

Research Statement

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I am an analyst working at the interface between analysis and probability theory. I aim to answer questions, which are motivated by probability, about objects in a framework extending beyond the reaches of probability. While group theory provides the setting for my work, my tools and framework are pulled from classical, complex and functional analysis. At hand, I have a few ongoing projects and many more in mind, some of which have concrete components and are well-fitted to undergraduate and graduate collaborative research. I am extremely motivated to learn new things, especially in analysis, partial differential equations and mathematical physics, and ideally, within a highly collaborative environment. While I give a detailed picture of my research in the sections to follow, the remainder of this introduction stands to describe the main theme of my doctoral work and is aimed at specialists and non-specialists alike.

In the most succinct terms, my current research centers on the question:

What happens when positivity is dropped?

As a simple illustration, consider a function $\phi : \mathbb{Z}^d \rightarrow \mathbb{C}$ which is, say, finitely supported and define its convolution powers iteratively by

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y)$$

where $\phi^{(1)} = \phi$. The study of convolution powers is central to probability theory for, in the case that ϕ is a probability distribution, i.e., ϕ is non-negative with unit mass, ϕ drives a random walk on the lattice whose n th-step transition kernels are given by $k_n(x, y) = \phi^{(n)}(y-x)$. Moreover, when this random walk is aperiodic and irreducible, the behavior of $\phi^{(n)}$ is well approximated by a single and appropriately scaled Gaussian density; this is the local (central) limit theorem. When such functions are allowed to be complex-valued, their convolution powers are seen to exhibit rich and disparate behavior, much of which never appears in the probabilistic setting. For instance, Figure 1 depicts the convolution powers of a complex-valued function ϕ supported on only six points in \mathbb{Z}^2 (details can be found in [RSC2]). Observe

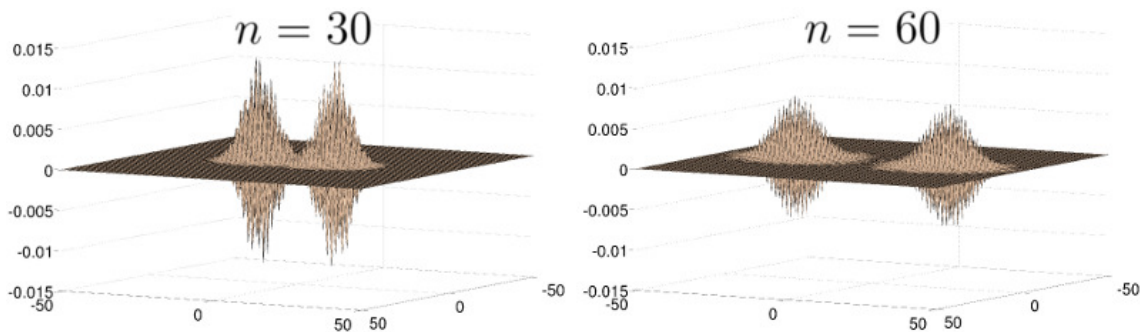


Figure 1: $\text{Re}(\phi^{(n)})$ for $n = 30$ and $n = 60$

that the convolution powers exhibit two distinct packets which drift apart as n increases; this behavior is impossible for the convolution powers of non-negative functions, c.f. [RSC2, Theorem 7.5]. It is this type of phenomenon, which lies just outside the view of classical probability, that I seek to understand.

The remainder of this statement is broken into two sections. The first describes my thesis work on convolution powers on \mathbb{Z} and \mathbb{Z}^d . This work can be found in [RSC1] and [RSC2] respectively. The second section describes my thesis work on heat kernels of “higher order” partial differential operators, especially their global space-time estimates. This work will be presented in two forthcoming articles [RSC3] and [RSC4]. All work is joint with my advisor, Professor Laurent Saloff-Coste.

