2525 = \frac{12{,}525}{10{,}000} = \frac{501}{400}$

If your students know (or can easily be told) what a prime number is, there is a natural follow-up line of questioning here, something like this:

- What prime numbers are factors of 10? of 100? of 1000? of any power of 10? (2 and 5 only)
- What does it mean to say that a fraction is in lowest terms? (The numerator and denominator have no factors in common.)
- If a fraction with denominator 10 is *not* in lowest terms, what must be a factor of the numerator? (2 and/or 5) What if the denominator is 100? (same answer) 1000? (same answer) any power of 10? (same answer)
- How do you know if a (whole) number is divisible by 2? (Its unit digit is even.) How do you know if it is divisible by 5? (Its unit digit is 5 or 0.)
- So how can you be sure that a fraction such as $\frac{501}{400}$ is really in lowest terms? (501 ends in an odd digit that is not 5, so it can't be divisible by 2 or 5.)
- [a more sophisticated question] Why do these divisibility tests work? (Because the rest of the number — without the units part — is divisible by 10, so it must be divisible by 2 or 5. Therefore, if the unit part is divisible by 2 or 5, too, the entire number must be.)

3. This question suggests the special role of 2 and 5 as factors in the decimal system. If you did not pursue the line of questioning above, you might bring out some of those ideas here. To focus on factors, have students do this part *without* using a calculator.

   (a) $\frac{4}{5} = 0.8$ (b) $\frac{13}{20} = 0.65$ (c) $\frac{121}{250} = 0.484$ (d) $\frac{17}{8} = 2.125$

4. Students might need some scrap paper for this. The common-fraction calculations in each part should be done without a calculator, and the answer should be expressed as a common fraction. You might also ask students to convert those answers to decimals as a check to see if they got the same result both ways. Responses to the "which was easier" questions are likely to vary, but it would not be surprising if most students find that the decimals are easier to add and subtract, but the common fractions are easier to multiply and divide.

   (a) $\frac{5}{8} + \frac{2}{5} = \frac{25+16}{40} = \frac{41}{40}$ $\quad 0.625 + 0.4 = 1.025$

   (b) $\frac{3}{4} - \frac{72}{125} = \frac{750-576}{1000} = \frac{174}{1000} = \frac{87}{500}$ $\quad 0.75 - 0.576 = 0.174$

   (c) $\frac{7}{8} \times \frac{3}{25} = \frac{7 \times 3}{8 \times 25} = \frac{21}{200}$ $\quad 0.875 \times 0.12 = 0.105$
   Decimal multiplication without a calculator might be unfamiliar to some students. It is good for them to know, but not essential for this example. These numbers have been chosen to make the hand calculation easy.

   (d) $\frac{3}{8} \div \frac{2}{5} = \frac{3}{8} \times \frac{5}{2} = \frac{3 \times 5}{8 \times 2} = \frac{15}{16}$ $\quad 0.375 \div 0.4 = 0.9375$
   As in the previous part, division without a calculator is a good skill to know, but not essential here. Again, these numbers are chosen to minimize tedious hand calculation.

5. When fractions are expressed as decimals, their relative sizes become immediately obvious. This illustrates a convenient feature of decimals. However, expressing common fractions as decimals without a calculator can be a tedious task in long division. Therefore, unless part (c) is done with a calculator, students will not see this as a convenience at all! Parts (a) and (b) are quick checks to see if students are thinking about decimals appropriately.

   (a) 0.3761

   (b) Look for the first place where they are different. The one with the larger digit is the larger number. (*Note:* This test works for all *finite* decimals. There is a small complication in the case of infinite decimals, which can be ignored here.)

   (c) $\frac{3}{5} < \frac{83}{137} < \frac{5}{8} < \frac{247}{391} < \frac{16}{25} < \frac{11}{17} < \frac{613}{942} < \frac{2}{3}$ $\quad$ As decimals:
   $0.6 < 0.605... < 0.625 < 0.631... < 0.64 < 0.647... < 0.650... < 0.666...$

6. (a) The data for this question comes from the web page

   www.unc.edu/rrowlett/units/custom.html,

   by Russ Rowlett at the University of North Carolina at Chapel Hill. The picture of the plowman next to the table mirrors the fact that a *furlong* was traditionally the length of the furrow plowed on Saxon farms.

furlong	chain	rod	yard	foot
1/8 mile	1/10 furlong	1/4 chain	2/11 rod	1/3 yard
0.125	0.1	0.25	0.182	0.333

(b) Thinking about how to approach this problem is a worthwhile exercise in reasoning. If you are using this material in class, you might ask students for suggestions. Perhaps the easiest way (but maybe not the most obvious one) is to calculate 1 foot as a fraction of a mile and then take the reciprocal of that. This calculation is just the product of the decimals or of the fractions. Use of a calculator is appropriate here.

By decimals: $0.125 \times 0.1 \times 0.25 \times 0.182 \times 0.333 = 1.8939375\text{E}^{-}4$ is the "number" of miles in a foot. Divide 1 by this number to get the number of feet in a mile, 5280.00528.

By fractions: $\frac{1}{8} \times \frac{1}{10} \times \frac{1}{4} \times \frac{2}{11} \times \frac{1}{3} = \frac{2}{10560} = \frac{1}{5280}$ is 1 foot as a fraction of a mile, so 5280 is the number of feet in a mile. The answers should agree, and they do, within a very small roundoff error.

(c) Decimals make this a straightforward calculator computation:
$(2 \times 0.125) + (4 \times 0.1 \times 0.125) + (3 \times 0.25 \times 0.1 \times 0.125) = 0.309375$ mi.
Multiply this by 5280 to get the number of feet, 1633.5 ft.

One issue raised only indirectly in this activity sheet is the question of which decimals are finite and which are infinite. This is related to "rounding off," of course, but it can run much deeper than that. Historically, Stevin as an engineer was not concerned with infinite decimals. He simply needed to get "close enough" to the exact value of any fraction, which meant getting within the margin of error of the tools of his time. Decimals, finite or infinite, will do that. To get a value for $\frac{1}{3}$ with 0.000001 tolerance, for example, it suffices to carry out the division just six places, to 0.333333.

If you want to pursue the question of finite versus infinite decimal fractions with your students, you might begin by asking if they noticed anything special about the denominators of all the fractions in activities 4 and 5. Help them see that they are all products of 2's and 5's only. Why is that? Well, the "easy" answer is that $2 \times 5 = 10$, so any power of 10 is a product of 2's and 5's. This means that all those examples are easily turned into finite decimals.

You can take this idea deeper by asking what would happen if a denominator had some other (prime) factor, such as 3 or 7 or 11. Students might experiment with turning such fractions into decimals. If they are not using calculators, 3 and 11 are more convenient factors than 7. In any case, dividing such a denominator into any (relatively prime) numerator shows that these decimals never terminate; they are always infinite. In fact, since *any* finite decimal must be expressible as a fraction with some power of 10 as its denominator, a reduced fraction with a denominator that has a prime factor other than 2 or 5 *must* convert to an infinite decimal.

Sheet 3-6: Percent

• Main Feature •
The special case of hundredths

The Latin word for "hundred" is *centum.* The term *per cent* ("for each hundred") as a name for fractions with denominator 100 began with the commercial arithmetic of the 15th and 16th centuries, for quoting interest rates in hundredths. This custom has persisted in business, reinforced in the U.S. by a monetary system based on dollars and *cents* (hundredths of dollars). The continued use of percents as a special branch of decimal arithmetic comes primarily from its connection with money. The percent symbol evolved over several centuries. It started as a handwritten abbreviation for "per 100" around 1425 and was gradually transformed into "per $\frac{0}{0}$" by 1650, then simply to "$\frac{0}{0}$," and finally to the % sign we use today.

The logic and the history of percents are somewhat at odds with each other. Logically, a "percent" is just a two-place decimal. For instance, 15% is just another way of writing 0.15. Historically, however, percents were in common use some 150 years or so before Stevin's popularization of decimal fractions.

Solutions

The main purpose of this activity sheet is to show students that percents are simply fractions with denominator 100. If they think of a percent as a number over 100, then the arithmetic of percents is just a special case of something familiar. The first two questions establish the meaning of the word, then the symbol for it.

1. (a) 6 percent of \$2.00 = 12 cents (6 cents out of every dollar)

   (b) 7 percent = $\frac{7}{100}$

2. These three parts move from a direct application of "out of every hundred" to the need for some proportional reasoning.

   (a) 7% of 400 = 28 (b) 10% of 250 = 25 (c) 5% of 40 = 2

   Students who have trouble with parts (b) and (c) can be encouraged to put the percent in fraction form and translate "of" as "times."

Here is a student research project that combines history and numbers:

> Look up at least four different types of money, from four different places in 15th-century Europe, and describe how they are related to each other. In particular, how could you fairly exchange each type for the others?

One useful source is http://www.treasurerealm.com/coinpapers/dictionary/. That site could easily be used as a starting point for this project, but we do not have any way of judging the factual accuracy of the many specific definitions given there.

3. (a) 4% of 300 florins is 12 florins each month.

   (b) 12×24 (months) = 288 florins in all.

4. These examples reinforce students' understanding of commonly used percentages.

   (a) $25\% = \frac{1}{4}$ (b) $50\% = \frac{1}{2}$ (c) $20\% = \frac{1}{5}$

   (d) $15\% = \frac{3}{20}$ (e) $33\% = \frac{33}{100}$ (f) $75\% = \frac{3}{4}$

   (The answer to part (e) is *not* $\frac{1}{3}$. The values are close, but not the same.)

5. These examples establish that whole-number percents are simply two-place decimals (hundredths).

   (a) 25% = 0.25 (b) 12% = 0.12 (c) 3% = 0.03

   (d) 86% = 0.86 (e) 33% = 0.33 (f) 5% = 0.05

6. (a) 1% of \$13.00 = \$0.13 (b) 5% of \$12.00 = \$0.60

   (c) 40% of \$32.00 = \$12.80 (d) 18% of \$2.50 = \$0.45

7. This idea is a natural extension of what has come before. In fact, many students will already have figured it out on their own, even if they have not already seen it in class. The use of calculators here will allow students to focus on the translation idea without getting bogged down in the routine multiplication.

   (a) $0.2 \times \$43.60 = \$8.72$ (b) $0.42 \times 7{,}956 = 3341.52$

   (c) $0.07 \times \$245 = \$17.15$ (d) $0.61 \times 1159 = 706.99$

   (e) $1.5 \times 360 = 540$

8. Students responses will vary, of course. Here is the main idea.
   (a) 100% is the total amount available, so 110% is more effort than the athlete can have.
   (b) He is trying to say that the athlete is giving effort beyond what could be expected.

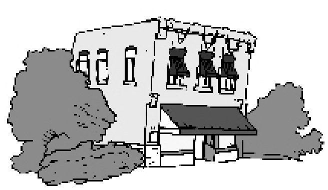
9. (a) $\$80 + 0.4 \times \$80 = \$80 + \$32 = \$112$
   (b) No. 40% off the selling price of \$112 is $0.4 \times \$112 = \$44.80$, so the sale price is $\$112 - \$44.80 = \$67.20$. This is \$12.80 less than the wholesale price she paid. (The sale price can be computed more efficiently by thinking of it as 60% of the tag price: $0.6 \times \$112 = \$67.20$.)

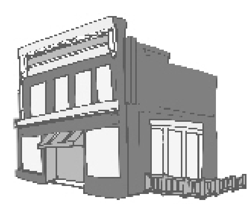
10. Yes and no. It doesn't make any difference in the total amount collected, because

$$([bill] \times 1.15) \times 1.07 = ([bill] \times 1.07) \times 1.15.$$

(Multiplication is associative and commutative.) However, more of the added amount goes for tax in the first case than in the second. If your students are having trouble seeing this, you might have them work out the amounts for a particular example, say a basic bill of \$100.

A useful follow-up question is, "Is this the same as charging a total of 22% to the basic bill?" In this case, the answer is no; the actual added percentage comes from the product 1.15×1.07, which is 1.2305. This means that the added charges total 23.05%. Again, you might find it better to have students try a particular example or two before coming up with a general solution.





4

Less than Nothing?

Mathematical Focus	Negative Numbers
Historical Connections	India, 7th & 12th centuries The Arab World, 9th century Europe, 15th–19th centuries

Did you know that negative numbers were not generally accepted, even by mathematicians, until a few hundred years ago? It's true. Columbus discovered America more than two centuries before negatives joined the society of numbers. And they didn't become first-class citizens until the middle of the 19th century, about the time of the American Civil War. Students who have trouble with negative numbers are often just reflecting the difficulties faced by some of the best mathematical minds of times gone by.

Numbers arose from counting and measuring things: 5 goats, 37 sheep, 100 coins, 15 inches, etc. Fractions were just a way of counting with smaller units: $\frac{5}{8}$ in. is five *eighths* of an inch, $\frac{3}{10}$ mi. is three *tenths* of a mile, and so on. If you're counting or measuring, the smallest possible quantity must be zero, right? After all, how can any quantity be less than nothing? It is not too surprising, then, that the idea of a negative number — a number less than zero — was difficult to accept.

Negative numbers first appeared when people began to solve problems that can be expressed as equations, such as:

"I am 7 years old and my sister is 2. When will I be exactly twice as old as my sister?"

This translates into solving the equation

$$7 + x = 2(2 + x),$$

where x is the number of years from now that this will happen. As you can see, in this case the answer is 3 (years from now). But the same kind of question can be asked for any ages. For instance, we could ask for the solution of

$$18 + x = 2(11 + x).$$

In this case, however, the solution is negative: $x = -4$.

The scribes of Egypt and Mesopotamia were able to solve such equations more than 3000 years ago, but they never considered the possibility of negative solutions. They would say that our second problem had no solution. Chinese mathematicians of that era seem to have been able to handle negative numbers in intermediate steps towards solving equations. But they didn't accept them as final answers. Our mathematics, like much of our Western culture, is rooted mainly in the work of ancient Greek scholars. Despite the depth and subtlety of their mathematics and philosophy, the Greeks ignored negative numbers completely.

