I.H.S. Senior Seminar:
Fourier Analysis

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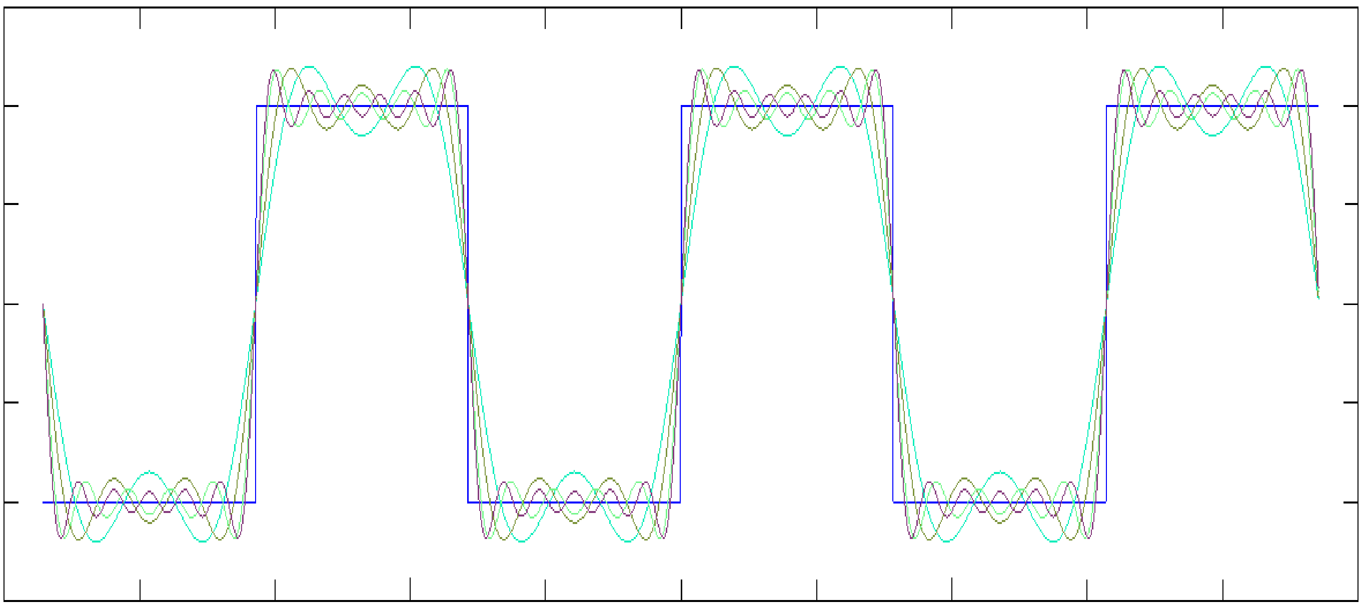

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1 Introduction

For the next 18 lectures, we will study a small piece of a beautiful subject called Fourier analysis. The small piece we'll look at is the study of Fourier series on the real line and by the end of this lecture you will have an idea of what it's about. Although we will sometimes look at some very technical details, my aim is to present a bird's eye view of Fourier series and, hopefully, share with you my love and fascination with it. I've sprinkled some exercises into these notes to give you a chance to try things on your own–to get your hands dirty, so to speak. If you find something confusing, need help with an exercise or simply want to share your thoughts, send me an email at *edr62@cornell.edu*. I'm here to help.

We will use some somewhat standard notation throughout. Let's write it down so it's always at our fingertips.

Notation 1.1. The following sets will be commonly used:

- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{Z}$ is the set of integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$.
- $\mathbb{N}_+$ is the set of positive real numbers $\{1, 2, 3, \ldots\}$.

1.1 Periodic Functions

Definition 1.2. *Let $f : \mathbb{R} \to \mathbb{R}($or , that is, a real or complex valued function on the real line. If for some $p > 0$,*

$$f(x + p) = f(x)$$

for all real numbers x, we say that f is periodic with period p.

Here are some periodic functions:

Example 1.3. The following functions are periodic with period 2π.

1. $\sin x, \sin 2x, \sin 3x, \ldots.$
2. $\cos x, \cos 2x, \cos 3x, \ldots.$
3. $\tan x, \tan 2x, \tan 3x, \ldots.$
4. Any constant function.
5. The "sawtooth" function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = (x - 2n\pi)$$

whenever $\pi(2n - 1) < x \leq \pi(2n + 1)$ for some integer n (See Figure 1).

Let's note that any periodic function f with period p is also periodic of period $2p$ because, for any x,

$$f(x + 2p) = f(x + p + p) = f(x + p) = f(x).$$

It follows (by induction) that if f is periodic with period p, then f is necessarily periodic with period np for any natural number n. So in this way we see that any periodic function has lots of periods. If a function f is periodic with period $p > 0$ and this is the smallest number for which f is periodic, we say that p is the *fundamental period* of f.

Exercise 1.4. If they exist, determine the fundamental periods of the functions in Example 1.3.

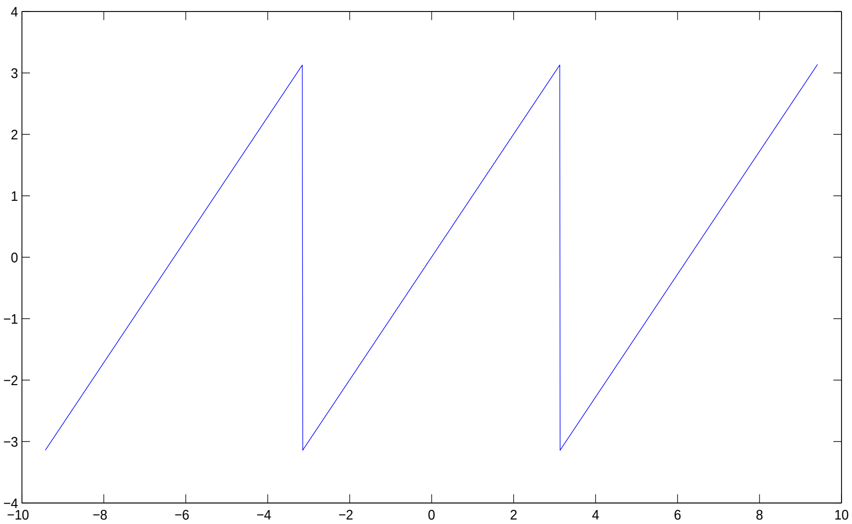


Figure 1: The sawtooth function

Okay, so we've seen some periodic functions. We should now ask: Why are periodic functions important? Why should we care? It turns out that physics, electronics, music, etc. are saturated by periodic functions. As an example, let's consider the (ordinary) differential equation,

$$ay'' + by' + cy = f(t). \tag{1}$$

where a, b and c are constants and f is a "forcing function" which is often a periodic function of time. This differential equation shows up in the study of oscillators wherein $a = m$ is the mass of the object, c is a spring constant and b is a constant depending on the resistance of the system. In this case, f is generally a force applied to the mass, possibly to agitate the system, say, by hitting it periodically with a sledge hammer. This equation also shows up in the study of RLC circuits (see any book on ODEs). As we shall see, when f is periodic, (1) can be solved by considering series of periodic functions, called *Fourier Series*. The solutions are fantastically interesting! For instance, if (1) models the oscillation of a bridge, one can find a frequency, called the critical frequency, at which the bridge will collapse if it is hit with a sledge hammer (even softly) at just that frequency (the frequency is $1/p$). This phenomenon is known as resonance and you should look up the most famous example, the Tacoma Narrows bridge disaster (search online for a video). Conversely, if you an an engineer, you might want to prevent me from breaking your bridge and so knowing Fourier analysis is important for this purpose too.

Another application of periodic functions and Fourier series is to the study of heat diffusion. In fact, this was Fourier's original motivation for considering them [7]. The first systematic treatment of Fourier series appeared in Fourier's main work, *Théorie analytique de la chaleur* (Analytic theory of heat), in 1822. We will discuss this application in depth in a few days.

1.2 Trigonometric Polynomials

We begin by studying sums of the form

$$P_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx). \tag{2}$$

Any such expression is known as a *trigonometric polynomial* of order n. We will soon let $n \to \infty$ and consider series of the form

$$a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx);$$

these are called *trigonometric series*. As you know from studying numerical sequences and series, we will need to talk (and worry) about convergence. Let's see what can be done with these polynomials.