1 Convolution powers

The study of convolution powers of complex-valued and finitely supported functions on the lattice has a long history dating back to a problem in statistical data smoothing known as De Forest’s problem. This problem, posed by Erastus L. De Forest (1834-1888) and subsequently investigated by I.J. Shoenberg and T.N.E. Greville in the 1950’s, was concerned with determining the behavior of a class of data smoothers given by the convolution powers of normalized symmetric and real-valued functions on \mathbb{Z} . Beyond data smoothing, iterative convolution was found to be extremely useful for producing numerical solutions to partial differential equations. This study saw explosive investigation in the 1960’s in parallel with advancements in scientific computing. The article [DSC] contains a more extensive discussion on this application-driven history and references to the literature (see also [RSC2]).

Continuing from the introduction, consider a finitely supported function $\phi : \mathbb{Z}^d \rightarrow \mathbb{C}$ and introduce the convolution powers $\phi^{(n)}$ for $n \in \{1, 2, \dots\} =: \mathbb{N}_+$. Much of my work has thus far concentrated on answering the following four basic questions:

1. What can be said about the decay of $\|\phi^{(n)}\|_\infty = \sup |\phi^{(n)}|$ as $n \rightarrow \infty$?
2. Is there a simple pointwise description of $\phi^{(n)}(x)$ analogous to the local (central) limit theorem that can be made for large n ?
3. Are global space-time pointwise estimates obtainable for $|\phi^{(n)}|$?
4. Under what conditions is ϕ stable in the sense that

$$\sup_{n \in \mathbb{N}_+} \|\phi^{(n)}\|_1 = \sup_{n \in \mathbb{N}_+} \sum_{x \in \mathbb{Z}^d} |\phi^{(n)}(x)| < \infty?$$

For brevity, here I’ll discuss only the first question. A thorough discussion of the others can be found in the introductory section of [RSC2]; it should however be noted that Question 4 is particularly interesting and its affirmative answer is both a necessary and sufficient condition for a finite difference scheme to a parabolic partial differential equation to converge to a classical solution (in an appropriate topology).

When ϕ is a probability distribution driving an aperiodic and irreducible random walk on \mathbb{Z}^d , it is well-known that there are positive constants C and C' for which

$$Cn^{-d/2} \leq \sup_x \phi^{(n)}(x) \leq C'n^{-d/2}$$

for all $n \in \mathbb{N}_+$. This inequality turns out to be quite useful in probability because, in particular, it settles the question of recurrence and transience for symmetric and finite range random walks on \mathbb{Z}^d . In the case that ϕ is complex-valued, Question 1 can be stated thus: When is there a monotonically decreasing function $f : \mathbb{N}_+ \rightarrow (0, \infty)$ and positive constants C, C' and A for which

$$Cf(n) \leq A^n \sup_x |\phi^{(n)}(x)| \leq C'f(n)$$

for all $n \in \mathbb{N}_+$? In the case that $d = 1$, the theorem below helps to give a complete answer.

Theorem 1 (Theorem 1.1 of [RSC1]). *Let $\phi : \mathbb{Z} \rightarrow \mathbb{C}$ have finite support consisting of more than one point. There exists an integer $m \geq 2$ and positive constants C, C' and A for which*

$$Cn^{-1/m} \leq A^n \sup_x |\phi^{(n)}(x)| \leq C'n^{-1/m}$$

for $n \in \mathbb{N}_+$.

When $d > 1$, the situation is more complicated and the question above remains open. A partial answer is provided by Theorem 1.1 of [RSC2] which pertains to a large but limited class of normalized functions $\phi : \mathbb{Z}^d \rightarrow \mathbb{C}$ whose Fourier transform behaves “nicely” near local extrema. To each ϕ in this class, the theorem gives a positive rational number ω_ϕ and positive constants C and C' for which

$$Cn^{-\omega_\phi} \leq \sup_x |\phi^{(n)}(x)| \leq C'n^{-\omega_\phi}$$

for all $n \in \mathbb{N}_+$.

1.1 Generalization and future directions

The paper [RSC1] gives a complete theory for convolution powers on \mathbb{Z} . In this one-dimensional theory, a number of smooth functions called *attractors* appear as the limiting objects in local limit theorems in the way that the Gaussian appears in the probabilistic setting. This class of attractors includes the Airy function and the heat kernel evaluated at imaginary times and both arise by taking scaled limits of oscillatory integrals. Not considered in [RSC2] is a large class of functions in \mathbb{Z}^d whose analysis will require oscillatory integral methods. A study considering this class remains to be done and, in conjunction with the results of [RSC2], stands to provide a much more (although not fully complete) theory on \mathbb{Z}^d . At this stage, even looking at specific examples is quite interesting and much is open.