Brahmagupta, a prominent Indian mathematician of the 7th century, treated positive numbers as possessions and negative numbers as debts. He also stated rules for adding, subtracting, multiplying, and dividing with negative numbers. Later Indian mathematicians continued in this tradition, but they regarded negative quantities with suspicion for a very long time. Five centuries later, Bhāskara, after stating that the two roots of the equation $x^2 - 45x = 250$ are 50 and -5, says: "Here, the second [answer] is not to be taken, because of its inapplicability. For people have no clear understanding in the case of a negative quantity."[6]

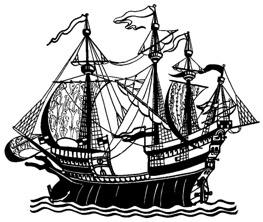
Indian mathematics first came to Europe through the Arabs, who did not use negative numbers. Al-Khwārizmī, for example, recognized that a quadratic equation can have two roots, but only when both of them are positive. This may have resulted from the fact that his approach to solving such equations depended on interpreting them in terms of areas and side lengths of rectangles, a context in which negative quantities made no sense.

The Arabs did understand how to expand products of the form

$$(x - a)(x - b).$$

They knew that in this situation negative times negative is positive, and negative times positive is negative. But they only applied this to problems involving subtractions whose answers are positive. So, while these "laws of signs" were known, they weren't understood as rules about how to operate with independent things called "negative numbers." Thus, European mathematicians learned from their predecessors a kind of mathematics that dealt only with positive numbers.

[6]From Bhāskara's *Bījagaṇita*; translation by Kim Plofker.



European mathematics made tremendous leaps after the Renaissance, motivated by astronomy, navigation, physical science, warfare, commerce, and other applications. In spite of that progress, or perhaps because of its utilitarian focus, there was continued resistance to negative numbers. In the 16th century, even such prominent mathematicians as Cardano in Italy, Viète in France, and Stifel in Germany rejected negative numbers as "fictitious" or "absurd." When negatives appeared as solutions to equations, they were called "fictitious solutions" or "false roots." But by the early 17th century, the tide was beginning to turn. As the usefulness of negative numbers became too obvious to ignore, some European mathematicians began to use them.

Sheet 4-1: What Are Negative Numbers?

• MAIN FEATURE •
Everyday occurrences of negative numbers

This activity sheet focuses on students' common-sense understanding of negative numbers. It also looks at how negatives can arise naturally in real-world situations.

Solutions

1. Students do not need to know any formal algebra to solve these simple equations. They can be done by guess-and-check. The only necessary algebra here is the idea that "equation" is a statement that requires the values on the two sides of the = sign to be the same. Of course, the algebraic solution process is more efficient for anyone who already knows how to use it.

   (a) The solution is 3. Check: $7 + 3 = 2 \times (2 + 3)$; $10 = 2 \times 5$.

   (b) $13 + \square = 2 \times (9 + \square)$. The solution is -5.
   Check: $13 + (-5) = 2 \times (9 + (-5))$; $8 = 2 \times 4$.

   (c) This part connects the calculations with their real-world meaning. The solution -5 means that Pedro's age was exactly twice Mia's age 5 years *ago*, when he was 8 and she was 4. (Thus, in this case the negative signals going backward, rather than forward, in time.)

2. This question takes a little thought about what the equation notation means. If your students have never seen equations involving an unknown x before, you might have to explain this answer to them. $4x + 20$ means "add something ($4x$) to 20." For Diophantus, the "something" had to be positive, so the result could never equal the smaller number, 4.

3. Here are some possible answers. Students may have other good suggestions.
   - Temperature. $-5°$ means $5°$ below zero.
   - Money (or accounting). $-\$10$ means a debt of \$10.
   - Sound balance (in car radios, etc). "-2 T" means 2 steps below balance in the treble (high) range.
   - Racing times (in sports). -1.2 sec. means 1.2 seconds less than the previous best time, for example.
   - Stock quotes (in newspapers or on TV). $-.02$ means that the stock price has dropped by 2 cents.

4. (a) Students may express this in different ways. The basic idea is that a negative number and its positive counterpart add up to 0, so they "cancel each other out" additively. For instance, $-2 + 2 = 0$.

   (b) 5 is the opposite of -5 (in this sense), but not of -3, because $-5+5 = 0$ but $-3+5 \neq 0$. (The wording here also gently suggests the symmetry of this relationship: 5 and -5 are inverses *of each other.*)

5. There is no single "right answer" here. Look for some visual representation of a negative number and its positive counterpart being on opposite sides of 0 and equidistant from it.

6. This is from the 1728 English translation, quoted on p. 192 of *Symbols, Impossible Numbers, and Geometric Entanglements: British Algebra Through the Commentaries on Newton's* Universal Arithmetick, by Helena M. Pycior (Cambridge University Press, 1997).

   (a) positive (b) less than zero

7. This question gets at a fundamental idea, which is called for in part (e). The earlier parts lead students to think about this general principle by a series of simple examples, the last of which may cause some students a bit of trouble.

   (a) 4 (b) -4 (c) -4 (d) -6

   (e) Students who answer parts (a)–(d) correctly still may not be able to articulate a general principle here. Stated informally, the idea is this: *A first number is less than a second if you have to add a positive amount to the first one to get the second.* In mathematicians' language and notation, $a < b$ if $a + p = b$ for some positive number p.

This principle (it's actually the definition of "less than") orders the negative numbers in a way that is consistent with the arithmetic of positive numbers. In particular, it explains the answer to part (d): $-6 < -4$ because $-6 + 2 = -4$.

(f) $-10 < -7 < -4 < -3 < -1 < 0 < 2 < 3 < 5 < 8 < 10$

8. This quote is from pp. 4–5 of the 1984 Springer-Verlag edition of Leonhard Euler's *Elements of Algebra.* It is noteworthy that even as late as 1770, about the time of the American Revolution, the most prominent mathematician in Europe felt it necessary to explain what negative numbers mean. These simple sums reinforce this common-sense interpretation of negative numbers.

(a) $-35 + 35 = 0$

(b) $-12 + 20 = 8$

(c) $-30 + 10 = -20$

(d) $14 + (-9) = 5$

(e) $22 + (-40) = -18$

(f) $-25 + (-50) = -75$

Sheet 4-2: Adding & Subtracting Negative Numbers

• Main Feature •
Rules for + and − with negatives

This activity sheet and the next highlight reasoning and sense making at its best. If negatives are to be accepted as legitimate numbers, then the rules of positive-number arithmetic must be extended to them in some consistent way. That is, we must *choose* how to define $+$, $-$, $\times$, and $\div$ on negative numbers so that the most important properties of whole-number arithmetic are carried over.

The easiest operation to extend is addition. Simple, common-sense examples should enable students to formulate their own rules for this, first in words and then in symbols. Subtraction is the natural next step. It is also a common-sense extension. The questions on this sheet proceed from example to verbal rule to symbolic rule, thereby giving students a little taste of using symbols as convenient abbreviations for their ideas. The final step for each operation asks students to model some real-world situations with the rules they set up.

Solutions

1. (a) $(-7)+(-3)=-10$ (b) $7+(-3)=4$ (c) $(-7)+3=-4$
   (d) $(-6.5)+(-2)=-8.5$ (e) $6.5+(-2)=4.5$ (f) $(-6.5)+2=-4.5$

2. Here are some typical answers that capture the right ideas.
   (d) If I borrow \$6.50 and then borrow \$2 more, I owe \$8.50 altogether.
   (e) If I have \$6.50 and pay a \$2 debt, I have \$4.50 left.
   (f) If I owe someone \$6.50 and pay them only \$2, I still owe \$4.50.

3. Here are some typical answers. This question gets at the use of negatives to indicate direction along a line or linear scale.
   
   (a) If it is 7° below zero and gets 3° colder, it will be 10° below zero.
   (b) If it is 7° above zero and gets 3° colder, it will be 4° above zero.
   (c) If it is 7° below zero and gets 3° warmer, it will be 4° below zero.

4. The wording here is less important than the ideas. The next question uses symbols to sharpen the meaning. These are the ideas:

   If both numbers are negative, add their values (meaning, of course, their absolute values) and make the sum negative.

   If one number is negative and the other is positive, the difference of their values is the value of the sum. That sum is positive or negative, depending on which of the original numbers has the larger value.

5. You might have to help your students see how these symbols capture the ideas of #4. The patterns match exactly the examples in #1. Because smaller must be subtracted from larger, $b-a$ cannot be used here.
   $(-a)+(-b)=-(a+b)$ $a+(-b)=a-b$ $(-a)+b=-(a-b)$

6. If you have already taught your students rules about changing signs when adding or subtracting, you might point out how these results conform to them.
   (a) $(-5)-(-2)=-3$ because $(-3)+(-2)=-5$
   (b) $5-(-2)=7$ because $7+(-2)=5$
   (c) $(-5)-2=-7$ because $(-7)+2=-5$

7. These questions connect the arithmetic to everyday life and to its historical roots. As before, student wording may vary, but the ideas should be these.
   (a) If I owe \$5 and pay back \$2, then I owe only \$3.
   (b) (This one is a bit tricky.) I have \$5 and have borrowed \$2 more. If that \$2 debt is forgiven, I now have \$7.

(c) I owe \$5 and spend \$2 more on my debit card. Now I owe \$7.

8. The patterns match exactly the examples in #6. Again, because smaller must be subtracted from larger, $b - a$ cannot be used here.

   $(-a) - (-b) = -(a - b) \qquad a - (-b) = a + b \qquad (-a) - b = -(a + b)$

9. There is more than one correct way to do these. Students should be able to explain why their answers make sense.

   (a) $(-\$600) + \$200 = -\$400$ or $(-\$600) - (-\$200) = -\$400$

   (b) $(-5^\circ) - 12^\circ = -17^\circ$ or $(-5^\circ) + (-12^\circ) = -17^\circ$

   (c) $8 + (-2.5) = 5.5$ (million dollars)

Sheet 4-3: Multiplying & Dividing Negative Numbers

• Main Feature •
Rules for × and ÷ with negatives

The most difficult idea here is the fact that the product of two negatives is positive. Students often are confused about this. A main purpose of these activities is to give them a common-sense rationale, so that they do not have to trust blindly in a memorized fact (which sometimes is misremembered).

We approach this idea by looking at the most natural ways to preserve and extend the essential properties of whole-number multiplication. Putting ourselves in Brahmagupta's shoes, we first must decide what properties are important to preserve, and then *define* multiplication of negative numbers to do that.

In other words, *we get to make up the rules here*! These "negative numbers" are new things that have been dumped into our number system. We have to tell the multiplication of that system how to handle these new things. We can do that in any way we please, but if we don't choose wisely, we may foul up some very useful properties of positive-number multiplication. The questions of this activity sheet help students to identify such properties and see how their patterns suggest how to extend multiplication to negative numbers.

Solutions

1. This question treats the product of a positive and a negative as repeated addition of the negative number, using the idea of multiple debts. Part (b)

also illustrates how multiplication remains commutative. If you are teaching your students that terminology, this is a natural place to bring it up.

(a) \$40 because $8 \times \$5 = \$40$

(b) \$150; yes; $30 \times -\$5 = 5 \times -\$30 = -\$150$

2. These parts, which should be done without a calculator, also provide some easy mental arithmetic practice.

(a) -50 (b) -50 (c) -56 (d) -72

(e) -8.6 (f) -1.1 (g) -0.9 (h) -7.07

3. Negative. This rule can be stated more succinctly as "The product of a positive number and a negative number is negative" if students automatically associate the term *product* with multiplication. However, some students may not make that connection easily yet.

4. This pattern is the key to defining the product of two negatives. Help students to focus on the fact that each time the multiplier decreases by 1, the product *increases* by 6. As the multiplier goes from 1 to 0 to -1, the product goes from negative to 0 to positive.

   (a) $5 \times (-6) = -30$; $4 \times (-6) = -24$; $3 \times (-6) = -18$; $2 \times (-6) = -12$

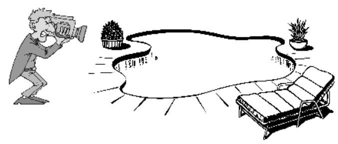
   As the first number gets smaller (by 1), the product gets larger (by 6). (Student wording may vary, but this is the idea that it should reflect.)

   (b) $1 \times (-6) = -6$; $0 \times (-6) = 0$; $(-1) \times (-6) = 6$; $(-2) \times (-6) = 12$

5. $4 \times (-3) = -12$; $3 \times (-3) = -9$; $2 \times (-3) = -6$; $1 \times (-3) = -3$; $0 \times (-3) = 0$; $(-1) \times (-3) = 3$; $(-2) \times (-3) = 6$; $(-3) \times (-3) = 9$; $(-4) \times (-3) = 12$

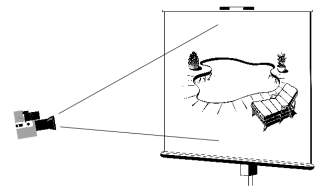
6. Students need to start by getting the first answer correct: $(-7) \times 4 = -28$. Now, if $(-7) \times (-4) = -28$, too, then $(-7) \times 4 = (-7) \times (-4)$. But then cancelling the common factor (-7) leaves $4 = -4$, which is nonsense. So the correct answer to the second question must be positive; $(-7) \times (-4) = 28$.

Sometimes students have trouble imagining a real-world illustration of the fact that the product of two negatives is positive. Here is a simple one.[7]



Pedro is a home-video buff, and his neighbor Sam has a swimming pool. When Sam fills his pool, the water *rises* at the rate of 2 inches per minute. Pedro videotapes the rising water.

[7]Adapted from page 293 of *The Mathematics of the Elementary Grades* by William P. Berlinghoff and Robert M. Washburn. New York: Ardsley House, 1990.



When he runs the tape *forward* for 3 minutes, he sees the water *rising* 6 inches. That is, $(+2) \times (+3) = +6$. When he runs the tape *backwards* for 3 minutes, he sees the water *dropping* 6 inches. That is, $(+2) \times (-3) = -6$.

When Sam drains his pool, the water *drops* at the rate of 2 inches per minute. Pedro videotapes this, too, and when he runs the tape *forward* for 3 minutes, he sees the water *dropping* 6 inches. That is, $(-2) \times (+3) = -6$. When he runs the tape *backwards* for 3 minutes, he sees the water *rising* 6 inches. That is, $(-2) \times (-3) = +6$.