Example 1.5. Let's return to our sawtooth function f from Example 1.3. There is a sequence of trigonometric polynomials P_n which approximates f nicely. These are the polynomials

$$P_n(x) = 0 + \sum_{k=1}^{n} 0 \cdot \cos(kx) + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k} \sin(kx) = \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sin(kx).$$

for $n = 1, 2, \ldots$. In a few lectures, I'll show you how I found the coefficients of these polynomials; they are called the Fourier coefficients for f. In Figure 2, I've plotted the graphs of f and P_n for $n = 1, 2, 3, 4, 5$ and 40. As n increases, do the graphs of P_n begin to look like the graphs of f?

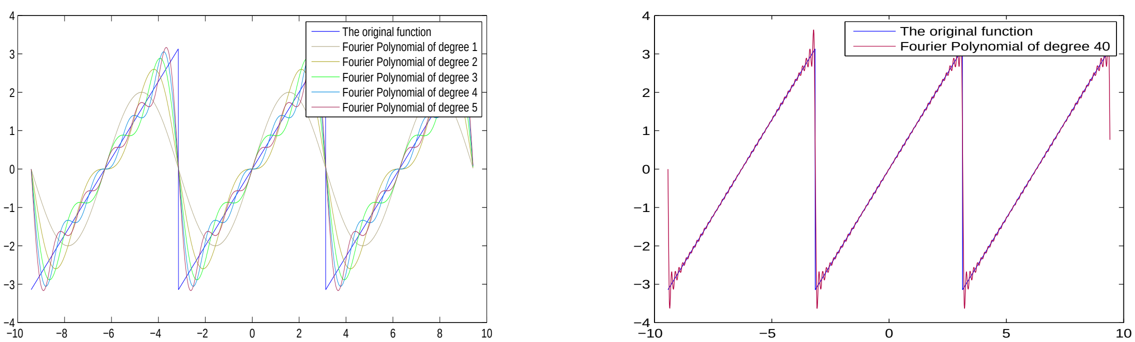


Figure 2: The graphs of f and P_n for $n = 1, 2, 3, 4, 5, 40$.

Taking the previous example as evidence, one of the main claims we will examine in this seminar can be phrased as follows:

Claim 1.6. *Let f be a "nice" periodic function of period 2π. Then f can be approximated by a sequence of trigonometric polynomials. That is, we can find coefficients $a_0, a_1, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that P_n, defined by* (2)*, is close to f.*

Over the last 250 years, the above claim occupied some of the best mathematicians including d'Alembert, Euler, Fourier, Bernoulli, Lagrange, Cauchy, Dirichlet, Du Bois-Reymond, Fejér, Cesáro, Weierstrass and Kolmogorov, to name a few [7]! It was only in the 1960's that the claim was completely sorted out by Lennart Carleson. Let's make one important note concerning Claim 1.6. In mathematics, our statements (theorems, propositions, corollaries, claims, etc.) require precision. In the above claim, we have not talked about what it means to be a "nice" function nor have we discussed the sense in which f is approximated by P_n. For instance, does approximate mean be within a mile of f? To address the claim satisfactorily and to understand what is known, we must make these notions precise.

2 Convergence

For today and the next lecture, we will study some basic mathematical machinery needed to talk about Fourier series. In the process, we will begin to make precise the notion of "approximate" in the claim from the previous lecture. This machinery centers on the notion of convergence and, in one way or another, everything done in your calculus class depends on it.

2.1 Numerical sequences and series

You already know what it means for a sequence of numbers $\{a_n\} \subseteq \mathbb{R}$ to converge to some number $a \in \mathbb{R}$. That is, you know what

$$\lim_{n\to\infty} a_n = a$$

means. Let's write down the definition for future reference.

Definition 2.1. *Let $\{a_n\} \subseteq \mathbb{R}$ be a sequence of real numbers. We say that $\{a_n\}$ converges to a number $a \in \mathbb{R}$ if the following property holds: For each $\epsilon > 0$ there is a natural number $N = N(\epsilon) \in \mathbb{N}$ such that*

$$|a_n - a| < \epsilon \text{ whenever } n \geq N. \tag{3}$$

In this case we write

$$\lim_{n\to\infty} a_n = a.$$

Example 2.2. Consider the sequence $\{a_n\}$ defined by $a_n = 1/n \in \mathbb{R}$ for $n \in \mathbb{N}$. I claim that this sequence converges to the number 0. To prove this claim I must be able to do the following: If you give me an ϵ, I should give you an N for which (3) holds.

Proof. Let ϵ be **any** number greater than 0. Note that $1/\epsilon$ is another positive number (although it might be huge) and so I can take next largest integer on the number line, $N_\epsilon = \lceil 1/\epsilon \rceil + 1$. Then observe that, for any $n \geq N_\epsilon$,

$$|a_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N_\epsilon} \leq \frac{1}{1/\epsilon + 1} < \epsilon.$$

□

Exercise 2.3. Consider the sequence $\{a_n\} \subseteq \mathbb{R}$ defined by $a_n = \cos(1/n)$ for $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} a_n = 1.$$

You may use the fact that, for all real numbers x,

$$|\cos(x) - 1| \leq |x|.$$

Let's now talk about series. Let $\{a_n\}$ be a sequence of real numbers and define

$$s_n = a_1 + a_2 + \cdots a_n = \sum_{k=1}^{n} a_k;$$

for each n, s_n is called the nth-partial sum of the sequence $\{a_n\}$. In this way, we have produced a new sequence of numbers $\{s_n\}$ and we can talk about its convegence. If

$$s = \lim_{n\to\infty} s_n$$

exists, we say that the series converges and write

$$\sum_{k=1}^{\infty} a_k = s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

In fact, we will usually use the shorthand: The series $\sum a_n$ converges.

Example 2.4 (Geometric series). Consider $\{a_n\}$ defined by $a_n = r^n$ for $n = 0, 1, 2, \ldots$ (we'll start at $n = 0$ this time) for some fixed real number $0 \leq r < 1$. We want to investigate the convergence of the series formed by the partial sums

$$s_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} r^k.$$

One can check (by induction) that

$$s_n = \frac{1 - r^n}{1 - r}$$

for all $n \in \mathbb{N}$. From this formula it is evident that $\lim_{n\to\infty} s_n$ exits (because $r < 1$) and

$$\sum_{k=0}^{\infty} r^n = \sum_{k=0}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - r^n}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r}.$$

Exercise 2.5. As you know, the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

does not converge because its sequence of partial sums grows without bound. This is seen with the help of the integral comparison test (look this up in your calculus book). Using the integral comparison test, investigate the convergence of the closely related series,

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

We will see soon that the convergence of this series will help us prove an important result about Fourier series. In Section 5, we will prove an exciting result about this series (see Example 5.13).

The main point that I want you to take away from this section is this: The definition and investigation of series depends on sequences.

2.2 Sequences and series of functions

Today, we will be instead talking about convergence of functions. Throughout this section we will consider functions and sequences of functions $f : [a, b] \to \mathbb{R}$ (or $\mathbb{C}$) where $[a, b]$ is an arbitrary interval in the real line. To this end we consider a sequence $\{f_n\}$ of functions and ask: what does it mean for $\{f_n\}$ to converge to a function f? That is, what is the appropriate way to generalize Definition 2.1? Let's consider some examples.

Example 2.6. Consider the sequence $\{f_n\}$ of continuous functions defined on the interval $[0, 1]$ by

$$f_n(x) = x^n$$

for $0 \leq x \leq 1$ and $n \in \mathbb{N}$.

The graphs of f_n are illustrated for $n = 1, 2, 3 \ldots, 20$ in Figure 3. We should observe that the sequence $\{f_n\}$ appears to converge to something strange, the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for all } 0 \leq x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

Note that this function isn't continuous.

Example 2.7. Now consider the sequence $\{f_n\}$ of functions defined on the interval $[-\pi, \pi]$ by

$$f_n(x) = \cos(x/n) - 1/2$$

for $-\pi \leq x \leq \pi$ and $n \in \mathbb{N}$. The graphs of f_n are illustrated for $n = 1, 2, \ldots 10$ in Figure 4. It appears that the sequence f_n are converging to the constant function $f(x) = 1/2$ as $n \to \infty$.