One interesting generalization is to consider the limiting behavior of the convolution powers $\mu^{(n)}$ of a complex-valued Borel measure μ on \mathbb{R}^d ; in this context the \mathbb{Z}^d theory appears by viewing $\phi : \mathbb{Z}^d \rightarrow \mathbb{C}$ as a purely atomic measure on \mathbb{R}^d . This work has already seen some investigation in which local limit theorems and strong convergence are replaced by generalized central limit theorems and weak convergence of measures [Her].

Another open direction arises by replacing \mathbb{Z}^d by a finitely generated subgroup G_0 of a homogeneous group G . Homogeneous groups are connected and simply connected nilpotent Lie groups that admit “scaling” limits; the simplest non-abelian example is the Heisenberg group. As the heat kernel arises as the attractor to the local limit theorems in the probabilistic setting, I expect local limit theorems for the convolution powers of $\phi : G_0 \rightarrow \mathbb{C}$ to involve convolution kernels corresponding to homogeneous operators. Due to the limited scope of Fourier transform techniques, such generalizations will be complicated, however, much recent work has been done to establish local limit theorems for probability distributions on homogeneous groups and this work will undoubtedly be useful to complex-valued generalizations (see [Br]). Global space-time estimates will be interesting as well.

2 Heat kernel estimates: Developing a unifying theory

In the study of convolution powers of complex valued functions on \mathbb{Z}^d , a large class of smooth functions is seen to arise naturally as a class of attractors in local limit theorems. The canonical example (appearing in the probabilistic setting) is the heat kernel

$$K_{(-\Delta)}^t(x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^2} e^{-ix \cdot \xi} d\xi$$

which also arises as a fundamental solution to the heat equation, $\partial_t + (-\Delta) = 0$. However, just as natural to local limit theorems is the appearance of the so-called bi-Laplacian heat kernel

$$K_{(-\Delta)^2}^t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^4} e^{-ix \cdot \xi} d\xi.$$

This function analogously arises as a fundamental solution to the higher-order “heat” equation $\partial_t + (-\Delta)^2 = 0$. Taking the above as motivation, the general class of attractors studied in [RSC2] are fundamental solutions to a class of partial differential equations on $[0, \infty) \times \mathbb{R}^d$ of interest in their own right.

To introduce this class, we take Λ to be a constant coefficient partial differential operator on \mathbb{R}^d and let P be its symbol, i.e., P is the Fourier multiplier of Λ and is necessarily a multivariate polynomial. For any real $d \times d$ matrix E , we define a one-parameter group $\{\delta_t^E\}_{t>0}$ which acts on functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ by $\delta_t^E(f)(x) = f(t^E x)$; here, t^E is the matrix exponential $\exp((\log t)E)$. The operator Λ is said to be *homogeneous with respect to* $\{\delta_t^E\}$ if

$$t\Lambda = \delta_t^{-E} \circ \Lambda \circ \delta_t^E$$

for all $t > 0$ and in this case we write $E \in \text{Exp}(\Lambda)$. We say that Λ is *positive-homogeneous* if the real part of its symbol, $R = \text{Re } P : \mathbb{R}^d \rightarrow \mathbb{R}$, is strictly positive away from the origin and if $\text{Exp}(\Lambda)$ contains a diagonalizable matrix. These two properties together guarantee that $\text{tr } E = \text{tr } E'$ for all $E, E' \in \text{Exp}(\Lambda)$ and this allows us to define the positive number

$$\omega_\Lambda = \text{tr } E$$

for any $E \in \text{Exp}(\Lambda)$ called the *homogeneous degree* of Λ .