7. For parts (d) - (g), students should think of forming the products two numbers at a time. (Since multiplication is associative and commutative, it doesn't matter which two numbers they choose first.)
   (a) positive (b) negative (c) positive (d) positive
   (e) negative (f) positive (g) negative

8. This question and the next look briefly at division of signed numbers, linking their rules back to multiplication. Division is revisited in the next activity sheet, which looks at fractions with signed numerators and denominators.
   (a) $(-20) \div 5 = -4$ Check: $(-4) \times 5 = -20$
   (b) $(-18) \div (-6) = 3$ Check: $3 \times (-6) = -18$
   (c) $24 \div (-4) = -6$ Check: $(-6) \times (-4) = 24$

9. (a) negative (b) negative (c) positive. It might help your students to think about the sign rules for multiplying or dividing *two* signed numbers this way:
   If the signs are different, the result is negative.
   If the signs are the same, the result is positive.

Sheet 4-4: Fitting In

• MAIN FEATURE •
Understanding rules for negatives

This activity sheet begins by looking at a historical distinction that we have only hinted at up to now. European mathematicians of the late Middle Ages and the Renaissance learned the rules for the arithmetic of signed numbers, but they used them *only* as formal manipulations for working with algebraic expressions for positive quantities. To be able to multiply $(x-2)(x-4)$ and get the correct answer, for instance, we need to know that -2 times -4 is 8.

The mathematicians of those centuries believed firmly that only positive numbers were legitimate. They would only consider $(x-2)(x-4)$ for values of x that were bigger than 4. Thus, the rules were understood, but were thought of as merely formal. When the idea of "negative numbers" as stand-alone objects was first considered, there was a lot of confusion about how to deal with them. Some of this confusion centered on ratios involving negative numbers, which leads us to examine the roles of signed numerators and denominators.

The pedagogical issue here is that the routine ways in which we manipulate signs conceal some potential confusions that took early mathematicians a long time to sort out. Thus, it is not surprising that they might raise similar difficulties for today's students, unless they are explored and put to rest conceptually. Along the way, students get to review some basic facts about negative numbers.

Solutions

1. This question lays the foundation for #2. Some students might need a little help understanding the idea or equal ratios. We have avoided starting with fractions because it is too easy for the formality to obscure the idea. If some students don't see that 4:6 and 6:9 represent the same ratio, you might ask whether "2 out of every 3 people like peanut butter" and "4 out of every 6 people like peanut butter" say essentially the same thing.

   (a) The smallest three are 6 : 9, 8 : 12, and 10 : 15, but there are infinitely many other correct answers.

   (b) 5 : 8 :: 15 : 24   1 : 4 :: 3 : 12   2 : 5 :: 8 : 20

   (c) $\dfrac{5}{8} = \dfrac{15}{24}$ $\dfrac{1}{4} = \dfrac{3}{12}$ $\dfrac{2}{5} = \dfrac{8}{20}$

   (d) Less. This sets up the issue in #2.

2. This question describes a real objection to negatives stated by Antoine Arnauld (1612–1694), a prominent French philosopher, theologian, and mathematician.

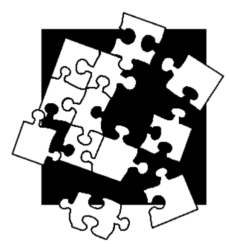
   (a) This part recalls a principle from sheet #4-1: A first number is less than a second if some positive number added to the first gives you the second. (We state this algebraically as $a < b$ if there is some $c > 0$ such that $a + c = b$. But students don't need this formality unless you're trying to teach them how to read algebraic symbolism.) In this case, $-1 < 1$ because $-1 + 2 = 1$.

(b) This part recalls a principle from sheet #4-3: When multiplying or dividing two numbers with different signs, the result is negative, regardless of order. Therefore, $1 \div (-1) = (-1) \div 1 = -1$.

(c) $1 : -1 :: -1 : 1$

(d) By 2(a), $-1 < 1$. By 2(b), $1 : -1 :: -1 : 1$; division represents the ratios. So larger is to smaller as smaller is to larger. But this makes no sense in terms of proportion. (See #1, for example.)

(e) This is an open-ended opinion question. Responses will depend heavily on the students' age and maturity level. In most cases, this question is best used for class discussion. Here is the underlying idea. When the number system is expanded to include negative numbers, some properties of positive numbers may not extend well, if at all. In deciding how the arithmetic of negative numbers "ought to work," we may have to discard some properties in favor of others. In this case, the idea that division represents proportionality does not extend well to negative numbers. We should use it only for positive numbers, because the law of signs for division is too closely tied to the logic of arithmetic to discard.



3. 17th century mathematicians were struggling with how to fit negative numbers into arithmetic without introducing inconsistencies. This paradoxical argument appears in John Wallis's *Arithmetica Infinitorum* of 1655. Some parts of this question are pretty sophisticated, but students should be able to see the main ideas. How much precision and clarity you should expect in responses depends heavily on the level and background of your students. Don't expect too much; these are difficult ideas.

(a) 3; 30; 300; 3000; 30,000; 300,000; 3,000,000

(b) The values of the fractions are getting larger and larger. They are "approaching infinity," so to speak.

(c) He would have said that it is infinite. By the pattern above, the closer the denominator gets to 0, the larger the number is. The "limiting case" is that the denominator is 0, so the value of that fraction must be greater than any finite number. (Don't worry too much about precise wording here, as long as students see the point of the pattern.)

(d) Yes, $-1 < 0$ (because $-1 + 1 = 0$ and 1 is positive). $\frac{3}{-1} > \frac{3}{0}$. By the pattern above, as the denominator gets smaller, the fraction gets bigger.

(e) Yes. $\dfrac{3}{-1} = \dfrac{-3}{1}$ (by the rule of signs for division) and any number divided by 1 is itself.

(f) This asks students to organize their thoughts about the preceding parts and put them into some clear, logical order. As the denominator get smaller, the value of a fraction with numerator 3 gets larger, until $\dfrac{3}{0} = \infty$. But $-1 < 0$, so $\infty = \dfrac{3}{0} < \dfrac{3}{-1} = \dfrac{-3}{1} = -3$; that is, $\infty < -3$.

(g) This part asks students to acknowledge that the example is typical, not special in any way. Yes, this argument can be used for any negative number, say $-n$, because the same patterns and reasoning would lead to $\dfrac{n}{0} = \infty$ and $\dfrac{n}{-1} = -n$.

4. This makes for a good class discussion. It is not an easy question! If it were, Wallis would never have posed his paradox in the first place; he wasn't stupid. However, he was writing at a time about a generation before Newton and Leibniz proposed the beginnings of calculus, a time when "infinite quantities" or limiting arguments were not well understood.

   Wallis's argument exemplifies the danger of treating infinity or something divided by 0 as if it were a number. The whole argument rests on the assumption that the pattern of smaller denominators yielding larger fractions "passes through" 0 and extends to negatives as numbers smaller than 0. But there is no reason to require that, particularly if it conflicts with the more basic rules for signed numbers.

Sheet 4-5: Powers and (Sometimes) Roots

• Main Feature •
Computing powers of signed numbers

The questions on this sheet give students practice with exponents. The first three questions establish the principle that even powers of negative numbers are positive and odd powers are negative.

Solutions

1. (a) $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$

(b) $(-7)^3 = (-7)\cdot(-7)\cdot(-7) = -343$

(c) $(-6)^2 = (-6)\cdot(-6) = 36$

(d) $(-2)^6 = (-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2) = 64$

(e) $(-2)^5 = (-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2) = -32$

(f) $(-5)^4 = (-5)\cdot(-5)\cdot(-5)\cdot(-5) = -625$

2. This question asks students to recognize and extend the pattern of #1.

   (a) positive (b) negative (c) negative (d) positive

   (e) positive (f) negative

3. Student explanations may be phrased in different ways, but the basic idea is this: Think of a repeated product of a negative number in pairs of factors. The product of each pair is positive. (Students have seen this before.) If there are an even number of factors, they all can be paired up, so the final product must be positive. If there are an odd number of factors, all but one can be paired up. The product of the pairs is positive, and multiplying that by the remaining negative factor yields a negative final product.

4. This is a routine exercise in finding easy square roots. The question is stated in equation form to give students a little practice in understanding the meaning of algebraic notation. Parts (d) – (f) provide a little extra practice with decimals.

   (a) 2 or -2 (b) 5 or -5 (c) 6 or -6

   (d) 0.5 or -0.5 (e) 0.3 or -0.3 (f) 0.01 or -0.01

Placing negatives on a number line was not obvious in the 16th and 17th centuries. For instance, René Descartes's coordinate system for the plane did not use negative numbers in the way that the familiar Cartesian coordinate system (named for him) does now. His coordinate axis — he used only one fixed axis — was for positive numbers only. The next activity gives students a visual sense of the relative positions of positive and negative square roots, as well as providing a little practice in common-sense approximation.

5. (a) 25, 36, 49, 64, 81, 100, 121, 144

   (b) The sequence along the line should look like this:

   $-6 < -\sqrt{30} < -5 < -\sqrt{19} < -4 < -\sqrt{11} < -3 < -\sqrt{7} < -2 < -\sqrt{3} <$
   $-1 < 0 < 1 < \sqrt{3} < 2 < \sqrt{7} < 3 < \sqrt{11} < 4 < \sqrt{19} < 5 < \sqrt{30} < 6$

6. This part sets up the next one. The idea here is that the square of any square root, positive or negative, is the number under the radical sign.

   $(\sqrt{6})^2 = 6$ $(-\sqrt{6})^2 = 6$ $(\sqrt{-6})^2 = -6$ $(\sqrt{a})^2 = a$

7. The square root of a number times itself should equal the number. But any positive number times itself must be positive and any negative number times itself must be positive.

8. The names Descartes gave to different kinds of numbers are still used, despite their misleading implications. *Real* numbers represent all the points on a number line — positive, negative, and zero. *Imaginary* numbers (which are just as authentic as real numbers)[8] are the even roots of negatives and their products with reals. *Complex* numbers are sums of real and imaginary numbers.

   (a) These depend on recalling that the product of two negatives is positive.

   $(\sqrt{-1})^4 = (\sqrt{-1} \cdot \sqrt{-1}) \cdot (\sqrt{-1} \cdot \sqrt{-1}) = (-1) \cdot (-1) = 1$

   $(-\sqrt{-1})^4 = ((-\sqrt{-1}) \cdot (-\sqrt{-1})) \cdot ((-\sqrt{-1}) \cdot (-\sqrt{-1})) = (-1) \cdot (-1) = 1$

   (b) $\sqrt{-3}$ and $-\sqrt{-3}$

[8]Alternatively, you could say that *all* numbers are imaginary because they are abstractions, rather than concrete objects.

Activity Sheet Set

for

Pathways from the Past

I: Using History to Teach Numbers, Numerals, & Arithmetic

William P. Berlinghoff
Fernando Q. Gouvêa

Oxton House Publishers
2010

Format Note: The activity sheets in this set are two-sided. That is reflected in the page numbering and in the border for each pair of pages. If you print this as a two-sided document, the sheets will come out the same as in the original print version.

Name:________________________________ Date:_____________

Activity 1-1

Egyptian Hieroglyphics

Nearly 4000 years ago, the ancient Egyptians wrote with picture symbols called *hieroglyphics* ("hy-row-**gliff**-ix"). When they carved numbers on monuments and other important things, they had a different picture for each power of ten:

1	10	100	1000	10,000	100,000	1,000,000
𓏺	𓎆	𓍢	𓆼	𓂭	𓆐	𓁨
stroke	heel bone	coiled rope	lotus flower	pointed finger	tadpole	astonished man

They used as many copies of a symbol as they needed for a number, but never more than nine of any one. (Why not?)

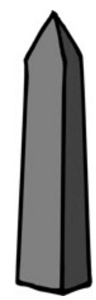
Their system did not use place value; the symbols meant the same thing in any order. For example, both 𓍢𓎆𓎆𓏺𓏺𓏺𓏺 and 𓏺𓎆𓏺𓍢𓏺𓎆𓏺 stand for 124.

1. Edgar the Explorer found these numerals carved into a stone monument. What numbers do they stand for?

   (a) 𓎆𓏺𓏺𓏺𓏺𓎆 ____________________

   (b) 𓆼𓁨𓆼𓁨𓆼 ____________________

   (c) 𓆐𓆐𓆼𓍢𓍢𓍢𓎆𓏺𓏺 ____________________



2. Write each number as an Egyptian numeral in two different ways.

   (a) 535 ____________________ or ____________________

   (b) 241,367 ______________________ or ______________________

   (c) 1,000,003 ______________________ or ______________________

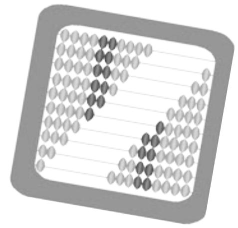
3. Three of these five numerals stand for the same number:

   (a) 𓆼𓍢𓎆𓎆𓂭𓏺𓏺 (b) 𓆼𓎆𓍢𓍢𓏺𓂭𓂭 (c) 𓎆𓎆𓍢𓂭𓏺𓏺𓆼

   (d) 𓏺𓆼𓎆𓏺𓍢𓂭𓎆 (e) 𓆐𓆼𓍢𓍢𓏺𓏺𓂭

   Which ones are they? ______________ What is that number? ___________

   What are the other two numbers? ____________ and ____________



4. Most Egyptians used an abacus or a counting board for arithmetic because even simple addition and subtraction were not easy in their system. To see how this is so, do each of the following calculations *without converting to our usual numeration system*.

   (a) Add 𓍢𓆼𓆼𓆼𓆼𓆼𓆼𓆼𓍢 and 𓍢𓆼𓆼||||𓆼𓆼𓍢

   ____

   (b) Add 𓍢𓍢𓍢𓍢𓍢∩∩∩∩∩∩||| and 𓍢𓍢𓍢𓍢∩∩∩∩∩||||

   ____

   (c) Subtract 𓍢𓍢∩∩∩||||| from 𓆼𓍢𓍢∩|||

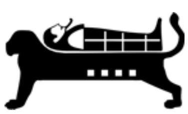
   ____

   Now convert these questions and your answers to our system. Did you get them right?