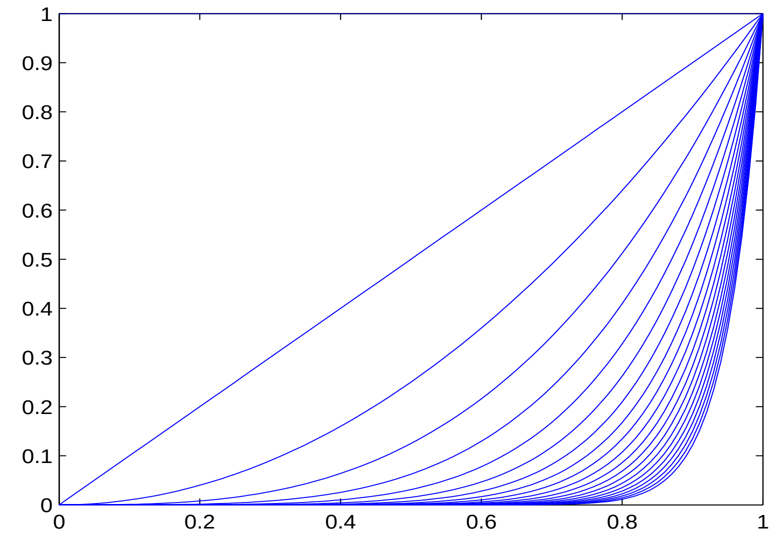


Figure 3: A famous picture: The graphs of $f_n(x) = x^n$ for $n = 1, 2, \ldots, 20$.

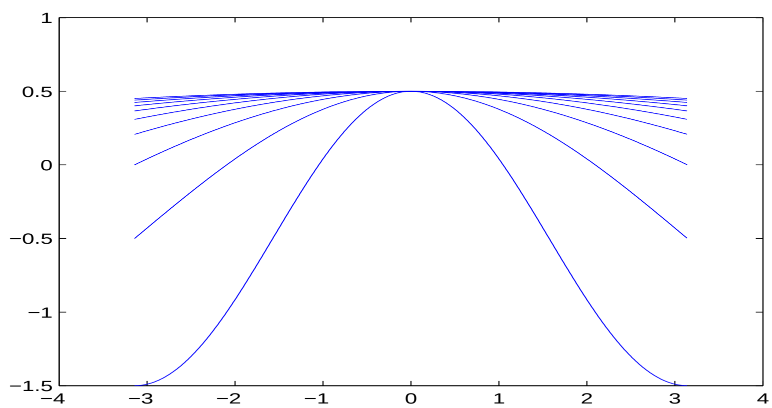


Figure 4: The graphs of $f_n(x) = \cos(x/n) - 1/2$ for $n = 1, 2, \ldots, 10$.

You should now compare and contrast Figures 2, 3 and 4. From my viewpoint, it seems that the graphs of the functions f_n from Example 2.7 approximate the limit function $f = 1/2$ more uniformly than the graphs of the trigonometric polynomials approximate the sawtooth function in Example 1.5. The sequences in these examples seem to converge in different ways and this suggests that, maybe, we should have more than one notion of convergence. Let's define a straightforward one.

Definition 2.8 (Pointwise convergence)**.** *Let $\{f_n\}$ be a sequence of functions mapping from $[a, b]$ into $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$. If, for some real number $x \in [a, b]$,*

$$\lim_{n\to\infty} f_n(x) = f(x),$$

we say that $\{f_n\}$ converges to f at x. If this happens for all $x \in [a, b]$, we say that the sequence $\{f_n\}$ converges to

f pointwise.

The preceding definition has the following equivalent formulation: The sequence $\{f_n\}$ converges to f at x if for each $\epsilon > 0$, there is $N = N(x, \epsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$. Notice the order of quantifiers in this statement. We first talk about a point x and then talk about a limit. The next definition, which is a stronger notion of the one above, moves the x-quantifier.

Definition 2.9 (Uniform convergence)**.** *Let $\{f_n\}$ be a sequence of functions mapping from $[a, b]$ into $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$. If for any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in \mathbb{R}$ (or for all x in the domains of f_n) and all $n \geq N$, we say that $\{f_n\}$ converges to f uniformly.

Uniform convergence is sometimes called "convergence in the graph". This is illustrated in Figure 5 in which we see the graph of a function f (in black) in the center of a "band" of radius ϵ (in red). For a sequence of functions $\{f_n\}$ to converge uniformly to f means that, for sufficiently large n, the graph of f_n is completely contained in the band of radius ϵ surrounding f; the blue line is an example of the graph of one such f_n.

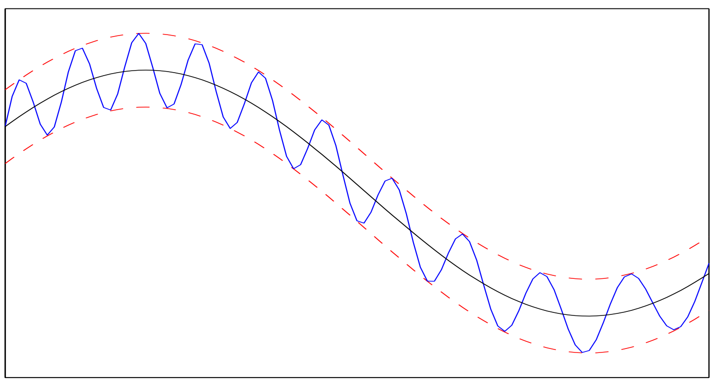


Figure 5: An illustration of uniform convergence

Example 2.10. Consider the sequence $\{f_n\}$ defined on the interval $[-\pi, \pi]$ by

$$f_n(x) = x - \sin(x/n)$$

for $-\pi \leq x \leq \pi$ and $n \in \mathbb{N}$. I claim that this sequence converges uniformly to $f(x) = x$.

Proof. Let $\epsilon > 0$ and observe that

$$|f_n(x) - f(x)| = |\sin(x/n)|$$

for all $-\pi \leq x \leq \pi$ and all $n \in \mathbb{N}$. You know that $|\sin y| \leq |y|$ for all $y \in \mathbb{R}$ and so,

$$|\sin(x/n)| \leq \frac{|x|}{n} \leq \frac{\pi}{n}$$

for all $-\pi \leq x \leq \pi$ and all $n \in \mathbb{N}$. So if I choose $N = N(\epsilon) = 4\lceil 1/\epsilon \rceil > \pi/\epsilon$, then

$$|f_n(x) - f(x)| = |\sin(x/n)| \leq \frac{\pi}{n} \leq \frac{\pi}{N} < \frac{\pi}{\pi/\epsilon} = \epsilon$$

and this holds for all $-\pi \leq x \leq \pi$ and $n \in \mathbb{N}$. That's a proof! $\square$

Exercise 2.11. Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \cos(x/n) - 1/2, \tag{4}$$

for $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$. Show that that $\{f_n\}$ converges to the function $f(x) = 1/2$ uniformly. You may use the fact that

$$|\cos(x) - 1| \leq |x|$$

for all $x \in \mathbb{R}$.

You should ask: why is uniform convergence important? When talking about convergence of functions (pointwise or otherwise) we are interested to know when nice things converge to nice things. For instance, note that in Example 2.10 and the exercise above, we took a sequence of continuous functions which converged uniformly to a limit and this limit function was also continuous. As the following fact shows, this is no coincidence.

Fact 2.12. *Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ (written $\{f_n\} \subseteq C([a, b])$). If $\{f_n\}$ converges to f uniformly, then f is continuous, i.e., $f \in C([a, b])$.*

Exercise 2.13. Using the definition of pointwise convergence, show that the sequence of functions $f_n(x) = x^n$ of Example 2.6 does indeed converge to the function f which is zero for all $x < 1$ and 1 when $x = 1$. Use the above fact (you'll use the contrapositive statement) to conclude that f_n does not converge to f uniformly.

Let's now introduce series of functions. Let $\{g_n\}$ be a sequence of functions defined on the interval $[a, b]$ and for each n define

$$S_n(x) = \sum_{k=1}^{n} g_k(x)$$

for $x \in \mathbb{R}$. Each S_n is called the nth-partial sum of the sequence $\{g_n\}$. In this way we form a new sequence $\{S_n\}$ of functions on $[a, b]$ and we can talk about convergence. If the sequence of partial sums converges pointwise, we say the the series $\sum g_k$ converges pointwise. If the sequence of partial sums converges uniformly, we say that the series $\sum g_k$ converges uniformly. Here is another big fact:

Fact 2.14 (The Weierstrass M-test)**.** *Let $\{g_n\}$ be a sequence of functions and let $\{M_n\}$ be a numerical sequence of non-negative numbers such that $|g_n(x)| \leq M_n$ for all $n \in \mathbb{N}$. If the series $\sum M_k$ converges, then the series*

$$\sum_{k=1}^{\infty} g_k(x) = \lim_{n \to \infty} \sum_{k=1}^{n} g_k(x)$$

converges uniformly.

Proposition 2.15. *Suppose that $\{g_n\} \subseteq C([a, b])$, i.e., it is a sequence of continuous functions such that $|g_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and for all $x \in [a, b]$. If $\sum_k M_k$ converges then $\sum g_k$ converges uniformly and*

$$S(x) = \sum_{k=1}^{\infty} g_k(x) = \lim_{n \to \infty} \sum_{k=1}^{n} g_k(x)$$

is continuous.

