Math 338: Homework 5

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Saturday, March 16th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1. Let $\{s_n\}$ be a sequence of real numbers. For each $n \in \mathbb{N}_+$, define

$$x_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

In this way, we form a new sequence $\{x_n\}_{n\in\mathbb{N}}$ called the *Cesáro* means of $\{s_n\}$.

1. Prove the following: If $\{s_n\}$ converges to s, then $\{x_n\}$ converges to s.
2. Given an example of a sequence which doesn't converge but its Cesáro means converge. Justify your answer.

Exercise 2 (All things Cauchy)**.** We recall:

Definition A. A sequence $\{s_n\}$ of complex numbers is said to be **Cauchy** if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}_+$, such that

$$|s_m - s_n| < \epsilon \tag{1}$$

whenever $n, m \geq N$.

It is not hard to see that the Cauchy condition (1) is equivalent to the following condition: For all $\epsilon > 0$, there exits $N \in \mathbb{N}_+$, such that

$$|s_{n+k} - s_n| < \epsilon \tag{2}$$

whenever $n \geq N$ and $k \geq 0$. (You should, at least, convince yourself that these are equivalent).

1. Show directly that $\{\frac{n+2}{n}\}_{n\in\mathbb{N}_+}$ is a Cauchy sequence of real numbers, i.e., prove that it satisfies (1) (or (2)).
2. Consider the sequence $\{\sqrt{n}\}_{n\in\mathbb{N}_+}$.
   (a) Show that this sequence is not convergent and therefore not Cauchy.
   (b) Show that, for any $k \geq 0$,
   $$\lim_{n\to\infty}\left(\sqrt{n+k} - \sqrt{n}\right) = 0.$$
   (c) Pertaining to the sequence $\{\sqrt{n}\}$, how does the limit established in the previous item differ from the condition (2)?

Now, consider the following definition

Definition B. A sequence $\{s_n\}$ of complex number is said to be **Super-Duper-Cauchy** if

$$|s_{n+1} - s_n| \leq 2^{-n} \tag{3}$$

for all $n \in \mathbb{N}_+$.

3. Give an example of a Super-Duper-Cauchy sequence. Give an example of a Cauchy sequence which is not Super-Duper-Cauchy.
4. With the help of the triangle inequality, show that if a sequence $\{s_n\}$ satisfies (3), then
   $$|s_m - s_n| \leq \sum_{k=n}^{m-1} 2^{-k}$$
   whenever n, m are natural numbers such that $n < m$.

5. Use the estimate established in the previous item to prove the following proposition:

   Proposition C. *Let $\{s_n\}$ be a Super-Duper-Cauchy sequence. Then $\{s_n\}$ is Cauchy (and therefore convergent).*

   Hint: The proof of Theorem 3.26 in the textbook might be helpful (which is the same argument as I gave in class on Friday).

Exercise 3. Please do Exercise 6 of Chapter 3 in Rudin.

Exercise 4. Please do Exercise 7 of Chapter 3 in Rudin. Hint: Cauchy-Schwarz will help your analysis of partial sums.

Exercise 5. Please do Exercise 9 in Chapter 3 in Rudin.

Exercise 6. (Abel's Test) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Prove the following statements

1. If $\{a_n\}$ is a bounded sequence and the series $\sum_n b_n$ is absolutely convergent, then the series $\sum_n a_n b_n$ is absolutely convergent (and therefore converges). **Hint:** Use the Cauchy criterion, Theorem 3.22.

2. If $\{a_n\}$ is a non-increasing sequence of non-negative numbers and $\sum b_n$ converges, then $\sum_n a_n b_n$ converges. **Hint:** By assumption, $a_1 \geq a_2 \geq a_3 \geq \cdots 0$. Abbott's exercises 2.7.12 and 2.7.14 outline a method that works.

3. Comment on whether or not these results hold if $\{a_n\}$ or $\{b_n\}$ is taken to be complex.