   (a) __________ + __________ = __________

   (b) __________ + __________ = __________

   (c) __________ – __________ = __________

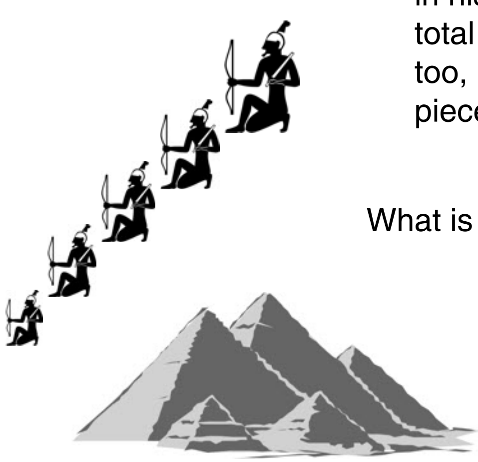


5. A carving on the tomb of King Nevvawaz says that he had 𓂭𓂭𓂭𓆼𓆼𓆼𓆼∩∩∩∩∩ soldiers. For each soldier, the king had ||| gold pieces put in his tomb as a tribute to the sun god. The total number of gold pieces was carved there, too, but it has worn away. How many gold pieces were put in the tomb?

   ____

   What is the hieroglyphic numeral that wore away?

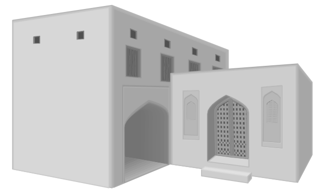
   ____



Name:________________________________ Date:_____________

Activity 1-2

Babylonian Numerals



Between 1900 and 1600 BCE, the Babylonians wrote numbers by making marks in soft clay with a special kind of stick. The clay pieces were often small enough to fit in one hand. The stick made two different wedge-shaped marks when pressed into the clay in two different ways, something like this:

one 𒁹 *ten* 𒌋

When the sun hardened the clay, these tablets became permanent records. Thousands of them still exist in museums around the world, nearly 4000 years after they were made.

To write the numbers from 1 to 59, they put together ones and tens. For instance, *twenty-three* was 𒌋𒌋𒁹𒁹𒁹.

1. What are these numbers?

(a) 𒌋𒁹𒁹 _____ (b) 𒌋𒌋𒌋 _____ (c) 𒌋𒌋𒁹𒁹𒁹𒁹𒁹𒁹 _____

2. Write each number in Babylonian.

(a) 17 ___________________

(b) 40 ___________________

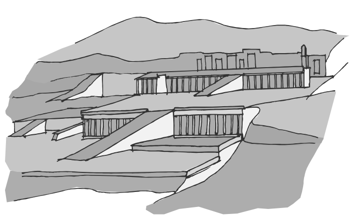
(c) 53 ___________________

For 60 to 3599, they put a second group of these symbols to the left of the first one, separated by a space. The value of the whole thing was the value of second group multiplied by 60 and added to the value of the first group. For instance,

𒁹𒁹 𒌋𒁹𒁹 is $2 \cdot 60 + 12 = 132$.

3. What are these numbers?

(a) 𒁹𒁹𒁹 𒌋𒌋𒁹 _____ (b) 𒌋𒁹 𒌋𒁹𒁹𒁹𒁹 _____ (c) 𒌋𒌋𒌋𒁹 𒌋𒌋𒌋𒁹 _____

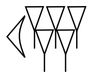


4. Write each number in Babylonian.

(a) 125 ___________________

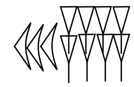
(b) 792 ___________________

(c) 3154 ___________________



fifteen

When writing numbers that used lots of wedges, the Babylonians often put them very close to one another, sometimes overlapping, as in the examples on the left and right. For easier reading, we have put all the wedges for a numeral on a single line and we have made the *ones* wedge a little simpler.



thirty-eight

5. In our notation, how much is 60^2? ________ How much is 60^3? ___________

Numbers from 3600 on were written by using more groups of the two wedge shapes farther to the left, multiplied by 60^2, 60^3, and so on.

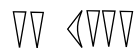
6. Explain how 7883 is  .

____

7. Write each number in Babylonian.

(a) 50,000 ____________________ (b) 72,723 _______________________

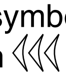
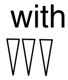
The Babylonians' place-value system let them write large numbers easily with only two symbols. But it had one big flaw: There was no way to show that a place had been skipped! For instance, to write 60 you need a  in the 60s place and nothing in the 1s place. But they had no symbol for nothing! So  could mean 1 or 60 or 3600 or something even bigger. The only way to know was to figure out what made sense for the situation.

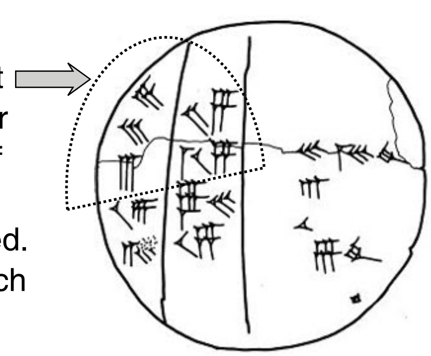
8. A clay tablet says that the total number of some things is , but the part that says what is being counted is broken off.

If this is a shepherd counting his sheep, what number is it likely to be?_______

If this is King Hammurabi counting his soldiers, what numbers are more likely? Give at least two.

______________ ______________

9. This is an example from an actual Babylonian tablet, copied here. The symbols in the upper left quadrant say that  with  is , and this answer combined with  is  . To make sense of this, you need to know how the numbers are being combined *and* which place values have been skipped. In both cases the numbers are being multiplied; which place values have been skipped? Explain.



A HAND TABLET

____

____

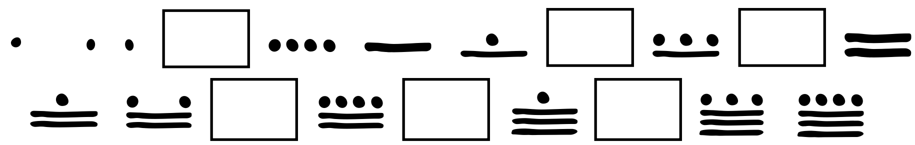
Name:__________________________________ Date:______________

Activity 1-3

Mayan Numerals

Halfway around the world from the Babylonians and Egyptians lived a people called the *Maya* ("**mī**-yuh"). They lived in Central America, on and near the Yucatan Peninsula, now part of southern Mexico and the countries of Belize, Guatemala, and Honduras. Between 1500 b.c. and 300 a.d. the Maya developed an advanced civilization, including a written language. By about 300 b.c. they had a place-value system for writing numbers that used two basic symbols, a dot "•" for *one* and a bar "——" for *five*.

1. Here are the Mayan numerals for 1 through 19. Fill in the missing entries.

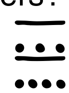
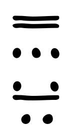


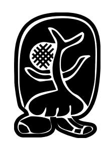
To write larger numbers, the Maya used these basic symbols as their "digits." They wrote the symbols vertically. Each digit was multiplied by an increasing place-value amount, from bottom to top. The lowest place value was 1; the next was 20; the third was 18×20 (surprise!); the fourth was 18×20^2. From there on, each place value was 18 times the next power of 20.

2. Explain why 948 is  . ____

____

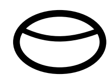
3. What are these numbers?

(a)  ______ (b)  ________ (c)  __________

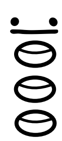


4. How would a Mayan write these numbers?

(a) 33 (b) 405 (c) 36,842

Unlike the Babylonians, the Maya used a place-holder symbol to show a skipped position. It was an oval that looked something like this: 

5. What are these numbers?

(a)  ______ (b)  ________ (c)  __________

6. How would a Mayan student write these numbers?

(a) 20 ☐   (b) 200 ☐   (c) 735 ☐   (d) 7350 ☐

7. How would a Mayan multiply each of these one-place Mayan numerals by 10 and by 20? Try it.

Numeral:	𝋤	𝋦	𝋪	𝋫	𝋮	𝋳
× 10						
× 20						

8. (a) What patterns do you see in the table for #7?

(b) Do your patterns work for two-place numerals? If so, explain why. If not, give examples that do not work.

9. How can you multiply 𝋣 𝋠 𝋠 by 20 and by 400 without converting to our system? Do it. Then check by writing the numbers in our system.

by 20: ☐   by 400: ☐

_______ number   _______ × 20   _______ × 400

10. Why should 𝋳 𝋠 and 𝋡 𝋡 𝋠 be the same number?

Name:______________________________ Date:____________

Activity 1-4

Roman Numerals

About 2000 years ago, from the first century BCE to the fifth century CE, the Romans ruled most of civilized Europe. Their system for writing numbers was used throughout Europe for many centuries after that.

Symbol	Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Roman numeration is additive. It does not use place value, with one exception. The table at right shows its basic symbols and their values. To get the value of a Roman numeral, you just add up the values of its basic symbols, like this:

$$\mathrm{CLXXII} = 100 + 50 + 10 + 10 + 1 + 1 = 172$$

To write larger numbers, the Romans put a bar over any symbol they wanted to multiply by 1000. For instance,

$$\overline{\mathrm{V}} = 5000 \text{ and } \overline{\mathrm{VII}}\mathrm{CLX} = 7000 + 100 + 50 + 10 = 7160.$$

1. (a) XXVIII = _____ (b) DCCCLXI = ______ (c) MMCXXXVII = _________

   (d) $\overline{\mathrm{V}}\mathrm{MCCCXIII}$ = ____________ (e) $\overline{\mathrm{MCC}}\mathrm{DLXXX}$ = ________________

2. Write the Roman numeral for each number.

   (a) 37 = __________ (b) 256 = ____________ (c) 2011 = ____________

   (d) 20,363 = ______________ (e) 2,000,001 = ______________

To avoid more than three copies of the same symbol in a numeral,the Romans used a subtraction rule. If a basic symbol had a smaller value than the next one to its right, the value of the pair was the larger one minus the smaller one. For instance, IV = 5 – 1 = 4. To make sure that there was only one way to write a number, only power-of-ten symbols could be subtracted, and they could only be paired with the next two larger values:

I could be paired with V or X, but not with L, C, D, or M.

X could be paired with L or C, but not with D or M.

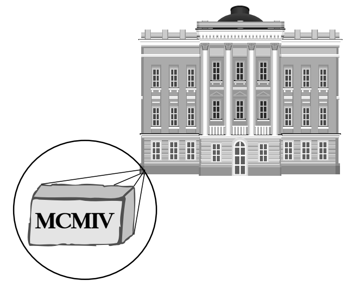
C could be paired only with D or M.

3. (a) CXLIV = ____ (b) MCMXCIX = ______ (c) MMMCDLXXIV = _______

4. Write each number in Roman numerals. (a) 324 ____________

(b) 489 _______________ (c) 2,396,944 ___________________________

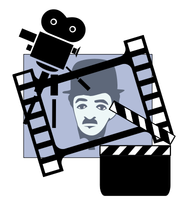
Roman numerals are still used today. They are legal for use in copyright notices and often appear that way in movies. If you look at the cornerstone of an important building—a library, a university hall, a state capitol—you will probably see the year it was built engraved in Roman numerals. You can find examples in other places, too, if you look hard enough.



5. (a) What is the date on the cornerstone shown in the drawing above? _______

(b) The Roman numeral on a courthouse in our town says that it was built in 1859. What is that numeral? _________________

6. (a) The copyright notice on the movie *Charade* says MCMLXIII. What year is that? _______



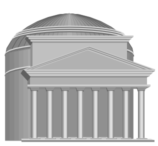
(b) The copyright for *Charlie Chan's Secret* says MCMXXXV. In what year was it filmed? ______

(c) *The Last Time I Saw Paris* was filmed in 1954, but its copyright notice says MCMXLIV. What's wrong?

____

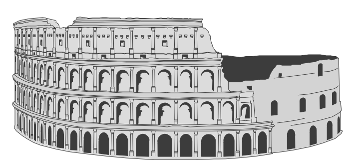
(d) The copyright notice on the Australian film *Mad Dog Morgan* says MCMDXXVI. What's wrong?

____



It is not easy to do arithmetic with Roman numerals. When people wrote numbers this way, the calculated with an abacus or a counting board. Do the following calculations *without translating them into our usual system*. (You'll need some scrap paper.) Be prepared to explain how you did them. Then chack your answers by translating.

7. Add DCCCXLVIII and CDXXXIV.



____

8. Subtract DCCCXLVII from MCCLXVI.

____

7. Multiply CCLXXXIV by XVI.

____

Name:_________________________________ Date:_______________

Activity 1-5

Hindu-Arabic Numerals

To write numbers today, we use the *Hindu-Arabic* system. It was invented by the Hindus in India about 600 CE. The Arabs learned it from them in the 7th and 8th centuries. The Europeans learned it from the Arabs several centuries later. It replaced the Roman system because it was easier to use.

The Hindu-Arabic system uses place value based on powers of ten:

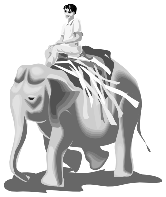
$$1,\ 10,\ 100 = 10^2,\ 1000 = 10^3,\ 10{,}000 = 10^4, \ldots.$$

Its basic symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The chart below shows basic symbols of the other systems you have learned about.

Hindu-Arabic	0	1	5	10	50	100	500	1000
Egyptian		𓏺		𓎆		𓍢		𓆼
Babylonian		𒁹		𒌋				
Mayan	𝋠	𝋡	𝋥					
Roman		I	V	X	L	C	D	M

1. Fill in the following table so that all the numerals in each row represent the same number.

Egyptian	Babylonian	Mayan	Roman	Hindu-Arabic
	𒁹 𒌋𒁹𒁹			
𓎆𓎆𓏺𓏺𓍢𓏺𓏺𓎆𓎆				
		𝋯 𝋠		
				620
			MCCCXX	



2. Rank the five systems of question 1 in order, from "easiest to use" to "hardest to use, " *based on your own opinions*. Be prepared to give reasons for your rankings.

(1) ____________________________ (easiest)

(2) ____________________________

(3) ____________________________

(4) ____________________________

(5) ____________________________ (hardest)

3. Arrange the letters for the following events in chronological order on the line below, from earliest to most recent.

(a) The Roman Republic was founded.

(b) The Egyptians began using hieroglyphic numerals.

(c) Rome fell to Odoacer, ending the Roman Empire.