Proof. Because $\sum M_k$ converges, an immediate application of the Weierstrass M-test (Fact 2.14) guarantees the the series $\sum g_k$ converges uniformly to a limit $S(x)$. That is, the sequence of partial sums $\{S_n\}$, defined for each n by

$$S_n(x) = \sum_{k=1}^{n} g_k(x),$$

converge uniformly to $S(x)$. Now, we know that a finite sum of continuous functions is continuous and so we can conclude that S_n is continuous. Thus we have a sequence of continuous functions which converges uniformly to a limit $S(x)$; in view of Fact 2.12, $S(x)$ must be continuous. $\square$

2.3 Integrals and limits

In this final subsection on convergence, we discuss the interplay between limits and integration. Our discussion here will be used heavily in the next section when we introduce Fourier coefficients and begin to address Claim 1.6. You have recently learned about Riemann integration and you've seen how good it is. Indeed, it is fantastically useful, however there are some essential problems with it. The problems we discuss today led to a complete revolution in mathematics around 1900. What emerged is called the Lebesgue theory of integration; without it, no complete treatment of Fourier series would be possible. Let's begin by asking a question: If $\{f_n\}$ is a sequence of Riemann integrable functions on $[a,b]$ which converge pointwise to a limit function f, i.e., for each $x \in [a,b]$,

$$f(x) = \lim_{n\to\infty} f_n(x),$$

does it follow that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx, \tag{5}$$

i.e., can you exchange integrals and limits? This is, at least, what my intuition leads me to believe. Let's see if we can answer this question by looking at some examples.

Example 2.16. Consider the sequence of function $f_n(x) = x^n$ on $[0,1]$ of Examples 2.6. We recall that the limit,

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{when } x = 1, \end{cases}$$

holds pointwise. To examine the validity of (5), let's first compute the Riemann integrals of the functions in the sequence. For each n, we have

$$\int_0^1 f_n(x)\,dx = \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}. \tag{6}$$

Let's now evaluate the integral of the limit. Because f is continuous and identically 0 except at $x = 1$, f is certainly Riemann integrable and the integral can be computed as follows:

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx + \int_1^1 1\,dx = 0 + 0 = 0. \tag{7}$$

Thus by combining (6) and (7), we see that

$$\int_0^1 f(x)\,dx = 0 = \lim_{n\to\infty}\frac{1}{n+1} = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx.$$

Thus (5) is valid for this sequence.

So maybe now we're hopeful that things work out. Unfortunately, as the next example shows, things can go very wrong.

Example 2.17. Let $\{r_n\}$ be an enumeration of the rational numbers in the interval $[0,1]$. This is, by definition, a sequence $\{r_n\}$ of **all** the rational numbers between 0 and 1. The fact that this sequence can be formed at all is non-trivial but we will take this for granted. For each n, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, \dots r_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus each function f_n is a function which is zero everywhere except at the rational numbers $r_1, r_2, \dots, r_n$ where it takes the value 1. You can check that, for each $x \in [0,1]$,

$$f(x) = \lim_{n\to\infty} f_n(x),$$

where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

This limit function f is called the characteristic function of the set $\mathbb{Q} \cap [0,1]$. As I said before, for each n, f_n is a function which is zero except at a finite number of points. Using what you know from the theory of Riemann integration (suppose for instance that $r_1 < r_2 < r_3 < \cdots r_n$),

$$\int_0^1 f_n(x)\,dx = \int_0^{r_1} f_n(x)\,dx + \int_{r_1}^{r_2} f_n(x)\,dx + \cdots + \int_{r_{n-1}}^{r_n} f_n(x)\,dx + \int_{r_n}^1 f_n(x)\,dx = 0 + 0 + \cdots + 0 + 0 = 0.$$

We see trivially then that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 0 = 0.$$

But you already know from your calculus class that f is not Riemann integrable on $[0,1]$ and so writing $\int_0^1 f(x)\,dx$ doesn't even make sense; in particular, (5) can't hold.

In view of the preceding example, we conclude that the answer to the question posed about (5) is no. Unfortunately, this issue is an essential one and cannot be overcome with the theory of Riemann integration. They needed to construct an entire new theory for that – the Lebesgue theory. If you want to learn the Lebesgue theory (and I encourage you to) a good start can be had by reading [5]. Another selling point of the Lebesgue theory, also called measure theory, is that it provided the long-needed mathematical foundation for Probability theory; this is mainly due to Kolmogorov. Let's return to (5). Fortunately for us, uniform convergence saves the day again:

Proposition 2.18. *Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. If the sequence $\{f_n\}$ converges to f uniformly, then f is Riemann integrable and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx. \tag{8}$$

Proof. There are two claims being made by in the proposition. The first claim is that f, the limit function, is Riemann integrable. The second claim is a claim about numbers, namely that the numbers $\{\int_a^b f_n(x)dx\}$ converge to the number $\int_a^b f(x)dx$. We shall not have time to prove the first statement; its proof can be found in [8]. Let's address the second claim in the slightly easier case that $a = 0$ and $b = 1$. We need to show that, for any $\epsilon > 0$, there is an $N = N(\epsilon)$ such that

$$\left| \int_0^1 (f_n(x) - f(x))\,dx \right| = \left| \int_0^1 f_n(x)\,dx - \int_0^1 f(x)\,dx \right| < \epsilon$$

whenever $n \geq N$ (note that we just used the fact that the difference of integrals is the integral of the difference). So, let $\epsilon > 0$ be fixed. Remember from class the following facts about the Riemann integral:

1. If g is Riemann integrable, then
$$\left| \int_0^1 g(x)\,dx \right| \leq \int_0^1 |g(x)|\,dx$$

2. If g and h are Riemann integrable and $g(x) \leq h(x)$ for all $x \in [0,1]$, then
$$\int_0^1 g(x) \leq \int_0^1 h(x)\,dx$$

3. For any number α,
$$\int_0^1 \alpha\,dx = \alpha\int_0^1 \,dx = \alpha(1-0) = \alpha.$$

What we haven't talked about yet is uniform convergence. It's one of our hypotheses so let's invoke it. Because $\{f_n\}$ converges uniformly to f, we can find N such that
$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$
whenever $n \geq N$ (here we used $\epsilon' = \epsilon/2$ instead of ϵ when invoking the definition of uniform convergence). Let's fix this N and combine all of the above stated facts to see that, whenever $n \geq N$,
$$\left|\int_0^1 (f_n(x) - f(x))\,dx\right| \leq \int_0^1 |f_n(x) - f(x)|\,dx \leq \int_0^1 \frac{\epsilon}{2}\,dx = \frac{\epsilon}{2} < \epsilon.$$

This is exactly what we needed to show! □

Exercise 2.19. By nearly repeating the above arguments, give a proof of (8) in the general case that a and b are arbitrary. Hint: You will again have to use two different epsilons. One of them, ϵ, will need to be arbitrary and the other, ϵ' will be proportional to ϵ and used when invoking the definition of uniform convergence.

Now let's talk about integrating series. Let $\{g_k\}$ be a sequence of Riemann integrable functions on $[a,b]$. You've learned in your calculus class that the integral of a finite sum is the sum of the integrals (this is called linearity). Therefore, we know that
$$\int_a^b \left(\sum_{k=1}^n g_k(x)\right)\,dx = \sum_{k=1}^n \int_a^b g_k(x)\,dx$$
for any $n \in \mathbb{N}$. Translating the above discussion about sequences to series, we arrive at the following question: If $\{g_k\}$ is a sequence of Riemann integrable functions on $[a,b]$ for which the series $\sum g_k$ converges (either pointwise or uniformly), when is
$$\begin{aligned}\int_a^b \left(\sum_{k=1}^\infty g_k(x)\right)\,dx &= \int_a^b \left(\lim_{n\to\infty}\sum_{k=1}^n g_k(x)\right)\,dx \\ &= \lim_{n\to\infty}\left(\int_a^b \sum_{k=1}^n g_k(x)\right)\,dx = \lim_{n\to\infty}\sum_{k=1}^n \int_a^b g_k(x)\,dx = \sum_{k=1}^\infty \int_a^b g_k(x)\,dx?\end{aligned}$$

Written more compactly, when is
$$\int_a^b \sum_{k=1}^\infty g_k(x)\,dx = \sum_{k=1}^\infty \int_a^b g_k(x)\,dx?$$