Given a positive-homogeneous operator Λ with symbol P , the function $K_\Lambda^{(\cdot)} : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C}$ defined by

$$K_\Lambda^t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-tP(\xi)} e^{-ix \cdot \xi} d\xi$$

is a fundamental solution to the “heat” equation $\partial_t + \Lambda = 0$ in the sense of [E]. Equivalently, K_Λ is the integral kernel of semigroup $\{e^{-t\Lambda}\}_{t>0}$ with infinitesimal generator $-\Lambda$, i.e.,

$$(e^{-t\Lambda} f)(x) = \int_{\mathbb{R}^d} K_\Lambda(t, x - y) f(y) dy$$

for f in a suitable class of functions. Using complex-analytic techniques, one can show that

$$|K_\Lambda(t, x - y)| \leq \frac{C}{t^{\omega_\Lambda}} \exp\left(-tMR^\# \left(\frac{x - y}{t}\right)\right) \quad (1)$$

for $x, y \in \mathbb{R}^d$ and $t > 0$ where C and M are positive constants and $R^\#$ is the Legendre-Fenchel transform of $R = \text{Re } P$ and is defined by

$$R^\#(x) = \sup_{\xi \in \mathbb{R}^d} \{x \cdot \xi - R(\xi)\}$$

for $x \in \mathbb{R}^d$ [RSC2]. In the elliptic (isotropic) setting, the estimate (1) is well-known: Given a positive integer m , $(-\Delta)^m$ is easily seen to be a positive-homogeneous operator with homogeneous degree $d/2m$ and symbol $|\xi|^{2m}$. Here we have $R^\#(x) = C_m|x|^{2m/(2m-1)}$ for $C_m > 0$ and so (1) takes the form

$$|K_{(-\Delta)^m}(t, x - y)| \leq \frac{C}{t^{d/2m}} \exp(-M|x|^{2m/(2m-1)}/t^{1/(2m-1)})$$

for $t > 0$ and $x, y \in \mathbb{R}^d$ —this estimate is ubiquitous in the theory of higher-order elliptic systems [E, D1]. Beyond the elliptic theory, the presence of the full d -dimensional Legendre transform in the global space-time estimate (1) is remarkable because it sharply captures the generally anisotropic nature of K_Λ . For instance, consider the operator $\Lambda = \partial_1^4 - i\partial_1^2\partial_2 - \partial_2^2$ on \mathbb{R}^2 ; this is easily seen to be a positive-homogeneous operator with homogeneous order $3/4$ and symbol $P(\xi_1, \xi_2) = \xi_1^4 + \xi_1^2\xi_2 + \xi_2^2$. In this case, the estimate (1) shows that $|K_\Lambda|$ decays on the order of $\exp(-M|\cdot|^{4/3})$ along the x_1 -coordinate axis and on the order of $\exp(-M|\cdot|^2)$ along the x_2 -coordinate axis for fixed $t > 0$.

From the perspective of partial differential equations, it is quite natural to consider the heat equation $\partial_t + H = 0$ where H is a variable coefficient partial differential operator. In the case that H is comparable to

$(-\Delta)^m$ for some m , it is usually given the moniker elliptic; such operators are commonplace in mathematical physics arising everywhere from fluid dynamics to quantum mechanics [D1]. Analogously, we consider a variable coefficient partial differential operator H that is comparable to a positive-homogeneous operator Λ (in the sense of symbols or in the sense that it satisfies a Gårding inequality). Realizing H as a densely defined, closed, sectorial operator on $L^2(\mathbb{R}^d)$, we can then study the strongly continuous semigroup $\{e^{-tH}\}_{t>0}$ with infinitesimal generator $-H$ and here, some fundamental questions arise:

1. Under what conditions does $\{e^{-tH}\}_{t>0}$ have an integral representation of the form

$$(e^{-tH}f)(x) = \int_{\mathbb{R}^d} K(t, x, y)f(y) dy$$

with heat kernel $K : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$?

2. If such an integral representation exists, how regular is K ?
3. If such an integral representation exists, does K satisfy an estimate analogous to (1)? Specifically, can we find positive constants C and M for which

$$|K(t, x, y)| \leq \frac{C}{t^{\omega_\Lambda}} \exp\left(-tMR^\# \left(\frac{x-y}{t}\right)\right) \quad (2)$$

for $0 < t < T \leq \infty$ and $x, y \in \mathbb{R}^d$?

