Math 338: Homework 4

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, March 6th at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

Exercise 1 (The Cantor Set)**.** In this exercise, you verify some details of the presentation of the Cantor set in Rudin (and that was done in class). Let's first recall the construction of the Cantor set: We started with the interval $C_0 = [0,1]$ and removed its middle third $(1/3, 2/3)$ thus producing $C_1 = [0,1/3] \cup [2/3,1]$. We continued this process inductively by removing all middle third of (intervals of) previous iterates. Precisely, we start with $C_0 = [0,1]$ and, for each $n \geq 1$, define

$$C_n = C_{n-1} \setminus \left(\bigcup_{k=0}^{k^*(n)} I_{n,k} \right). \tag{1}$$

where $k^*(n) := 3^{n-1} - 1$ and, for each $0 \leq k \leq k^*(n)$,

$$I_{n,k} := \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right).$$

The Cantor set is

$$C = \bigcap_{n=1}^{\infty} C_n$$

and is necessarily compact in view of the intersection theorem. In class, I claimed that C contains no intervals. Rudin gives a little more detail (a hint at the proof, per say). In this exercise, you will verify all of the details rigorously. In particular, you will prove the following proposition.

Proposition A. *For any* $0 < \alpha < \beta < 1$,

$$(\alpha, \beta) \not\subseteq C.$$

Please do the following.

1. First, just to check that things make sense, verify that the definition (1) gives the right picture for C_1, C_2 and C_3 (i.e., that it works).
2. Now, fix $0 < \alpha < \beta < 1$. Prove that there exists an $n \in \mathbb{N}_+$ and an integer k with $0 \leq k \leq k^*(n)$ with $I_{n,k} \subseteq (\alpha, \beta)$. **Hint:** You can use Rudin's choice of m in the second displayed equation on Page 42. Challenge: Can you do better (i.e., will any m possibly smaller than Rudin's hint also work)?
3. Use the above fact to complete the proof of the proposition.

Solution 1. 1. Let m be such that

$$\frac{5}{3^m} < \beta - \alpha.$$

Let k be the smallest natural number for which

$$\alpha < \frac{k}{3^{m-1}};$$

such integers exists because $\mathbb{N}$ is unbounded (and, in fact, you should should think about why there must be only one to satisfy this condition – there are several ways to show it). By construction, observe that

$$\frac{k-1}{3^{m-1}} \leq \alpha < \frac{k}{3^{m-1}} < \frac{3k+1}{3^m}.$$

Also, observe that

$$\frac{3k+2}{3^m} = \frac{3(k-1)+5}{3^m} = \frac{k-1}{3^{m-1}} + \frac{5}{3^m} \leq \alpha + \frac{5}{3^m} < \beta.$$

With this, we also note that $3k+2 < 3^m\beta < 3^m$ so that

$$k < 3^{m-1} - \frac{2}{3} < 3^{m-1} - 1 = k^*(m).$$

Hence, for the m chosen above, there is a natural number $k \leq k^*(n)$ with $I_{m,k} \subseteq (\alpha, \beta)$.

2. $(\alpha, \beta) \subseteq I_{m,k} \subseteq C_m^c$, there must be $x \in (\alpha, \beta)$ which does not belong to C_m and hence cannot belong to C.

Exercise 2. (logic and negation) When doing real/mathematical analysis, it is very natural to deal with compound logical statements – it's simply natural to the subject. We will start seeing such statements very soon and I this exercise gives you some practice working with them. In what follows, A is a non-empty set of real numbers and $\{s_n\}$ is a sequence of real numbers. Negate the following statements.

1. Puppies are dogs.
2. For all $a \in A$, $a > 0$.
3. There exists a real number $a \in A$, such that $a > 0$.
4. There exits a real number s such that, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ for which

$$|s_n - s| < \epsilon$$

for all $n \geq N$.

Hint: It might be helpful to write them in "If P, then Q" form.

Exercise 3. Determine whether or not the following sequences converge. Justify your answers, i.e., prove your claim in each case (either using the definition directly or invoking a result from Rudin).

1. $\left\{\frac{1}{2n}\right\}_{n\in\mathbb{N}_+}$
2. $\{(-1)^n\}_{n\in\mathbb{N}}$
3. $\{2^{-n}\}_{n\in\mathbb{N}}$
4. $\left\{\frac{2n-1}{3n+1}\right\}_{n\in\mathbb{N}}$

In the third item, you may use the fact that $0 < 2^{-n} \leq 1/n$ for all $n \in \mathbb{N}$ (but you should also be able to prove this by induction). Hint: Four of your proofs, directly or indirectly, should make use of the Archimedean property.

Exercise 4. In what follows, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers. Please do the following:

1. Prove the statement: If $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
2. Prove the statement: If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and, for some real number s, $\lim_{n\to\infty} a_n = s = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = s$.
3. For the first statement, would it still be true if $\{a_n\}$ and $\{b_n\}$ were complex sequences? Would the second statement still be true if all sequences were complex? Explain your reasoning.
4. Can you weaken the hypotheses of the second statement? If so, please write down and prove the most general statement you can. If not, please demonstrate the necessity of hypotheses with an example(s).

Exercise 5. This exercise provides a constructive approach to the "$\limsup$" of Definition 3.16 and gives another (better, in my opinion) proof of (a) of Theorem 3.17 of Rudin in the case of a bounded sequence. Let $\{s_n\}$ be a bounded sequence of real numbers and form a new sequence $\{t_n\}$ by setting

$$t_n = \sup\{s_k : k \geq n\}$$

for each $n \in \mathbb{N}$.

1. Show that $\{t_n\}$ converges. **Hint:** First, if A and B are non-empty subsets of real numbers with $A \subseteq B$, then $\sup A \leq \sup B$ – can you prove this? Armed with this fact, you can show that $\{t_n\}$ is decreasing using the fact that, for each $n \in \mathbb{N}$, $\{s_k : k \geq n+1\} \subseteq \{s_k : k \geq n\}$.

2. In view of the previous item, let $t = \lim_{n\to\infty} t_n$. Prove the following: If s is a subsequential limit of $\{s_n\}$, i.e., there is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ which converges to s, then

$$s \leq t.$$

3. Construct a subsequence of $\{s_n\}$ that converges to t. **Hint:** Here is a start: For $k = 1$, set $\epsilon_1 = 1$, $m_1 = 1$ and choose s_{n_1} for which $t_{m_1} - 1 = t_{m_1} - \epsilon_1 < s_{n_1} \leq t_{m_1}$. For $k = 2$, set $m_2 = n_1 + 1$, $\epsilon_2 = 1/2$ and choose s_{n_2} for which $t_{m_2} - \frac{1}{2} = t_{m_2} - \epsilon_2 < s_{n_2} \leq t_{m_2}$. Continuing the algorithm, for every k, you will produce a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ and a sequence $\{t_{m_k}\}$ of $\{t_n\}$ for which $t_{m_k} - \frac{1}{k} < s_{n_k} \leq t_{m_k}$ for all $k \in \mathbb{N}$. All of these properties, of course, should be rigorously verified.

4. Let E be the set of subsequential limits of $\{s_n\}$. Prove that

$$\lim_{n\to\infty} t_n = \max E = \limsup_{n\to\infty} s_n.$$

Exercise 6. Please do Problem 16 in Chapter 3 of Rudin.