(d) The Babylonians began using wedge-shaped numerals.

(e) Columbus landed in America.

(f) The first universities were founded in Europe.

(g) Alexander the Great conquered much of the Near East.

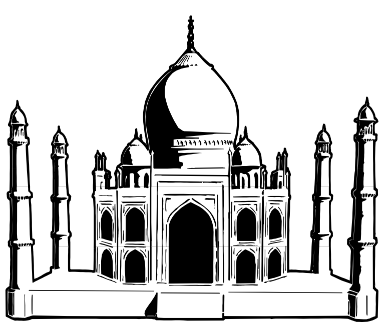
(h) Charlemagne ruled the Frankish Empire in Europe.

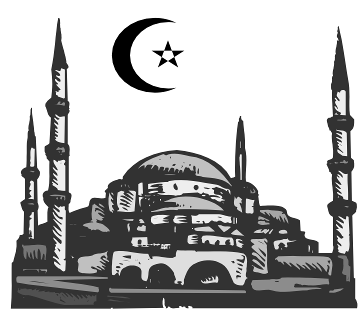
(i) The first European printing press was put into use.

(j) The Arabs formed the Hindu-Arabic numeration system.

____

earliest — most recent





Name:______________________________ Date:____________

Activity 2-1

Using a Place Holder

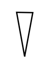
Most people think of zero as "nothing." But it's not. Zero is a very important something, not just in mathematics but whenever people use numbers. It started out as a place holder, a notice that a particular location was vacant. Even then it was very important.

A system for writing numbers is called a **numeration system**, and the written form of a number is called a **numeral**. If the value of a symbol in a numeral depends on where it is, the system uses **place value**. In our usual system, for instance, 131 and 311 stand for different numbers because the 3 means 3 tens in the first numeral, but it means 3 hundreds in the second.

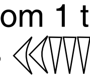
1. What does the 5 mean in each of these numerals?

(a) 251 __________ (b) 3615 __________ (c) 5432 __________

The earliest place-value system we know about was used by the Babylonians almost 4000 years ago. It was based on multiplying by 60. They had wedge-shaped symbols for the numbers *one* and *ten*, something like the ones shown here.

one 

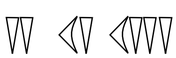
ten 

To write the numbers from 1 to 59, they just put together ones and tens. For instance, *twenty-three* was  .

2. What are these numbers?

(a)  ______ (b)  ______ (c)  ______

For numbers from 60 on, they put more groups of the two basic symbols to the left. The next group was multiplied by 60, the one after that by 60^2, and so on. For instance,

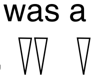
 meant $2 \cdot 60^2 + 11 \cdot 60 + 13 = 7873$.

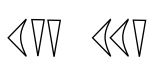
$60^2 = 3600$

$60^3 = 216{,}000$

$60^4 = 12{,}960{,}000$

⋮

But there was a problem. They had no way to show when a place was skipped. For example,  could mean $2 \cdot 60 + 1$ or $2 \cdot 60^2 + 1$ or $2 \cdot 60^2 + 1 \cdot 60$.

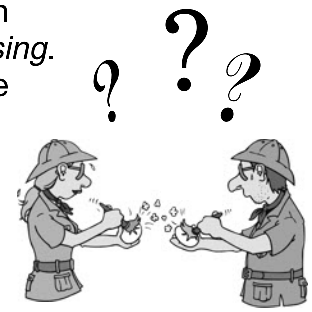
3. Interpret  in three different ways. Calculate the numbers.

(a) _________________ (b) _________________ (c) _________________

In the 19th century, archaeologists found hundreds of ancient Babylonian clay tablets. Many of them appeared to be using these wedge-shaped numerals for arithmetic. (Activity sheet 1-2 shows a drawing of one of these tablets.) But the calculations did not always seem to make sense, until people realized that sometimes places were being skipped!

Using a Place Holder

The next few problems will give you some idea of what the archaeologists had to deal with. They are calculations written with our usual numerals, *but with the zero place-holders missing*. Turn each one into a correct statement by putting zeros in the right places. For example:



1 2 + 1 3 = 2 2 3

is correct if you put in zeros like this:

120 + 103 = 223.

4. Now it's your turn. Try these:

(a) 2 3 + 1 7 = 3 1 ______________________________

(b) 2 3 – 1 7 = 1 2 3 ______________________________

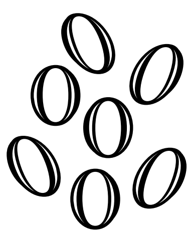
(c) 2 3 + 1 7 = 2 2 ______________________________

(d) 5 5 + 5 5 = 1 5 5 ______________________________

(e) 5 5 + 5 5 = 6 5 ______________________________

(f) 5 5 – 5 5 = 4 5 in two different ways.

______________________________ & ______________________________



5. In these multiplication examples, the 4 stands for 4 ones. Where are the missing zeros in the other numerals?

(a) 2 7 5 × 4 = 8 3 ________________________

(b) 2 7 5 × 4 = 1 8 2 ________________________

(c) 2 7 5 × 4 = 8 2 8 2 ________________________

6. There's nothing special about using 0 for the place-holder symbol. It's just a tradition that came from the original Hindu system. We could use any other symbol, such as #, for instance. If we did, then 2#3 would mean 2 hundreds and 3 ones. What would each of these numerals mean? (Write the place values of your answers in words.)

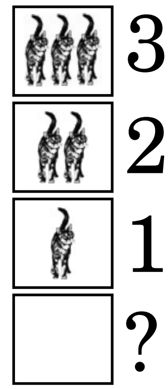
(a) 7# ____________________ (b) 4## ________________________

(c) 5#6 ____________________ (d) 2#1# ________________________

Name:___________________________________ Date:______________

Activity 2-2

The Number Zero



The Hindus' idea of a place holder was called *sunya*. The Arabs who learned the Hindu system in the 9th century called it *sifr*. When the Europeans learned it from the Arabs in the 12th century, they Latinized the word in two different ways, *cifra* and *zephirum*. These became our English words *cipher* ("**sy**-fur") and *zero*.

But the Indian mathematicians of the 9th century took a big step forward that the Arabs didn't see at the time. They realized that, just as 3 was an idea, whether it was counting cats or chickens or countries, so 0 was an idea, even though it wasn't counting anything. That is,0 is not just a place holder, it's a *number*. This meant that they had to decide how 0 should work when it is combined with other numbers by the operations of arithmetic.

1. (a) If you add 0 to a number, what should the result be? Why?

____

____

(b) 5 + 0 = ___ (c) 0 + 107 = _____ (d) 0 + 0 = ___

2. (a) Sometime around 850 CE, the Indian mathematician Māhavīra wrote that 0 subtracted from a number leaves the number unchanged. Explain why Māhavīra's idea makes sense if you are counting cats (or anything else).

____

(b) 3 – 0 = ___ (c) 1907 – 0 = _____ (d) 0 – 0 = ___

(e) Why do you think Māhavīra didn't mention subtracting a number from 0?

____

3. Māhavīra also wrote that a number multiplied by 0 results in 0.

a) How does this fit in with the arithmetic of whole numbers? (Think of 3 × 4 as 4 + 4 + 4, for instance.)

____

____

b) 7 × 0 = ____ (c) 0 × 25 = ____ (d) 0 × 0 = ____

0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + ... + 0 = ___ ?

The Number Zero

4. Māhavīra also claimed that a number divided by 0 remains unchanged. However, about 1100 CE, the Indian mathematician Bhāskara said that a number divided by 0 is an infinite quantity.

$\frac{37}{0}$???

   (a) Explain how Māhavīra's idea does not match the way division works for other numbers. (Think about $5 \times (3 \div 5)$, for example.)

   ____

   ____

   (b) Why was Bhāskara's idea reasonable? ______________________________

   ____

   (c) Suppose $7 \div 0$ equals some very big number; we'll call it B for "big." What goes wrong? (Think about how $\div$ and $\times$ "undo" each other.)

   ____

   ____

5. Dividing a number by 0 and dividing 0 by a number are two *very* different things.

   (a) $0 \div 17 =$ ____ (If you chop 0 yards of rope into 17 pieces, how big is each piece?)

   (b) $0 \div 5 =$ ____ (c) $0 \div 83{,}496 =$ ____

6. (a) In 7^5, what does the 5 mean? ______________________________

   (b) Write 7^5 and 7^3 as repeated products: __________ and __________

   (c) If $7^5 \times 7^3 = 7^N$, what is N? ___ Why? ______________________________

   (d) What should $7^5 \times 7^0$ be? ___ What should 7^0 be? ___ Why? ____________

   ____

7. Did you get question 6(d)? If you did, congratulations! If not, don't feel bad; it's not obvious. Think about it this way: A whole-number exponent counts the number of repeated factors in a product. In 6(c), the 7^3 says that 7^5 is multiplied by 3 more 7's. In $7^5 \times 7^0$, the 7^0 says that 7^5 is multiplied by no more 7's. Therefore:

   (a) $7^5 \times 7^0 =$ ___ ? (b) So $7^0 =$ __ ? (c) $2^0 =$ __ ? (c) $586^0 =$ __ ?

8. True or False: If $a^0 = b^0$, then $a = b$. Explain: ______________________________

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Name:________________________________ Date:_____________

Activity 2-3

Zero in Equations

1. Can you find two nonzero numbers whose product is 0? If so, do it. If not, explain why you think it can't be done.

____

2. What can you say about numbers a and b in each case?

   (a) $3a = 3b$ __________ (b) $47a = 47b$ __________ (c) $0a = 0b$ __________

3. Use your answer to #1 to justify your answer to #2(a). ____________________

____

Question 1 points to an important fact:

If the product of two numbers is 0, then at least one of them must be 0.

In the early 17th century, Thomas Harriot turned this fact into a powerful tool for solving equations. Harriot was a geographer, a naturalist, and a mathematician. In 1585 he was sent by Sir Walter Raleigh to help found the first English colony in the New World. That colony was on Roanoke Island, in an area the British called Virginia. It did not survive, but some of Harriot's writings about it still exist. He was its surveyor and its historian. Harriot also wrote about algebra, an important field of study in 17th-century England. He was the first person to use a simple, powerful principle that made solving polynomial equations much easier than it had been up to then.

4. Which state is Roanoke Island in today? __________________________

> **Harriot's Principle:** Move all the terms of the equation to one side of the equal sign, so that the equation has the form
>
> [some polynomial] = 0.

5. Rewrite each of these equations according to Harriot's Principle.

   (a) $x^2 + 3 = 4x$ ________________________

   (b) $2x^3 - 6x + 3 = 4x^2 + 2x - 9$ ________________________

   (c) $7x + 1 = 4x^2 - x + 5$ ________________________

6. Use the "important fact" from question 1 to rewrite equations 5(b) and 5(c) so that the leading coefficient (the coefficient of the highest-power term) is 1.

   (5b) ________________________ (5c) ________________________

Zero in Equations

By the 16th century, European mathematicians knew a lot about factoring polynomials. For example, they knew that $x^2 - 4x + 3 = (x - 3)(x - 1)$. When Harriot's Principle told them that $x^2 + 3 = 4x$ was the same as $x^2 - 4x + 3 = 0$, they could immediately solve that equation. Do you see how?

7. Solve $x^2 + 3 = 4x$. That is, find two numbers for x that make it a true statement.

   Show how you did it. $x =$ ___ or $x =$ ___

   ____

8. Solve these equations.

   (a) $0 = 2x^2 + x - 15 = (2x - 5)(x + 3)$ $x =$ ___ or ___

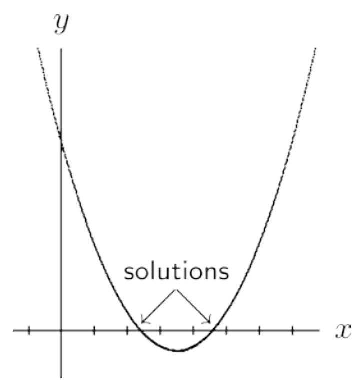
   (b) $0 = 3x^2 + 19x - 14 = (x + 7)(3x - 2)$ $x =$ ___ or ___

In 1637 the French mathematician René Descartes described a way to picture algebraic expressions by graphing them on coordinate axes. Using Harriot's Principle, the solutions to an equation can be approximated by graphing without factoring.
For instance:

To solve $x^2 + 11 = 7x$, first rewrite it as $x^2 - 7x + 11 = 0$.

Now graph $x^2 - 7x + 11 = y$.

The solutions are where the graph crosses the x-axis.



9. The picture above is the graph of $x^2 - 7x + 11 = y$. Use it to approximate the solutions to $x^2 + 11 = 7x$. Check by substituting into the original equation. See if you can make the difference between the two sides less than 0.01.

   $x =$ _____; difference: _____ $x =$ _____; difference: _____

10. What's wrong with this "proof" that 2 = 1?

Suppose a and b are nonzero and $a = b$.	
Multiply both sides by a:	$a^2 = ab$
Subtract b^2 from both sides:	$a^2 - b^2 = ab - b$
Factor both sides:	$(a + b)(a - b) = b(a - b)$
Divide by $(a - b)$:	$a + b = b$
Substitute b for a: (They are equal.)	$2b = b$
Divide by b:	**2 = 1**

____

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Activity 3-1

Unit Fractions

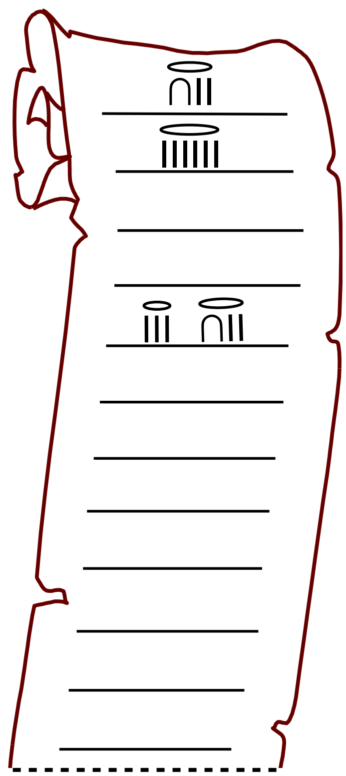
In the 17th century BCE, the Egyptians had a strange way of dealing with fractions. They worked only with "parts." We would call these *unit fractions*, fractions with numerator 1. For instance, "the eighth part" is $\frac{1}{8}$. To show that a symbol represented a part, they put a dot or an oval above the symbol for the size, like this:

ten: ∩ the tenth: $\dot{\cap}$ twelve: ∩ II the twelfth: $\overset{\frown}{\cap\text{ II}}$

They used only one copy of each size part in any single number. For example, they would call $\frac{3}{8}$ "the fourth and the eighth."