Example 2.20. Returning to Example 1.5, we saw that the trigonometric polynomials
$$P_n(x) = \sum_{k=1}^n \frac{(-1)^{k+1}}{2n}\sin(kx)$$
approximated the function $f(x) = x$ pretty nicely on the interval $[-\pi,\pi]$. But now we can quantify "approximate". It is a fact that the sequence $\{P_n\}$, which is a sequence of partial sums, converges pointwise to $f(x) = x$ on the interval $(-\pi,\pi)$, i.e.,
$$x = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{2k}\sin(kx) = \lim_{n\to\infty}\sum_{k=1}^n \frac{(-1)^{k+1}}{2k}\sin(kx)$$

for all $-\pi < x < \pi$. This convergence isn't uniform due to the Gibb's phenomenon (we will discuss this in Section 5) and, in fact, there is a minor issue concerning convergence at the points $-\pi$ and π but we won't worry about this now. However, we can still ask if

$$\int_{-\pi}^{\pi} x\,dx = \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \frac{(-1)^{k+1}}{2k} \sin(kx)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{-\pi}^{\pi} \frac{(-1)^{k+1}}{2k} \sin(kx)\,dx?$$

Simply do the calculation and you'll see that it's true:

$$\int_{-\pi}^{\pi} x\,dx = \frac{x^2}{2}\Big|_{-\pi}^{\pi} = 0$$

and, for each k,

$$\int_{-\pi}^{\pi} \frac{(-1)^{k+1}}{2k} \sin(kx)\,dx = \frac{(-1)^{k+1}}{2k} \frac{-\cos(kx)}{k}\Big|_{-\pi}^{\pi} = 0.$$

Consequently

$$\int_{-\pi}^{\pi} x\,dx = 0 = \sum_{k=1}^{\infty} 0 = \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \frac{(-1)^{k+1}}{2k} \sin(kx)\,dx.$$

We end this section with a proposition. I'll leave its proof as an exercise.

Proposition 2.21. *Let $\{g_k\}$ be a sequence of Riemann integrable functions on $[a,b]$ and suppose that the series $\sum g_k$ converges uniformly on $[a,b]$. Then $\sum_{k=1}^{\infty} g_k(x)$ is Riemann integrable on $[a,b]$ and*

$$\int_a^b \left(\sum_{k=1}^{\infty} g_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

Proof. As we noted before, for each n, the partial sum S_n is Riemann integrable and

$$\int_a^b S_n(x)\,dx = \int_a^b \left(\sum_{k=1}^{n} g_k(x) \right) dx = \sum_{k=1}^{n} \int_a^b g_k(x)\,dx.$$

Of course, we know that uniform convergence of the series means uniform convergence of the partial sums and so we can use Proposition 8 to see that

$$\sum_{k=1}^{\infty} g_k(x) = \lim_{n\to\infty} S_n(x)$$

is Riemann integrable and

$$\int_a^b \left(\sum_{k=1}^{\infty} g_k(x) \right) dx = \lim_{n\to\infty} \int_a^b S_n(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b g_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

□

Another useful result is the following corollary.

Corollary 2.22. *Let $\{g_k\}$ be a sequence of Riemann integrable functions on $[a,b]$ and suppose that $\{M_k\}$ is a sequence of positive numbers for which $|g_k(x)| \le M_k$ for all $k \in \mathbb{N}$. If the series $\sum M_k$ converges, then the series $\sum g_k$ converges uniformly, the limit*

$$\sum_{k=1}^{\infty} g_k(x)$$

is Riemann integrable, and

$$\int_a^b \left(\sum_{k=1}^{\infty} g_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

Exercise 2.23. Prove the corollary above. Use Proposition 2.14 and Proposition 2.21.

3 Fourier coefficients

3.1 Coefficients of a trigonometric series

In this subsection, we will will focus our attention on functions which are defined by convergent trigonometric series. Thus, we consider $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=a_0+\sum_{k=1}^{\infty}(a_k\cos(kx)+b_k\sin(kx))=\lim_{n\to\infty}\left(a_0+\sum_{k=1}^{n}(a_k\cos(kx)+b_k\sin(kx))\right), \tag{9}$$

where we will assume that, for now, the series converges pointwise on $\mathbb{R}$ and hence pointwise on $[-\pi,\pi]$.

Proposition 3.1. *Suppose that the series defining f, (9), converges pointwise. Then f is periodic of period 2π. If, additionally, the series converges uniformly, then f is continuous on the interval $[-\pi,\pi]$.*

Exercise 3.2. Give a proof for the proposition above. Hint: Each of the partial sums are periodic with period 2π, you just need to show that the limit must be. Assuming uniform convergence, note that all partial sums are continuous and so you can use Proposition 2.15.

We now make a key observation by integrating f from $-\pi$ to π:

$$\begin{aligned}\int_{-\pi}^{\pi}f(x)\,dx &= \int_{-\pi}^{\pi}\left(a_0+\sum_{k=1}^{\infty}(a_k\cos(kx)+b_k\sin(kx))\right)dx\\ &= \int_{-\pi}^{\pi}a_0\,dx+\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty}(a_k\cos(kx)+b_k\sin(kx))\right)dx\\ &= 2\pi a_0+\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty}(a_k\cos(kx)+b_k\sin(kx))\right)dx.\end{aligned}$$

If we assume that the series defining f is uniformly convergent, we can integrate the series term-by-term in view of Proposition 2.21 and so

$$\begin{aligned}\int_{-\pi}^{\pi}f(x)\,dx &= 2\pi a_0+\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty}(a_k\cos(kx)+b_k\sin(kx))\right)dx\\ &= 2\pi a_0+\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}(a_k\cos(kx)+b_k\sin(kx))\,dx\\ &= 2\pi a_0+\sum_{k=1}^{\infty}a_k\int_{-\pi}^{\pi}\cos(kx)\,dx+b_k\int_{-\pi}^{\pi}\sin(kx)\,dx\\ &= 2\pi a_0+\sum_{k=1}^{\infty}0+0\\ &= 2\pi a_0.\end{aligned}$$

In other words,

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx.$$

So, we've made a curious observation: The a_0 of the coefficient of the series (9) can be solved for by simply integrating f. It turns out that we can find the other coefficients in a similar way. This will make use of the following and relatively straightforward fact.

Fact 3.3. *For any $n, m \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0.$$

Exercise 3.4. Confirm that the fact is true for $m = n = 1$ (check all the cases and remember your trig identities). The general result can then be shown by induction using the trigonometric identities involving angle addition.

So if we want to solve (9) for, say, b_6, the idea is to multiply both sides of (9) by $\sin(6x)$ and integrate from $-\pi$ to π while making use of Proposition 2.21. We have

$$\int_{-\pi}^{\pi} \sin(6x)f(x)\,dx = \int_{-\pi}^{\pi} a_0 \sin(6x)\,dx + \sum_{k=1}^{\infty}\left(a_k \int_{-\pi}^{\pi} \sin(6x)\cos(kx)\,dx + b_k \int_{-\pi}^{\pi} \sin(6x)\sin(kx)\,dx\right)$$

and, in view of Fact 3.3, all of the terms are zero except the $k = 6$ term. Thus,

$$\int_{-\pi}^{\pi} \sin(6x)f(x)\,dx = a_6 \int_{-\pi}^{\pi} \sin(6x)\cos(6x)\,dx + b_6 \int_{-\pi}^{\pi} \sin(6x)\sin(6x)\,dx = 0 + b_6\pi$$

and therefore

$$b_6 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(6x)f(x)\,dx.$$

In general, we find that

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,} \tag{10}$$

$$\boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)f(x)\,dx} \tag{11}$$

and

$$\boxed{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx)f(x)\,dx} \tag{12}$$

for $k = 1, 2, \dots$. A sufficient, but far from necessary, condition for the validity of these formulas is the following:

Proposition 3.5. *If the series defining f in* (9) *converges uniformly, then the coefficients $a_0, a_1, \dots$ and $b_1, b_2, \dots$ respectively satisfy* (10), (11), *and* (12).

