# Math 338: Homework 2

Please complete the following exercises below and write up your solutions consistent with the directions in the syllabus. Your solutions are due on Thursday, February 22nd at 10:00AM in the appropriate box outside my office door. If you get stuck on any part of the homework, please come and see me. More importantly, have fun!

**Exercise 1.** Let $A$ and $B$ be non-empty sets of real numbers, both of which are bounded above. Define

$$A+B=\{y\in\mathbb{R}\mid y=a+b \text{ for } a\in A \text{ and } b\in B\}.$$

1. Prove that
$$\sup(A+B)=\sup A+\sup B.$$
2. Is the above generally true in any ordered field $F$ that doesn't necessarily satisfy the least upper bound property? Answer by giving a counter example and/or proving the most general thing you can.

**Exercise 2** (Checking Some Details)**.** In Friday's class, I introduced various concepts in a metric space and I gave examples of these concepts for certain points and subsets of the metric space $\mathbb{R}$ (equipped with the "usual" metric $d(p,q)=|p-q|$). In this exercise, you check the details. Recall, for real numbers $a<b$,

$$(a,b)=\{x\in\mathbb{R}\mid a<x<b\}$$

and

$$[a,b]=\{x\in\mathbb{R}\mid a\leq x\leq b\}.$$

For a real number $p$ and a radius $r>0$, observe that

$$N_r(p)=(p-r,p+r).$$

If this equality isn't immediately clear, it's worth spending a minute thinking about. For the following, please rigorously justify (prove) all of your answers/assertions.

1. Determine the set of limit points of the sets (all taken as subsets of $\mathbb{R}$ equipped with its usual metric)
   (a) $(1,2)$
   (b) $[1,2]$
   (c) $[1,2]\cup\{100\}$
   (d) $\{1/n\mid n\in\mathbb{N}_+\}$.
   (e) $\mathbb{Q}$
2. Determine the set of isolated points of all subsets above.
3. Determine the set of interior points of all subsets above.
4. Which sets are open? Which sets are closed? Which sets are perfect?

**Exercise 3.** On $\mathbb{R}^n$ consider the following metrics:

1. The metric $d_1$ defined by
$$d_1(x,y)=\sum_{j=1}^{n}|y_j-x_j|$$
for $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ in $\mathbb{R}^n$.

2. The metric $d_2$ defined by
$$d_2(x,y) = \|x-y\| = \sqrt{\sum_{j=1}^{n}(y_j - x_j)^2}$$
for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$.

3. The metric $d_\infty$ defined by
$$d_\infty(x,y) = \max_{j=1,2,\ldots,n} |y_j - x_j|.$$
for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$. Note: The subscript $\infty$ appears for reasons that will come up if you take topics in analysis.

Observe that $d_2$ is the so-called Euclidean metric which we discussed in class on Friday.

1. For $n = 2$, please draw a picture of the neighborhoods with center $(0,0)$ and radius 1 yielded by the metrics $d_1$, $d_2$, and $d_\infty$.

2. Prove that $d_1$ and $d_\infty$ are indeed metrics. You can use the fact that the absolute value satisfies the triangle inequality if it is helpful.

3. Prove that
$$d_\infty(x,y) \leq d_2(x,y) \leq d_1(x,y).$$
for all $x, y \in \mathbb{R}^n$. **Hint:** For the upper inequality, the key step is gotten by by squaring $|a| + |b|$ and obtaining an inequality between it and the square root of the sum of squares. Then, use induction.

4. It turns out that a (modified) reverse inequality also holds: There are positive constants $\alpha$ and $\beta$ (both dependent on $n$) such that
$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_\infty(x,y)$$
for all $x, y \in \mathbb{R}^n$. Find these constants and prove the above inequality. **Hint:** The upper inequality is not too difficult; one simply notices that the sum of things is less than $n$ times the maximum of those things. For the lower inequality, use the Cauchy-Schwarz inequality with $a_j = |y_j - x_j|$ and $b_j = 1$ for all $j = 1, 2, \ldots, n$.

**Exercise 4.** Let $X$ be a set and let $d$ and $d'$ be metrics on $X$. We say that the metrics $d$ and $d'$ are equivalent and write $d \sim d'$ if there are positive real numbers $A$ and $B$ numbers such that
$$Ad'(x,y) \leq d(x,y) \leq Bd'(x,y)$$
for all $x, y \in X$.

1. Prove that $\sim$ is an equivalence relation.

2. Are the metrics $d_1$, $d_2$ and $d_\infty$ in the previous exercise equivalent?

For the remainder of this exercise, assume that $d$ and $d'$ are equivalent metrics on $X$ and $E \subseteq X$.

2. Show that $p \in X$ is a limit point of $E$ in the $d$ metric if and only if it is a limit point of $E$ in the $d'$ metric.

3. Prove that $E$ is open in the $d$ metric if and only if $E$ is open in the $d'$ metric[1].

[1] It is helpful to denote the neighborhoods in the $d$ metric by $N$ and the neighborhoods in the $d'$ metric by $N'$.

**Exercise 5.** In this exercise, we will discuss compactness in the metric space $\mathbb{R}$ equipped with the usual metric. Proceeding only by the definition of compactness (and not invoking the Heine-Borel Theorem), determine whether or not the following sets are compact and prove your assertion.

1. $\mathbb{Z}$
2. $[0, 1)$
3. $\{1/n \,|\, n \in \mathbb{N}_+\}$
4. $\{0\} \cup \{1/n \mid n \in \mathbb{N}_+\}$.

**Exercise 6.** Please do Baby Rudin's Exercise 9 of Chapter 2.

**Exercise 7.** Please do Baby Rudin's Exercise 16 of Chapter 2.

**Exercise 8.** Please do Baby Rudin's Exercise 22 of Chapter 2.