1. Ahmose is having trouble with his homework. His teacher asked him to fill in a table for the first twelve multiples of *the twelfth*. Can you help him? His table is on the scroll. Next to it is a matching table with our modern fraction symbols. Fill in both of them.

(*Remember*: You can't use the same size part more than once for each number.)



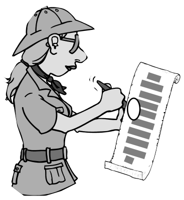
$\frac{1}{12}$
$\frac{2}{12} = \frac{1}{6}$
$\frac{3}{12} =$
$\frac{4}{12} =$
$\frac{5}{12} = \frac{1}{3} + \frac{1}{12}$
$\frac{6}{12} =$
$\frac{7}{12} =$
$\frac{8}{12} =$
$\frac{9}{12} =$
$\frac{10}{12} =$
$\frac{11}{12} =$
$\frac{12}{12} =$

2. What sum of unit fractions would Ahmose need for each of these numbers? Use our modern fraction symbols to write your answers.

(a) $\frac{4}{7}$ = ______________________ (b) $\frac{11}{16}$ = ____________________________

(c) $\frac{13}{27}$ = ______________________ (d) $\frac{23}{50}$ = ____________________________

3. Alyssa the Explorer found this papyrus scroll. Help her translate its numbers and find a pattern that connects them to each other.



What pattern do you see?

____

____

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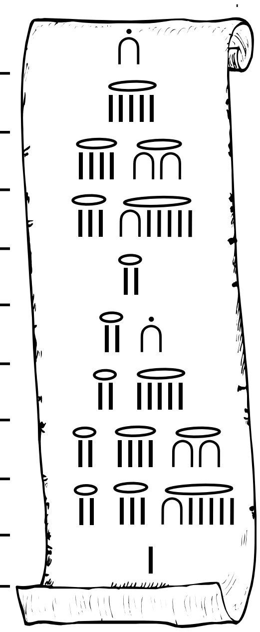

4. A lot of ancient Egyptian arithmetic was based on repeated doubling. Doubling "parts" is not always easy. Try these. You may use modern fraction symbols, but remember that the numerator can only be 1!

part	× 2	× 4	× 8
$\frac{1}{16}$			
$\frac{1}{28}$			
$\frac{1}{18}$			
$\frac{1}{13}$			

5. The opposite of doubling is halving. Write half of each of these "parts."

(a) $\frac{1}{16}$ ________ (b) $\frac{1}{15}$ ________ (c) $\frac{1}{350}$ ________

6. Why is it easy to divide a "part" by any whole number? Give an example to illustrate your answer.

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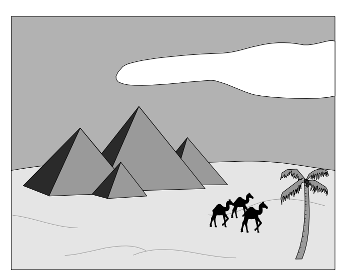

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Activity 3-2

Place-Value Fractions

Almost 4000 years ago the Babylonians wrote numbers using a place-value system based on *sixty*. They wrote in soft clay with a stick that made two different wedge-shaped marks, 𒁹 for *one* and 𒌋 for *ten.* To write the numbers from 1 to 59, they put together ones and tens. For instance, *thirty-two* was 𒌋𒌋𒌋𒁹𒁹 .

1. What are these numbers?

   (a) 𒌋𒌋𒁹𒁹𒁹𒁹 _____ (b) 𒌋𒁹𒁹𒁹𒁹𒁹𒁹𒁹 _____ (c) 𒌋𒌋𒌋𒌋𒁹 _____

2. Write each number in Babylonian.

   (a) 13 _______ (b) 22 _______ (c) 50 ________

For 60 to 3599, they put a second group of symbols to the left of the first one. The value of the whole thing was the value of second group multiplied by 60 and added to the value of the first group. Numbers from 3600 on were written by using more groups farther to the left, multiplied by 60^2, 60^3, and so on. For instance,

𒌋𒁹 𒁹𒁹 𒌋𒌋𒁹 is $11 \cdot 60^2 + 2 \cdot 60 + 21 = 39{,}741$.

3. What are these numbers?

   (a) 𒁹 𒌋𒁹𒁹𒁹 _______ (b) 𒌋 𒌋𒁹𒁹 𒌋𒌋𒁹𒁹𒁹𒁹 ___________

4. Write each number in Babylonian.

   (a) 812 ___________ (b) 11,425 ______________

A small notation change will make it easier to see how the Babylonians wrote numbers less than 1. Instead of using their symbols for the numbers 1 through 59, we'll use our own, with commas to separate the groups. For example, we will write 𒌋𒁹 𒁹𒁹 𒌋𒌋𒁹, which means $11 \cdot 60^2 + 2 \cdot 60 + 21$, as 11, 2, 21.

5. Write each of these Babylonian numerals using our symbols. Then write it as a sum of powers of 60, as in the example above.

   (a) 𒁹𒁹 𒌋𒌋𒁹𒁹𒁹𒁹 𒌋𒌋𒌋𒁹 __________ ___________________________

   (b) 𒌋𒌋𒌋 𒁹 𒌋𒁹𒁹𒁹𒁹𒁹𒁹𒁹 __________ ___________________________

   (c) 𒁹 𒌋𒁹 𒌋𒌋𒌋𒌋𒌋 𒌋𒁹𒁹 __________ ___________________________

To write fractions, the Babylonians started with 60ths. Nobody knows for sure why their system is based on 60, but some historians think it is related to their system of money:

60 shekels = 1 mina; 60 minas = 1 talent

Place-Value Fractions

This is not very different from our own base-ten system for money:

10 pennies = 1 dime; 10 dimes = 1 dollar

One big advantage of using 60 is that it has a lot of factors (numbers that divide it without remainder), so many different fractions can be expressed as multiples of 1/60.

6. List all the factors of 60. ____

Fractions were written by putting symbol groups of to the right of the *ones* place, just as we do with decimals. The first group was for 60ths, the next for 3600ths, etc. (If you think of the units as *hours*, then the first place to the right would be *minutes* and the next would be *seconds*.) For example,

$$7\tfrac{1}{4} = 7; 15 \quad \text{and} \quad 5\tfrac{3}{8} = 5; 22, 30.$$

7. Explain the two examples above. (Convert to fractions and simplify.)

____

____

8. How would we write these numbers today?

(a) 1; 20 __________ (b) 2; 30, 30 ____________

(c) 3; 24, 36 ______________ (d) 4; 1, 1, 1 _______________

A place-value system makes adding and subtracting fractions easy. The Babylonians just added the whole numbers place by place. If a sum was more than 60, they "exchanged" it for a 1 in the next place to the left. For example,

1; 45 + 1; 20 = 3; 5

This is just like "carrying" 10 when we add decimals in our system.

9. (a) Check the example above by converting it to common fractions.

____

(b) Add 2; 40 and 3; 50 in the Babylonian system. Then check by converting to common fractions.

Sum: _________ Check: _______________________________

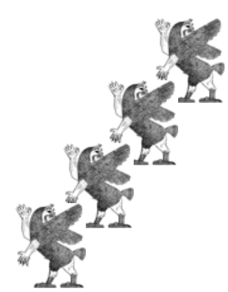
10. Do these *without converting to common fractions.*

(a) 2; 25, 17, 42 + 3; 40, 13, 50 = _____________________

(b) 7; 33, 28, 44 + 3; 29, 10, 21 = _____________________

(c) 8; 40, 29, 23 – 1; 35, 12, 15 = _____________________

(d) 5; 21, 24, 17 – 4; 21, 30, 47 = _____________________



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Activity 3-3

Name and Count

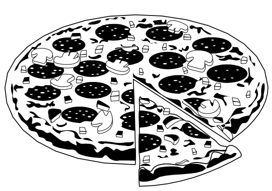
Today we describe the size of a piece of something by counting copies of a single, small enough part. That is, we choose a small part (a "unit part") that can be counted enough times to get exactly the amount we want. Then two numbers tell us the total amount: the size of the part, and the number of times we count it.

The size of the part is given by the ***denominator***, which is Latin for "namer."

The number of copies of that part is called the ***numerator***, Latin for "counter."

The earliest evidence of this approach to fractions comes from about 100 BCE, in a Chinese manuscript called *Nine Chapters on the Mathematical Art*. Its notation for fractions is a lot like ours. The one difference is that the Chinese did not use "improper fractions." Instead of $\frac{7}{3}$ they would write $2\frac{1}{3}$.

1. Find an English word with the same root as *denominator* that means something about naming. ____________________ Then find an English word with same root as *numerator* that means something about counting. _________________



2. If you have $\frac{3}{8}$ of a pizza, what is the size of the unit part? ____

   How many copies of that part do you have? ____

   If you double the denominator, how much pizza is that? ____

   Is that more or less than $\frac{3}{8}$? _______ Explain. ____________

   ____

Hindu manuscripts as early as the 7th century CE show fractions as one number over another. The size of the part was below the number of times it was to be counted, but there was no line between them. This form became common in Europe a few centuries later. The Arabs added a bar between the top and bottom numbers sometime around the 12th century.

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4

3. The denominator "names" the size of the equal pieces (the unit parts) by telling you how many there are in one whole thing. The more pieces there are, the smaller each one is. Put these fractions in size order, from smallest to largest:

$\frac{1}{9}$ $\frac{1}{64}$ $\frac{1}{2}$ $\frac{1}{8}$ $\frac{1}{6}$ $\frac{1}{12}$ $\frac{1}{3}$ $\frac{1}{256}$

____ < ____ < ____ < ____ < ____ < ____ < ____ < ____

Name and Count

4. The numerator "counts" the number of equal pieces you have. *If the pieces are the same size,* the more you have, the larger the number is. Put these fractions in size order, from smallest to largest:

$$\frac{1}{9} \qquad \frac{5}{9} \qquad \frac{8}{9} \qquad \frac{2}{9} \qquad \frac{7}{9} \qquad \frac{4}{9}$$

$$\_\_\_\_ < \_\_\_\_ < \_\_\_\_ < \_\_\_\_ < \_\_\_\_ < \_\_\_\_$$

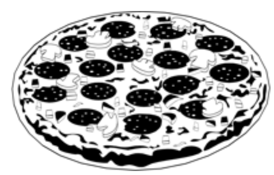
5. Circle the larger fraction in each pair. Be prepared to justify your choices.

(a) $\frac{4}{7}$ $\frac{5}{7}$ (b) $\frac{4}{7}$ $\frac{4}{8}$ (c) $\frac{5}{8}$ $\frac{3}{8}$

(d) $\frac{4}{6}$ $\frac{4}{5}$ (e) $\frac{2}{5}$ $\frac{2}{9}$ (f) $\frac{23}{35}$ $\frac{18}{35}$

(g) $\frac{53}{91}$ $\frac{53}{92}$ (h) $\frac{14}{37}$ $\frac{15}{37}$ (i) $\frac{45}{83}$ $\frac{45}{91}$

6. You and three friends go out to share a large pizza.
   (a) You decide to share it equally. What fraction will you have? _____
   (b) If the pizza is cut into 8 pieces, how many will you have? _____
   (c) If it is cut into 12 pieces, how many will you have? _____
   (d) If it is cut into 16 pieces, how many will you have? _____
   (e) Fill in the missing numerators: $\frac{1}{4} = \frac{\_\_}{8} = \frac{\_\_}{12} = \frac{\_\_}{16}$



7. Generalize the idea of the previous question to fill in the missing numerators.

(a) $\frac{1}{5} = \frac{\quad}{20}$ (b) $\frac{4}{7} = \frac{\quad}{21}$ (c) $\frac{2}{3} = \frac{\quad}{15}$ (d) $\frac{9}{12} = \frac{\quad}{4} = \frac{\quad}{20}$

When fractions have different numerators and denominators, it may be hard to say which is larger. That's because different-size pieces are being counted. For instance, $\frac{5}{7}$ and $\frac{8}{11}$ are easily compared using the same size pieces:

$$\frac{5}{7} = \frac{55}{77} \text{ and } \frac{8}{11} = \frac{56}{77}\text{, so } \frac{8}{11} \text{ is } \frac{1}{77} \text{ larger than } \frac{5}{7}.$$

8. Compare each pair of fractions as in the previous example.

(a) $\frac{4}{7} = \square$ and $\frac{5}{9} = \square$, so $\square$ is $\square$ larger than $\square$.

(b) $\frac{5}{6} = \square$ and $\frac{9}{11} = \square$, so $\square$ is $\square$ larger than $\square$.

Name:_________________________________ Date:_____________

Activity 3-4

Working with Parts

Nine Chapters on the Mathematical Art, an ancient Chinese text, shows that the Chinese of 100 BCE thought of a fraction as some number of copies of a small part of a whole thing. The bottom number was the size of the part; the top number told them how many copies to take.

1. (a) To measure out $\frac{3}{5}$ of a bushel of rice, how many equal parts of the bushel should you make? ____ How many of them should you take? ____

   (b) If you separate the bushel into 10 equal parts and take 3 of them, what fraction of it do you have? _____ Is that more or less than $\frac{3}{5}$? _______ How much more or less? ________________

The main idea of the *Nine Chapters* approach to fraction arithmetic is this:

Find a small enough part for all the fractions to be whole-number multiples of it. This turns any problem into a whole-number problem.

2. Write each of these sums as a number of copies of a single unit part.

   (a) a quarter and an eighth _______ (b) a half and a third _________

   (c) a third and a quarter _________ (d) a fourth and a fifth __________

   (e) Can (a) – (d) each have more than one correct answer?______ Explain.