Proof. The uniform convergence of (9) isn't affected by multiplying by $\sin mx$ or $\cos mx$ for any $m \in \mathbb{N}$. In other words, for any $n \in \mathbb{N}$,

$$\cos(mx)f(x) = a_0\cos(mx) + \sum_{k=1}^{\infty}(a_k\cos(mx)\cos(kx) + b_k\cos(mx)\sin(kx))$$

and

$$\sin(mx)f(x) = a_0\sin(mx) + \sum_{k=1}^{\infty}(a_k\sin(mx)\cos(kx) + b_k\sin(mx)\sin(kx)),$$

where these series converge uniformly for $x \in [-\pi, \pi]$. Noting that everything in sight is continuous and therefore Riemann integrable, Proposition 2.21 guarantees that we can integrate term-by-term. The equations, (10), (11), and (12), follow then from doing this integration and employing Fact 3.3. Only one term survives! □

3.2 Fourier coefficients of a periodic function

Let's see where we stand. In the last section, we started with a function f which was defined by a convergent trigonometric series. We concluded that it was periodic with period 2π; this was the first part of Proposition 3.1. We also found that, if the series happened to converge uniformly, then f was necessarily continuous and its coefficients satisfied the formulas (10), (11), and (12). In the present section, we flip that perspective. Let's make a definition.

Definition 3.6. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic of period 2π. Define the collection of numbers $a_0, a_1, \dots, b_1, b_2, \dots$ by (10), (11) and (12) for $k = 1, 2, \dots$, respectively. These are called the Fourier coefficients of f.*

In the above definition, we assume f is an arbitrary continuous and periodic function, i.e., it isn't necessarily given by a trigonometric series, and define a sequence of numbers by the formulas (10), (11), and (12). From this viewpoint, these are simply collection of numbers associated to f. Let's now define a new function $\tilde{f}$ by

$$\tilde{f}(x) = a_0 + \lim_{n\to\infty}\sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx)) = a_0 + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx)) \tag{13}$$

for any x for which the limit exists. We call $\tilde{f}$ the *Fourier series* for f. If the limit defining $\tilde{f}$ exists, at least uniformly, something magical happens:

Theorem 3.7. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic of period 2π. Define the Fourier coefficients of f, $\{a_0, a_1, \dots, b_1, b_2, \dots\}$ by Definition 3.6 and let $\tilde{f}$ be defined by (13). If, the series defining $\tilde{f}$ converges uniformly, then $f(x) = \tilde{f}(x)$ for every $x \in \mathbb{R}$. This is to say that f is equal to (or represented by) its Fourier series.*

This theorem is our first attempt to address Claim 1.6 from Section 1. In words, it says the following: If we start with a continuous and periodic function f and we form its Fourier series (13) and if this series, for any reason, happens to converge uniformly, then it must be equal to f. In other words, f can be approximated by trigonometric polynomials and, if we take the limit of these trigonometric polynomials, we recover f. In the next section, we will give much physical intuition for this idea. Let's now focus on proving the theorem and give an important corollary. In proving the theorem we will take only one fact for granted. Here it is:

Fact 3.8. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is continuous and periodic of period 2π. If all of the Fourier coefficients of f are zero, i.e., $0 = a_0 = a_1 + a_2 + \cdots + b_1 + b_2 + \cdots$, then f is the zero function.*

I'm asking you to buy the above fact. It isn't actually that complicated to prove but it would take us too far afield to do it. For a proof using Fejér's theorem, see Chapter 2 of [7]. An immediate consequence of this fact can be phrased as follows.

Proposition 3.9. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous periodic functions of period 2π. If f and g have the same Fourier coefficients, then $f = g$, i.e, $f(x) = g(x)$ for all $x \in \mathbb{R}$.*

Proof. Let f and g be as above and consider the new function $h:\mathbb{R}\to\mathbb{R}$ defined by the difference, $h(x) = f(x) - g(x)$ for all $x \in \mathbb{R}$; note that h must be continuous. Let's also label the Fourier coefficients of f, g and h respectively by

$$\underbrace{\{a_0(f), a_1(f), \dots, b_1(f), b_2(f), \dots\}}_{\text{Fourier coefficients of } f} \qquad \underbrace{\{a_0(g), a_1(g), \dots, b_1(g), b_2(g), \dots\}}_{\text{Fourier coefficients of } g} \qquad \underbrace{\{a_0(h), a_1(h), \dots, b_1(h), b_2(h), \dots\}}_{\text{Fourier coefficients of } h}.$$

Our hypothesis guarantees that $a_k(f) = a_k(g)$ for $k = 0, 1, \dots$ and $b_k(f) = b_k(g)$ for $k = 1, 2, \dots$. Let's now observe that, for any k,

$$\begin{aligned}
a_k(h) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kx)h(x)\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\cos(kx)f(x)-\cos(kx)g(x)\right)\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kx)f(x)\,dx-\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kx)g(x)\,dx\\
&= a_k(f)-a_k(g)=0.
\end{aligned}$$

If you look closely, you'll see that the only fact we used here is that the Riemann integral is linear, i.e., $\int f-g\,dx = \int f\,dx-\int g\,dx$. Therefore $a_k(h)=0$ for $k=1,2,\dots$. By that exact same reasoning we can conclude that $a_0(h)=0$ and $b_k(h)=0$ for all $k=1,2,\dots$. Putting all of this together, we have concluded that all of the Fourier coefficients of h are zero. By virtue of Fact 3.8, $f(x)-g(x)=h(x)=0$ for all $x\in\mathbb{R}$ and thus $f=g$. $\square$

We now have enough to prove Theorem 3.7.

Proof of Theorem 3.7. Let f be continuous and periodic of period 2π and let $\{a_0,a_1,\dots,b_1,b_2\}$ be the Fourier coefficients for f given by Definition 3.6. Also, we suppose that the Fourier series for f, $\tilde{f}$ defined by (13), converges uniformly. Then we can apply our theory of the previous subsection, in particular, Theorem 3.1 to conclude that the Fourier coefficients of $\tilde{f}$ are also $\{a_0,a_1,\dots,b_1,b_2,\dots\}$. Thus, f and $\tilde{f}$ have the same Fourier coefficients. Since they are both periodic and continuous, they must be equal by Proposition 3.9. We have proved the theorem! $\square$

The chain of logic used in the proof of Theorem 3.7 is illustrated in the following diagram:

$$\begin{tikzcd}
 & f \arrow[dr, bend left, "\text{Compute Fourier coef. of } f"] & \\
\tilde{f} \arrow[ur, dashed, "\tilde{f} \text{ converge unif. implies } f = f"] & & \{a_0,a_1,b_1,\dots\} \arrow[ll, "\text{Form } \tilde{f}"']
\end{tikzcd}$$

As useful as Theorem 3.7 is, its hypotheses are somewhat cumbersome in that they require you to examine all of the Fourier coefficients of f. Our next result, which is a corollary to Theorem 3.7, gives a much easier condition to check to ensure that f is equal to its Fourier series.

Corollary 3.10. *Let f be continuous and periodic of period 2π. If f is also twice continuously differentiable, i.e., f'' exists and is continuous, then f is equal to its Fourier series.*

Proof. We aim to apply Theorem 3.7 and so it is necessary to compute to f's Fourier coefficients and conclude something about them. Here is the trick: we will integrate by parts and remember that $(-\sin(kx))'=k\cos(kx)$. For any $k=1,2,\dots$,

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(kx)\,dx\\
&= \frac{1}{k\pi}\int_{-\pi}^{\pi}f(x)k\cos(kx)\,dx\\
&= \frac{1}{k\pi}\int_{-\pi}^{\pi}f(x)(\sin(kx))'\,dx\\
&= \frac{1}{k\pi}\left(f(x)(\sin(kx))\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}f'(x)(\sin(kx))\,dx\right).
\end{aligned}$$

Recalling that f is periodic of period 2π, $f(\pi) = f(-\pi)$ and because this is also true of $\sin(kx)$, the first term in the equation above must vanish. Therefore,

$$a_k = -\frac{1}{k\pi}\int_{-\pi}^{\pi} f'(x)\sin(kx)\,dx$$

So we've only taken one derivative of f and we can integrate by parts again to see that

$$\begin{aligned} a_k &= -\frac{1}{k\pi}\int_{-\pi}^{\pi} f'(x)\sin(kx)\,dx \\ &= \frac{1}{k^2\pi}\int_{-\pi}^{\pi} f'(x)(-k\sin(kx))\,dx \\ &= \frac{1}{k^2\pi}\int_{-\pi}^{\pi} f'(x)(\cos(kx))'\,dx \\ &= \frac{1}{k^2\pi}\left(f'(x)(\cos(kx))\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f''(x)(\cos(kx))\,dx\right). \end{aligned}$$

But because f is periodic of period 2π, it follows that f' is periodic of period 2π (you should check this using the definition of the derivative). Consequently $f'(\pi) = f'(-\pi)$ and the first term in the last line above is necessarily 0. It follows then that

$$a_k = -\frac{1}{k^2\pi}\int_{-\pi}^{\pi} f''(x)\cos(kx)\,dx.$$

Now, because we assume that f'' is continuous, it is Riemann integrable and we can use what you know about Riemann integration to see that

$$|a_k| = \frac{1}{k^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos(kx)\,dx.\right| \leq \frac{1}{k^2\pi}\int_{-\pi}^{\pi}|f''(x)\cos(kx)|\,dx \leq \frac{1}{k^2\pi}\int_{-\pi}^{\pi}|f''(x)|\,dx.$$

But the integral in the last term is simply a positive (fixed) number. We can therefore define the number $M = \int_{-\pi}^{\pi}|f''(x)|\,dx/\pi < \infty$ and observe that

$$|a_k| \leq \frac{M}{k^2}$$

for all $k \in \mathbb{N}$. Similarly, and you should work out the details, $|b_k| \leq M/k^2$ for $k \in \mathbb{N}$.