The semigroup $\{e^{-tH}\}$ and its infinitesimal generator $-H$ enjoy exceptional properties when Question 3 has an affirmative answer. For instance, the validity of (2) guarantees that the semigroup extends to a strongly continuous semigroup $\{e^{-tH_p}\}$ on $L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$ and, what's more, the respective infinitesimal generators $-H_p$ have spectra independent of p [D2]. Further, the estimate (2) is key to establishing the boundedness of the Riesz transform, it is connected to the resolution of Kato's square root problem and it provides the appropriate starting point for uniqueness classes of solutions to $\partial_t + H = 0$ [AHMT, Ou].

My own research centers on the above formulation of theory in terms of the full d -dimensional Legendre-Fenchel transform and the above three questions (the appearance of the 1-dimensional Legendre-Fenchel transform was previously recognized in [BaD, BlKu]). The results of my thesis pertain to two large classes of variable coefficient partial differential operators and will be presented in two forthcoming articles [RSC3] and [RSC4]. The article [RSC3] resolves the above three questions under the assumption that the coefficients of H are Hölder continuous and the proofs follow Levi's parametrix method adapted to parabolic systems by A. Friedman and S. D. Eidel'man. The results therein are consistent with those found in [EIK] pertaining to $2\vec{b}$ -parabolic systems. The article [RSC4] considers operators whose coefficients are only required to be bounded and measurable and, as necessary, the validity of the results are subject to a dimension-order restriction which, in the above formulation, comes neatly packed as " $\omega_\Lambda < 1$ ". The results of [RSC4] extend the results of E. B. Davies (see [D2]) beyond the elliptic theory and the method of proof is purely functional analytic and follows [D2] and [BaD].

2.1 Generalizations and future directions

The results of [RSC4] are stated in terms of an arbitrary Euclidean domain $\Omega \subseteq \mathbb{R}^d$ (and not simply \mathbb{R}^d). In this setting, the domain of the operators H are those corresponding to Dirichlet boundary conditions and, in general, the properties of the semigroup $\{e^{-tH}\}$ differ from those in which H has domain consistent with Neumann or mixed boundary conditions. When H is a second order elliptic operator with real coefficients (when a probabilistic interpretation exists), there is a theory of semigroup domination which helps to sort out the relationship between these semigroups [Ou]. However, in the general case of higher order operators, the relationship is less straightforward and much remains to be explored.

The most exciting open direction I see is to investigate heat kernel estimates for positive Rockland operators on a homogeneous group G ; these can be seen to generalize symmetric positive-homogeneous

operators. Heat kernel estimates for positive Rockland operators have been investigated previously by several authors [DH, Heb, AER]. Given a positive Rockland operator Λ on homogeneous group G , the best known estimate for the heat kernel K of Λ , due to Auscher, ter Elst and Robinson, is of the form

$$|K(t, g, h)| \leq \frac{C}{t^{\omega_\Lambda}} \exp \left(-M \left(\frac{|h^{-1}g|^{2m}}{t} \right)^{1/(2m-1)} \right) \quad (3)$$

where ω_Λ is the homogeneous degree of Λ , $|\cdot|$ is a homogeneous norm on G (consistent with Λ) and $2m$ is the highest order derivative appearing in Λ . In the context of \mathbb{R}^d , given a symmetric and positive-homogeneous operator Λ with symbol P , the structure $G_D = (\mathbb{R}^d, \{\delta_t^D\})$ for $D = 2mE$ where $E \in \text{Exp}(\Lambda)$ is a homogeneous group on which Λ becomes a positive Rockland operator. On G_D , it is quickly verified that $|\cdot| = R(\cdot)^{1/2m}$ is a homogeneous norm (consistent with Λ) and so the above estimate is given in terms of $R(\cdot)^{1/(2m-1)}$ which is, in general, dominated by the Legendre-Fenchel transform of R and hence the estimate (3) is suboptimal. In light of this, I am interested to see if (3) can be sharpened and, if so, a new question arises: What is to replace Legendre-Fenchel transform on G ?

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