   ____

All the usual rules for fraction arithmetic appear in the *Nine Chapters*: how to reduce a fraction that is not in lowest terms, how to add fractions, and how to multiply them. For instance, this is their rule for addition:

> *Each numerator is multiplied by the denominators of the other fractions. Add them as the dividend, multiply the denominators as the divisor. Divide; if there is a remainder, let it be the numerator and the divisor be the denominator.*

3. Use the *Nine Chapters* rule for addition to do these sums. Show your work.

   (a) $\frac{2}{3} + \frac{3}{5}$ ____

   (b) $\frac{1}{2} + \frac{4}{5} + \frac{7}{8}$ ____

   (c) $\frac{1}{3} + \frac{3}{4} + \frac{5}{8} + \frac{1}{6}$ ______________________________________

4. How (if at all) is the *Nine Chapters* rule for addition different from the way you usually add fractions? ______________________________

____

5. IBy replacing “Add” with “Subtract” in the *Nine Chapters* addition rule, you get a rule for for subtraction. Use it to answer these questions. Show your work.

(a) $\frac{2}{3} - \frac{3}{5} =$ ______________________________

(b) $\frac{7}{8} - \frac{1}{6} =$ ______________________________

(c) $\frac{3}{4} - \frac{5}{8} =$ ______________________________

The *Nine Chapters* also shows that the Chinese of 100 BCE knew how to reduce fractions that are not in lowest terms and how to multiply fractions.

6. (a) Write a rule for reducing a fraction that is not in lowest terms.

____

(b) Reduce each of your answers in #3 to lowest terms:

(3a) ____________ (3b) ____________ (3c) ____________

(c) Reduce to lowest terms: $\frac{54}{90} =$ ____________ $\frac{462}{495} =$ ____________

7. The *Nine Chapters* rule for multiplying two fractions is just like ours:

Multiply the numerators to get the numerator of the product;
multiply the denominators to get the denominator of the product .

Use it to find these products. Show your work.

(a) $\frac{6}{7} \times \frac{5}{8} =$ ____________ (b) $\frac{11}{12} \times \frac{7}{9} =$ ____________

(c) $\frac{5}{6} \times \frac{1}{6} =$ ____________ (d) $\frac{12}{100} \times \frac{3}{100} =$ ____________

8. To divide, the *Nine Chapters* first finds a common part size (denominator) for both fractions. Then the answer is just the quotient of the numerators; e.g.,

$$\frac{2}{3} \div \frac{4}{5} = \frac{10}{15} \div \frac{12}{15} = \frac{10}{12}$$

(a) Divide by the *Nine Chapters* method:

$\frac{5}{8} \div \frac{3}{7}$ ____________ $\frac{3}{10} \div \frac{4}{9}$ ____________

(b) Now divide by the “invert and multiply” method.

$\frac{5}{8} \div \frac{3}{7}$ ____________ $\frac{3}{10} \div \frac{4}{9}$ ____________

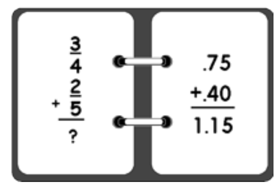
Name:________________________________ Date:_____________

Activity 3-5

Decimals

The Latin word for "ten" is *decem*, so "decimals" are numerals based on ten. You already know how the decimal system works for whole numbers. Starting from units, each place to the left goes up in value by a power of ten: tens, hundreds, thousands, etc. For instance, 456 is 6 units, 5 tens, and 4 hundreds.

Places to the right can be used in the same way. The Chinese and the Arabs knew this more than 1000 years ago. But in Europe, the main source of our mathematics, this system was not used for writing fractions until almost 100 years after Columbus discovered America. In his 1585 book called *The Tenth*, Simon Stevin, a Flemish engineer, showed how writing fractions as decimals allows them to be handled by the simpler processes of whole number arithmetic. The use of decimal fractions by prominent scientists during the next few decades paved the way for general acceptance of decimal arithmetic.



1. Name the place values for the first three places to the right of the units place.

   _units ____________ ____________ ____________

2. Write each decimal as a common fraction. Then reduce it to lowest terms (if it is not already in that form).

   (a) 0.7 ____________________ (b) 0.75 ____________________

   (c) 0.008 ____________________ (d) 0.33 ____________________

   (e) 2.12 ____________________ (f) 1.2525 ____________________

3. Write as decimals without using a calculator. (Stevin didn't have one in 1585.)

   (a) $\frac{4}{5}$ = ______ (b) $\frac{13}{20}$ = _______ (c) $\frac{121}{250}$ = _______ (d) $\frac{17}{8}$ = _______

4. (a) Add: $\frac{5}{8} + \frac{2}{5}$ = ________________________ . Now write these two

   fractions as decimals and add again: ______ + ______ = _________.

   Circle the way that was easier for you: fraction decimal

   (b) Subtract: $\frac{3}{4} - \frac{72}{125}$ = ____________________ . Now write these two

   fractions as decimals and subtract again: ______ – ______ = _________.

   Circle the way that was easier for you: fraction decimal

(c) Multiply: $\frac{7}{8} \times \frac{3}{25}$ = ___________________________ . Now write these two fractions as decimals and multiply again: ______ x ______ = _________.

Circle the way that was easier for you: fraction decimal

(d) Divide: $\frac{3}{8} \div \frac{2}{5}$ = _________________________ . Now write these two fractions as decimals and divide again: ______ – ______ = _________.

Circle the way that was easier for you: fraction decimal

Writing fractions as decimals was a big step forward. It opened the door to many powerful tools, including the metric system in the 18th century and electronic calculators in the 20th century. Calculators, in turn, make it even easier to use fractions as decimals.

5. (a) Which of these decimals is larger, 0.37498 or 0.3761? _______

(b) How can you tell which of two decimals is larger? ____________________

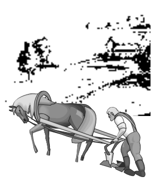
____

(c) Use a calculator to help you order these fractions from smallest to largest.

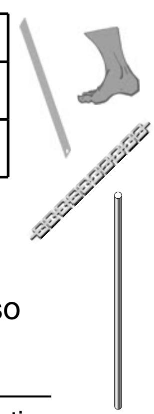
$\frac{3}{5}$ $\frac{2}{3}$ $\frac{5}{8}$ $\frac{16}{25}$ $\frac{11}{17}$ $\frac{83}{137}$ $\frac{247}{391}$ $\frac{613}{942}$

_____ < _____ < _____ < _____ < _____ < _____ < _____ < _____

6. People in 16th century England used many different units to measure distances. This table shows how some of them are related to each other.



furlong	chain	rod	yard	foot
1/8 mile	1/10 furlong	1/4 chain	2/11 rod	1/3 yard
0.125				



(a) Fill in the bottom row with decimals for each fraction. Round to three places, if necessary.

(b) Calculate the length of a mile in feet using your decimals and also by using the common fractions.
(Do your answers agree? Should they?)

______________ by decimals   ______________ by common fractions

(c) What fraction of a mile is the total of 2 furlongs, 4 chains, and 3 rods?

____________ mi. How many feet is that? ____________ ft.

Name:______________________________ Date:____________

Activity 3-6

Percent

The Latin word for "hundred" is *centum*. A penny is called a *cent* because it's a hundredth of a dollar. Our word "percent" is a short form of the Latin phrase *per centum*, meaning "for each hundred" or "out of every hundred." A 5 percent tax takes 5 cents out of every dollar.

1. (a) What is 6 percent of $2.00? ________

   (b) What common fraction equals 7 percent? _______

The symbol for percent evolved slowly over several centuries, from "per 100" in the 1400s to "per $\frac{0}{0}$" by 1650, then simply to "$\frac{0}{0}$," and finally to the % or % symbol we use today.

2. (a) 7% of 400 = _____ (b) 10% of 250 = _____ (c) 5% of 40 = _____

Europe of the 13th to 15th centuries gradually shifted from feudalism to more widespread commerce. Money became essential for doing business. Of course, there were no dollars and cents then. Countries and even cities developed their own money types — francs in France, guldens in Germany, florins in Florence, and so on. As money became more important, so did loans. Most loan payment rates were stated *per hundred* of whatever the local currency was; that is, they were stated as a *per cent* of the loan amount.

3. Marco the Merchant borrowed 300 florins to invest in a trading expedition to India. He had to pay the lender 4% every month until the ship returned two years later.

   (a) How much interest did he pay each month?____________

   (b) How much interest did he pay in all? ______________

4. Write each of these percents as a common fraction in lowest terms.

   (a) 25% = ________ (b) 50% = ________ (c) 20% = ________

   (d) 15% = ________ (e) 33% = ________ (f) 75% = ________

5. Decimals were not widely used in Europe until the 17th century, 150 years or more after percents appeared in business. They made working with percents much easier. Write each of these percents as a decimal.

   (a) 25% = ________ (b) 12% = ________ (c) 3% = ________

   (d) 86% = ________ (e) 33% = ________ (f) 5% = ________

6. When money is measured in dollars and cents, *percents* become even easier to use. 1% of a dollar is a cent (a penny), so a percentage just tells you how many pennies for each dollar. What are each of these amounts?

   (a) 1% of $13.00 = ________ (b) 5% of $12.00 = ________

   (c) 40% of $32.00 = ________ (d) 18% of $2.50 = ________

7. An easy way to calculate some percent of a number is to write the percent as a decimal and translate "of" as "times." For instance,

   12% of $15.00 is 0.12 x $15.00 = $1.80.

   Rewrite these and then use your calculator to compute the answers.

   (a) 20% of $43.60 ______________________________

   (b) 42% of 7,956 ______________________________

   (c) 7% of $245.00 ______________________________

   (d) 61% of 1159 ______________________________

   (e) 150% of 360 ______________________________

8. Some sportscasters will describe an athlete's effort as "110%."

   Why doesn't this make literal sense? ________________________

   ____

   What point is the sportscaster trying to make? ________________

   ____

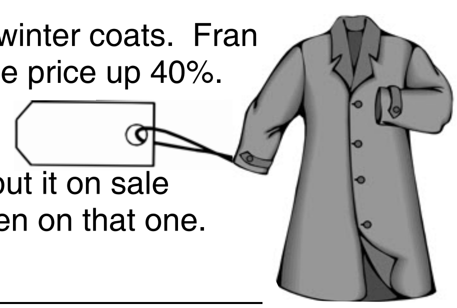
9. In the fall, Fran's Fashions bought a new line of winter coats. Fran paid $80 wholesale for each coat and marked the price up 40%.

   (a) Put her selling price in the price tag.

   In the spring, Fran had one coat left over. She put it on sale for 40% off, figuring she would at least break even on that one.

   (b) Would she break even? _____ Explain. __________________

   ____



10. A restaurant includes a 15% tip in their bills for big parties, and then charges 7% sales tax on the total. A restaurant across the street charges the 7% sales tax first and then a 15% tip on the total. Does it make any difference? Explain.

   ____

   ____

Name:________________________________ Date:_____________

Activity 4-1

What Are Negative Numbers?

Did you know that negative numbers were not commonly used until a few hundred years ago? It's true. Columbus discovered America more than two centuries before negatives were truly accepted as numbers.

1. Negative numbers arose when people began to solve problems like this:

   "Pedro is 7 years old and his sister Mia is 2.
   When will he be exactly twice as old as his sister?"

   This translates into solving the equation $7 + \square = 2 \times (2 + \square)$.

   (a) Solve this equation. That is, put the same number in both boxes so that the two sides are equal.

   (b) Suppose, instead, that Pedro is 13 years old and Mia is 9. Write the same kind of equation for this. Then solve it. ____________________________

   (c) What does your solution mean about Pedro's and Mia's ages?__________

   ____

2. The Ancient Greeks ignored negative numbers completely. For example,

   Diophantus, who wrote a book about solving equations in the 3rd century, looked at the equation $4x + 20 = 4$ and said, "This is absurd because 4 is smaller than 20." What did he mean?

   ____

   ____

3. Name three uses of negative numbers in the real world. For each one, give an example of what a negative number means.

   (1) ____

   (2) ____

   (3) ____

4. (a) Sometimes people call a negative number the "opposite" of a positive number. What does that mean? Give an example.

   ____

   (b) Is 5 the opposite of –5? _____ Is 5 the opposite of –3? _____ Explain.

   ____

What Are Negative Numbers?

5. Draw a diagram to illustrate how positive and negative numbers are "opposites" of each other. Label it to make your meaning clear.

6. In his 1707 book, *Universal Arithmetick*, Sir Isaac Newton said,

"Quantities are either Affirmative, or greater than nothing, or Negative, or less than nothing."

(a) What word would we use instead of "Affirmative" today? _____________

(b) What did he mean by "less than nothing"? ________________________

7. (a) Which is less (smaller), 4 or 6? ____

(b) Which is less, –4 or 4? ____

(c) Which is less, –4 or 0? ____

(d) Which is less, –4 or –6? ____

(e) What does it mean to say that one number is less than another?

____

(f) The symbol for "less than" is <. The smaller number is on the pointed side. Arrange these numbers in order from smallest to largest, left to right:

–7, 5, 3, –3, 0, 8, –4, –10, 10, 2, –1

___ < ___ < ___ < ___ < ___ < ___ < ___ < ___ < ___ < ___ < ___

8. In 1770, Leonhard Euler wrote,

"Since negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing. Thus, when a man has nothing of his own, and owes 50 crowns, it is certain that he has 50 crowns less than nothing; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point nothing, though really richer than before."

A *crown* was a denomination of money in Euler's time. Calculate these sums. Be prepared to explain them in terms of money.

(a) –35 + 35 = _____ (b) –12 + 20 = _____ (c) –30 + 10 = _____

(d) 14 + (–9) = _____ (e) 22 + (–40) = _____ (f) –25 + (–50) = _____

Name:________________________________ Date:_____________

Activity 4-2

Adding & Subtracting Negative Numbers

Some of the first people to think of negatives as numbers lived in India. Brahmagupta, an Indian scholar of the 7th century, thought of positive numbers as possessions and negative numbers as debts. But he also saw a problem: If negatives are included as numbers, then we must be able to combine them with each other and with positive numbers using +, –, ×, and ÷. So Brahmagupta *made up* the arithmetic rules for negatives.