Looking back at Proposition 2.15 for motivation, let's write

$$\tilde{f}(x) = a_0 + \lim_{n\to\infty}\sum_{k=1}^{n} g_k(x) = \sum_{k=1}^{\infty} g_k(x),$$

where $g_k(x) = a_k\cos(kx) + b_k\sin(kx)$. Considering the inequalities proved in the previous paragraph,

$$|g_k(x)| \leq |a_k\cos(kx)| + |b_k\sin(kx)| \leq |a_k| + |b_k| \leq \frac{2M}{k^2} := M_k$$

and, in view of Exercise 2.5, we know that the series $\sum M_k$ converges! Consequently, Proposition 2.15 ensures that the series defining $\tilde{f}$ converges uniformly and so we have met the hypotheses of Theorem 3.7. Thus $f = \tilde{f}$ as was asserted. $\square$

3.3 Arbitrary periods and periodic extensions

It isn't surprising that natural periodic phenomena are rarely periodic of period 2π. Here, we consider functions of period $p = 2L$ and their Fourier series. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period $2L$. We define its Fourier coefficients by

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,} \tag{14}$$

$$\boxed{a_k = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi k x}{L}\right) f(x)\,dx} \tag{15}$$

and

$$\boxed{b_k = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi k x}{L}\right) f(x)\,dx} \tag{16}$$

for $k = 1, 2, \dots$. The Fourier series for f is then defined by

$$\tilde{f}(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{\pi k x}{L}\right) + b_k \sin\left(\frac{\pi k x}{L}\right)\right)$$

for any $x \in \mathbb{R}$ for which the series converges. By simply writing the series for $\tilde{f}$, I do not mean to assert that the series converges in any way at any point. However, if we translate the results of the last section into our new $2L$-periodic situation, if the series converges, then f is a good candidate for the limit. That is, if the series defining $\tilde{f}$ converges at x, we have good reason to believe that $\tilde{f}(x) = f(x)$.

Exercise 3.11. Adjust the statement of Corollary 3.10 to account for periodic functions of period $2L$. Write it down. What are the hypotheses and what is the conclusion?

As we noted before, when a periodic function f is represented by its Fourier series, the trigonometric polynomials,

$$P_n(x) = a_0 + \sum_{k=1}^{n}\left(a_k \cos\left(\frac{\pi k x}{L}\right) + b_k \sin\left(\frac{\pi k x}{L}\right)\right),$$

approximate $f(x)$ for $x \in \mathbb{R}$. In practice, you will usually only be interested in approximating $f(x)$ for $-L < x \le L$ (or some nearby interval). Of course, when $\{P_n(x)\}$ converges to $f(x)$ for all $-L < x \le L$, then $\{P_n(x)\}$ converges to $f(x)$ for all $x \in \mathbb{R}$ by virtue of periodicity. So it suffices to only pay attention to convergence on the domain $-L < x \le L$. Let's zoom in:

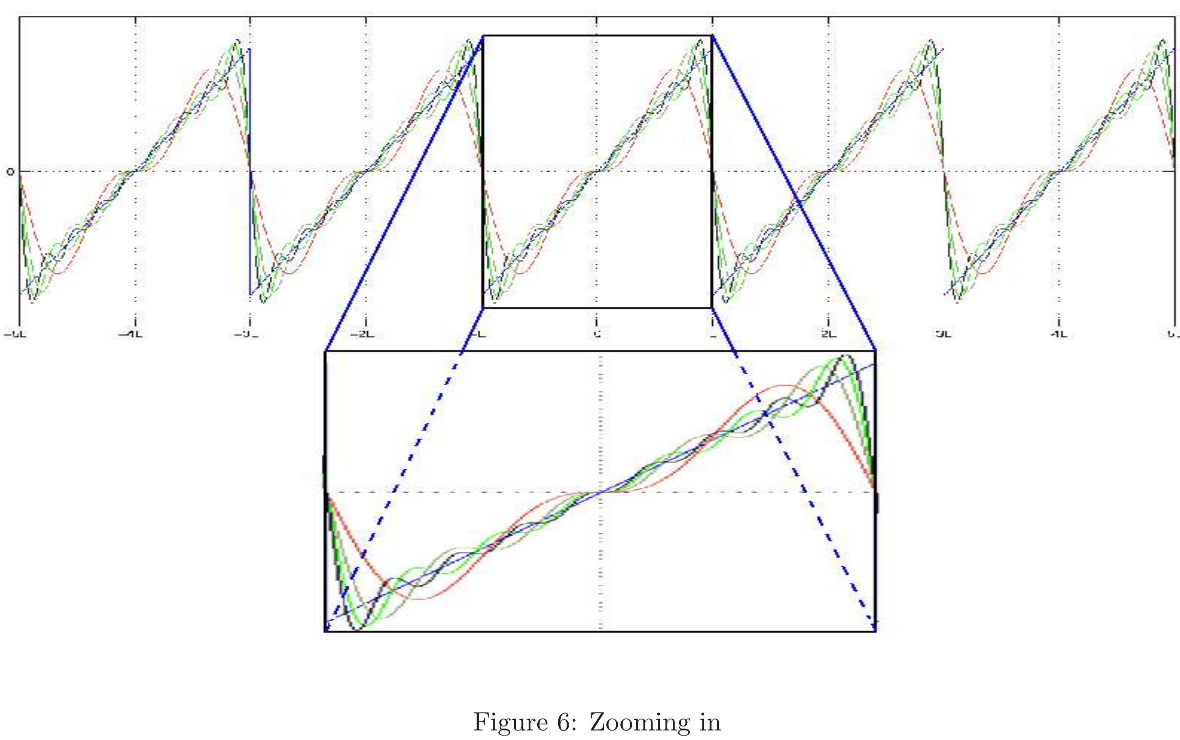


Figure 6: Zooming in

When looking at the zoomed-in window, periodicity becomes irrelevant. So we can change our perspective and consider a continuous function $f : (-L, L] \to \mathbb{R}$. Can we approximate f by trigonometric polynomials on $(-L, L]$? Sure, we can simply forget about periodicity and proceed as usual. Define the Fourier coefficients of f by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_k = \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{\pi k x}{L}\right) f(x)\,dx \;\text{ and }\; b_k = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{\pi k x}{L}\right) f(x)\,dx$$

for $k = 1, 2, \dots$ and its Fourier series by

$$\tilde{f}(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{\pi k x}{L}\right) + b_k\sin\left(\frac{\pi k x}{L}\right)\right)$$

for any x for which the series converges. If the series converges, say uniformly, then $\tilde{f}$ is a periodic function of period $2L$. In fact, the series defining $\tilde{f}$ converges to the *periodic extension of* f, $f_p : \mathbb{R} \to \mathbb{R}$ defined by the rule:

$$f_p(x) = f(x - n2L)$$

when $n \in \mathbb{Z}$ is such that $-L < x - n2L \le L$. In particular, the series defining $\tilde{f}$ converges to f on the interval $(-L, L]$. Because $(-L, L]$ was the only interval we cared about in the first place, the converges on $\mathbb{R}$ isn't really of interest nor is the periodic nature of $\tilde{f}$.

Exercise 3.12. Prove that f_p is an extension of f by showing that $f_p(x) = f(x)$ whenever $-L < x \le L$. Also, show that f_p is periodic of period $2L$.