1. If you were helping Brahmagupta, how would you tell him to answer each of these questions?

   (a) (–7) + (–3) = _____ (b) 7 + (–3) = _____ (c) (–7) + 3 = _____

   (d) (–6.5) + (–2) = _____ (e) 6.5 + (–2) = _____ (f) (–6.5) + 2 = _____

2. Explain your answers to parts *d*, *e*, and *f* of #1 as possessions or debts.

   (d) ____

   ____

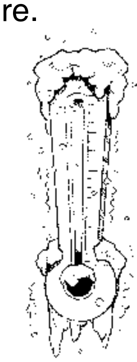
   (e) ____

   ____

   (f) ____

   ____

3. Explain your answers to parts a, b, and *c* of #1 in terms of temperature.



   (a) ____

   ____

   (b) ____

   ____

   (c) ____

   ____

4. Write in words a rule for adding numbers when at least one is negative.

   ____

   ____

   ____

5. Did you have trouble writing your general rule in words? Brahmagupta faced that problem, too. The Indians did all their mathematics in words. But a few symbols can make your ideas much clearer, like this: Suppose a and b are (positive) whole numbers and a is greater than b. Fill in the blanks below. (In Brahmagupta's time, subtraction only worked if a smaller positive number was taken from a larger one.)

   $(-a) + (-b) =$ _________ $a + (-b) =$ _________ $(-a) + b =$ _________

6. Brahmagupta knew that subtraction should undo addition. For instance,

   $$7 - 4 = 3 \text{ because } 3 + 4 = 7.$$

   How would he have answered each of these questions?

   (a) $(-5) - (-2) =$ ____ Why? ______________________________

   (b) $5 - (-2) =$ ____ Why? ______________________________

   (c) $(-5) - 2 =$ ____ Why? ______________________________

7. Explain your answers to #6 in terms of possessions and debts.

   (a) ____

   ____

   (b) ____

   ____

   (c) ____

   ____

8. Suppose a and b are whole numbers and a is greater than b. Fill in the blanks below to make Brahmagupta's subtraction rules. (Remember: Subtraction of whole numbers only worked if a smaller number was taken from a larger one.)

   $(-a) - (-b) =$ _________ $a - (-b) =$ _________ $(-a) - b =$ _________

9. Write an expression using either addition or subtraction of negative numbers to describe each of these situations.

   (a) Fran borrows \$600 for insulation and uses a \$200 tax credit to help pay for it. What does she still owe? __________________

   (b) At noon the temperature in Alaska was 5° below zero, 12° higher than it was at dawn. What was it at dawn? __________________

   (c) Bee Co. is worth \$8 million. It buys Aye Co., which is \$2.5 million in debt. What is Bee Co. worth now? ___________________

Name:________________________________ Date:_____________

Activity 4-3

Multiplying & Dividing Negative Numbers

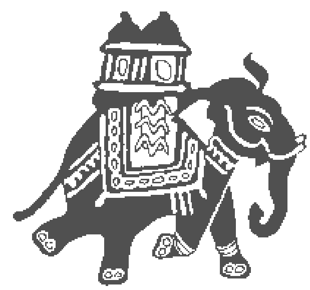
Let's use Brahmagupta's idea of possessions and debts to see how he might have handled multiplication.

1. (a) If I owe 8 people $5 each, how much money do I owe altogether? ______

   How did you get your answer? _________________

   (b) How much is my total debt if I owe 30 people $5 each? ______ Is the answer the same if I owe 5 people $30 each, instead? ______ Write this situation using negative numbers. ________________________________

2. Find each product:



   (a) 10 × (–5) = _____ (b) (–5) × 10 = _____

   (c) 8 × (–7) = ______ (d) (–12) × 6 = _____

   (e) 4.3 × (–2) = _____ (f) (–0.5) × 2.2 = _____

   (g) 0.25 × (–3.6) = _____ (h) (–7) × 1.01 = _____

3. When a positive number and a negative number are multiplied together, the answer is _________________ (*positive*, *negative* — choose one).

What about the product of two negatives? Should it be positive or negative? This confuses many people even today. But Brahmagupta knew the right answer 1400 years ago. He said then that the product of two negatives *must be positive*. To see why, think about the pattern in the next two questions.

4. (a) 5 × (–6) = ____; 4 × (–6) = ____; 3 × (–6) = ____; 2 × (–6) = ____; ...

   As the first number in these products gets smaller, what's happening to the product? ____

   (b) Continue this pattern for four more steps:

   __ × (–6) = ___; __ × (–6) = ___; ___ × (–6) = ___; ___ × (–6) = ___

5. Fill in the multipliers of –3 from 4 to –4, and then fill in the products.

   4 × (–3) = –12 ; ____ × (–3) = ____; ____ × (–3) = ____ ;

   ____ × (–3) = ____; 0 × (–3) = ____; ____ × (–3) = ____;

   ____ × (–3) = ____; ____ × (–3) = ____; (–4) × (–3) = ____

One important property of multiplication is called *cancellation*. For instance, if 3 x n = 3 x 5, we ought to be able to say that n = 5. That is, we ought to be able to "cancel" the common factor 3. Use this idea to answer the next question.

6. Terry and Jake took a math quiz. They both answered (–7) x 4 = ? correctly, but answered (–7) x (–4) = ? differently. Terry said 28 and Jake said –28. Use the idea of cancellation to show them which one is correct.

____

____

____

____

7. Complete each sentence with either *positive* or *negative*.

   (a) The product of two positive numbers is _______________.

   (b) The product of a positive and a negative number is ________________.

   (c) The product of two negative numbers is ________________.

   (d) The product of three positive numbers is ________________.

   (e) The product of three negative numbers is ________________.

   (f) The product of one positive and two negative numbers is _______________.

   (g) The product of one negative and two positive numbers is_______________.

8. As you probably know, division "undoes" multiplication. For example, 6 ÷ 3 = 2 because 2 x 3 = 6. This connection told Brahmagupta how division of signed numbers should work. See if you can answer these division questions as he would have. Then check by multiplying.

   (a) (–20) ÷ 5 = _____. Check: ____________________

   (b) (–18) ÷ (–6) = _____. Check: ____________________

   (c) 24 ÷ (–4) = _____. Check: ____________________

9. Complete each sentence with either *positive* or *negative*.

   (a) A positive number divided by a negative number is __________________.

   (b) A negative number divided by a positive number is __________________.

   (c) A negative number divided by a negative number is __________________.

Name:_________________________________ Date:_____________

Activity 4-4

Fitting In

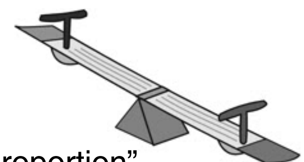
The Indian rules for signed numbers were learned by the Arabs, who brought them to Europe in the late Middle Ages. But the Arabs and early Europeans *always* thought of numbers as positive or zero. They applied these rules *only* to subtractions that ended up with positive numbers.

By the 17th century, the usefulness of negative numbers was becoming too obvious to ignore. They had to be included as part of the number system. But how and where? Here are two examples of how mathematicians of the 1600s struggled to understand negative numbers.

1. A *ratio* compares two numbers. For instance, 2 : 3 and 4 : 6 are ratios. In fact, they are the same ratio; they are "in proportion." This is traditionally written as 2 : 3 :: 4 : 6, and read "2 is to 3 as 4 is to 6."

   (a) Write three other ratios in proportion to 2 : 3. _______, _______, _______

   (b) Complete each statement.

   5 : 8 :: ___ : 24   1 : ___ :: 3 : 12   2 : 5 :: 8 : ___

   (c) These days, we usually write ratios as fractions and "in proportion" as equality of fractions. Write each statement of (b) in fraction form.

   _______________, _______________, _______________

   (d) If 5 : 7 :: *k* : 50, is *k greater* or *less* than 50? ______________

2. (a) Which is larger, 1 or –1? ___ How do you know? ____________________

   ____

   (b) True or false: 1 ÷ (–1) = (–1) ÷ 1. _________ How do you know? _______

   ____

   (c) Rewrite the equation in (b) as a proportion: _______________________
   17th century French scholar Antoine Arnauld argued that this proportion is nonsense because it says that a larger number is to a smaller as a smaller number is to a larger.

   (d) Use your answers from this sheet to explain Arnauld's argument.

   ____

   ____

   (e) How would you reply to Arnauld? ______________________________

   ____

3. Are negative numbers really less than 0? In 1655, British mathematician John Wallis wrote that, if negative numbers are less than 0, they must be greater than infinity! His paradox depends on the following ideas.

   (a) Write the value of each fraction beneath it:

   $\frac{3}{1}$ $\frac{3}{0.1}$ $\frac{3}{0.01}$ $\frac{3}{0.001}$ $\frac{3}{0.0001}$ $\frac{3}{0.00001}$ $\frac{3}{0.000001}$

   ___ ___ ___ ___ ___ ___ ___

   (b) What is happening to the value of these fractions as the denominators get closer to 0?____

   (c) What would John Wallis have said about the size of $\frac{3}{0}$? ______________

   Why? ____

   (d) Is $-1 < 0$?________ How did Wallis relate this to the size of $\frac{3}{-1}$?

   ____

   (e) Is $\frac{3}{-1} = -3$? ______ Explain. ______________________________

   (f) Summarize Wallis's argument that –3 is greater than infinity.

   ____

   ____

   ____

   (g) Can this argument be used for any negative number, or is –3 special in some way? Explain.____

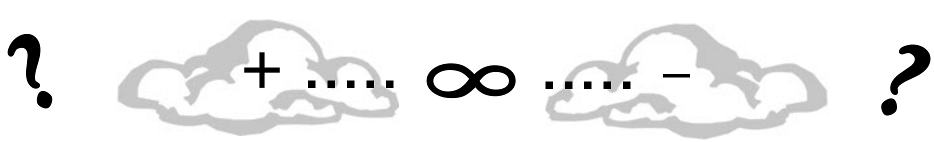
   ____

4. How would you resolve Wallis's paradox that negative numbers are both less than 0 and greater than infinity?

   ____

   ____

   ____



Name:______________________________ Date:______________

Activity 4-5

Powers and Roots

One reason why people of Shakespeare's time resisted negative numbers was related to finding powers and roots. To understand their suspicions, we begin with a quick review of powers and roots.

1. A ***power*** of a number is a repeated product of the number. Its ***exponent*** counts how many copies of the number are multiplied together. For example,

   $$5^3 = 5 \cdot 5 \cdot 5 = 125$$

   Write each power as a repeated product and calculate it.

   (a) 3^4 = __________ = ____　　(b) $(-7)^3$ = __________ = ____

   (c) $(-6)^2$ = ________ = ____　　(d) $(-2)^6$ = ______________ = ____

   (e) $(-2)^5$ = ____________ = ____　　(f) $(-5)^4$ = ___________ = ____

2. Answer each question *positive* or *negative*.

   (a) $(-10)^4$ is ____________.　　(b) $(-10)^5$ is ____________.

   (c) $(-24)^{15}$ is ____________.　　(d) $(-37)^{42}$ is ____________.

   (e) Any even power of a negative number is ____________.

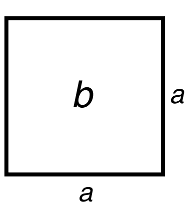
   (f) Any odd power of a negative number is ____________.

3. Justify your answers to 2(e) and 2(f).________________________________

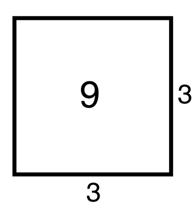
   ____

   ____

   ____



If $a^2 = b$, then b is called the ***square*** of a, and a is called a ***square root*** of b. For instance, 9 is the square of 3, and 3 is a square root of 9. We write $9 = 3^2$, and $\sqrt{9} = 3$.



Nowadays we would say that 9 has *two* square roots, 3 and –3, because $(-3)^2 = 9$, too. But 16th and 17th century mathematicians did not accept negative answers. They thought of solving $x^2 = 9$ as finding the side length of a square with area 9, so negative answers made no sense to them. They called an answer such as –3 a *false root* (or *false solution*).

4. Find both square roots in each case.

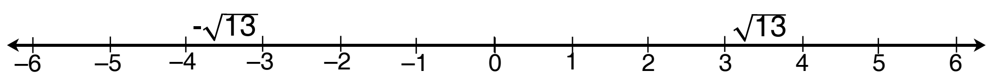
   (a) If $x^2 = 4$, $x =$ ___ or ___
   (b) If $x^2 = 25$, $x =$ ___ or ___
   (c) If $x^2 = 36$, $x =$ ___ or ___
   (d) If $x^2 = 0.25$, $x =$ ____ or ____
   (e) If $x^2 = 0.09$, $x =$ ____ or ____
   (f) If $x^2 = 0.0001$, $x =$ _____ or _____

The two square roots of a number often are written with a ± sign. For instance, $\pm\sqrt{9}$ stands for +3 and –3. Once 17th and 18th century mathematicians began to place negative numbers on a number line to the left of 0, it was easy to see that the two square roots of a number should always be the same distance from 0, on opposite sides.

5. (a) A number that is the square of an integer is called a ***perfect square***. The first four perfect squares are 1, 4, 9, 16. List the next eight.

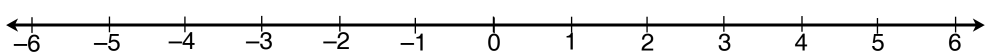
   _____, _____, _____, _____, _____, _____, _____, _____

   (b) To roughly estimate the square roots of a number, locate it between perfect squares, like this: 13 is between 9 and 16, so



   Place each of these ten square roots on the number line below.

   $\pm\sqrt{30}$ $\pm\sqrt{19}$ $\pm\sqrt{3}$ $\pm\sqrt{11}$ $\pm\sqrt{7}$



6. $(\sqrt{6})^2 =$ ___ $(-\sqrt{6})^2 =$ ___ $(\sqrt{-6})^2 =$ ___ $(\sqrt{a})^2 =$ ___

7. One property of negative numbers troubled the mathematicians of the 1500s and 1600s: Square roots of negatives could not be either positive or negative! Explain why not.

   ____

   ____

8. The equation $x^4 = 1$ has 4 solutions: 1, –1, $\sqrt{-1}$, and $-\sqrt{-1}$.

   (a) Show how $\sqrt{-1}$ and $-\sqrt{-1}$ are solutions. ____________________________

   ____

   (b) In the 1700s, French mathematician René Descartes called square roots of negative numbers ***imaginary***, and they are still called that today.

   The two imaginary solutions for $x^4 = 81$ are _______ and _______.