We make one final remark. The theory developed in the previous two subsections relied on the periodic function of interest being continuous on the entire real line. When we consider a continuous function $f : (-L, L] \to \mathbb{R}$, the periodic extension f_p (defined above) will necessarily be continuous on the open set

$$\bigcup_{n\in\mathbb{Z}}((n-1)L, (n+1)L) = \cdots \cup (-5L, -3L) \cup (-3L, -L) \cup (-L, L) \cup (L, 3L) \cup (3L, 5L) \cup \cdots$$

but it will often be discontinuous at the breakpoints $\{\cdots, -5L, -3L, -L, L, 3L, 5L, \cdots\} = \{(2n+1)L\}_{n\in\mathbb{Z}}$. This happens whenever

$$\lim_{x\to -L^+} f(x) \ne \lim_{x\to L^-} f(x) = f(L).$$

Hence the periodic extension that the Fourier polynomials (which are all continuous) are approximating is often discontinuous (even when f wasn't). This is precisely the situation that's going on in Figure 6. In the figure, $f(x) = x$ for $-L < x \le L$ and since

$$-L = \lim_{x\to -L^+} f(x) \ne \lim_{x\leftarrow L^-} f(x) = f(L) = L,$$

the periodic extension of f is discontinuous and so the Fourier polynomials for f are trying to converge to a discontinuous function. In these situations, our results about convergence (Theorem 3.7 and Corollary 3.10) don't apply. We will have to develop more theory to handle this behavior. Fortunately for us, there are interesting things going on at these points of discontinuity. One of these behaviors is called the Gibbs phenomenon and we will study in at the end of the course.

Exercise 3.13 (Half-range expansions). Let $f : [0, L] \to \mathbb{R}$ be continuous and consider the odd extension of f, $f_o : (-L, L] \to \mathbb{R}$, defined by

$$f_o(x) = \begin{cases} f(x) & \text{when } 0 \le x \le L \\ f(-x) & \text{when } -L < x < 0. \end{cases}$$

You should check that f_o is an odd function, i.e., $f_o(-x) = -f_o(x)$ for all $-L < x \le L$. Use the formulas (14), (15) and (16) to show that the Fourier coefficients of f_o satisfy $a_k = 0$ for all $k = 0, 1, 2, \dots$ and

$$b_k = \frac{2}{L}\int_0^L \sin\left(\frac{\pi k x}{L}\right) f_o(x)\,dx = \frac{2}{L}\int_0^L \sin\left(\frac{\pi k x}{L}\right) f(x)\,dx$$

for all $k = 1, 2, \dots$. You will need to use the fact that f_o is odd. Thus the Fourier series for f_o, which we denote by $\tilde{f}$ by an abuse of notation because we are only interested in what's going on on the interval $[0, L]$, is given by

$$\tilde{f}(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k x}{L}\right);$$

this is called the *half-range* Fourier expansion of f.

4 Applications

I originally planned to fill this section with three applications of Fourier series. This winter proved to be rougher than, I think, anyone expected and the seminar was canceled quite a few times due snow and subzero (F) temperatures. Thus, we were I was only able to cover one application of Fourier Series: Solving the heat equation. I hope (and expect) that you will see much more applications of Fourier series in the future; nature is brimming with such applications!

4.1 A taste of multivariable calculus

Up to this point in calculus, you have studied real-valued functions of one real variable, i.e., functions $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is often an interval. In the real world, you will more often be confronted with real-valued functions of many variables. For instance, the study of the flow of heat is concerned with a function $u(t, x, y, z)$ mapping from the set

$$[0, \infty) \times \mathbb{R}^3 = \{(t, x, y, z) : t \geq 0 \text{ and } x, y, z \in \mathbb{R}\},$$

often called space-time, to the set of real numbers $\mathbb{R}$. The function $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ gives you the temperature $u(t, x, y, z)$ at time t and at spatial location (x, y, z). You can think of (x, y, z) as the coordinates of a point in the room in which you are now sitting and t is the number of seconds since you started reading this paragraph.

To simplify things, we will focus here on functions of two variables. Thus we will consider functions $f(t, x)$ mapping from some subset A of $\mathbb{R}^2$ into $\mathbb{R}$. Here are some concrete examples:

- $f(t, x) = 4t + 5x$ defined for $(t, x) \in \mathbb{R}^2$.
- $f(t, x) = (x - t)^2 + \tan(t)$ defined for $(t, x) \in \mathbb{R}^2$.
- $f(t, x) = e^{-4t} \sin(2x)$ defined for $(t, x) \in [0, \infty) \times \mathbb{R}$.

We can plot them too. Below is the graph of the function $f(t, x) = e^{-4t} \sin(2x)$ for $(t, x) \in [0, 1] \times [0, 2\pi]$.

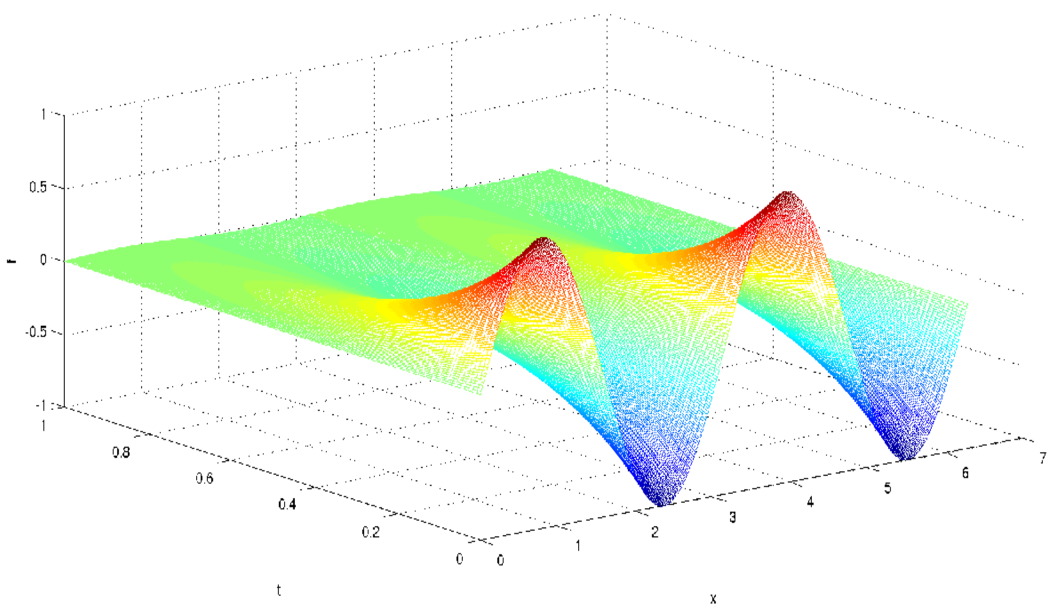


Figure 7: $f(t, x) = e^{-4t} \sin(2x)$

Let's talk about derivatives. Remember, for a function g of one variable, the derivative of g at x was defined by

$$\frac{dg}{dx}(x) = g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

when the limit existed. In this case we said that g was differentiable at x. You also learned nice rules for differentiating functions. In the multivariable world, we can also differentiate functions. We won't discuss this process in its full generality, you will learn this in Calc 3, we will only talk about partial derivatives. To this end, let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ where we will assume that A is a set of the form $A = (a, b) \times (c, d)$ called a *rectangle*. At a point $(t_0, x_0) \in A$, let's consider the limit

$$\lim_{h\to 0} \frac{f(t_0 + h, x_0) - f(t_0, x_0)}{h}.$$

This limit is analogous to that defining g' above; here we simply keep x_0 fixed and then differentiate with respect to the variable t at t_0. When this limit exists we call it the *partial derivative of f with respect to t at* (x_0, t_0) and write

$$\frac{\partial f}{\partial t}(t_0, x_0) := \lim_{h\to 0} \frac{f(t_0 + h, x_0) - f(t_0, x_0)}{h}.$$

Similarly, the *partial derivative of f with respect to x at* (x_0, t_0) is defined by

$$\frac{\partial f}{\partial x}(t_0, x_0) = \lim_{h\to 0} \frac{f(t_0, x_0 + h) - f(t_0, x_0)}{h}$$

when the limit exists. If both limits exist, we say that f is *differentiable at* (t_0, x_0). Partial derivatives are computed exactly like the derivatives of single-variable calculus. When say computing $\partial f/\partial t$, you simply need to keep the x variable fixed.

Let's consider an example: Again, let's work with the function $f(t, x) = e^{-4t}\sin(2x)$. Observe that

$$\frac{\partial f}{\partial t}(t, x) = \lim_{h\to 0} \frac{e^{-4(t+h)}\sin(2x) - e^{-4t}\sin(2x)}{h} = \left(\lim_{h\to 0} \frac{e^{-4(t+h)} - e^{-4t}}{h}\right)\sin(2x).$$

Noting that the term in parentheses is simply the derivative of $g(t) = e^{-4t}$, we have

$$\frac{\partial f}{\partial t}(t, x) = -4e^{-4t}\sin(2x).$$

We could have avoided direct computation of the limit if we had noted that, for fixed x, $\sin(2x)$ is simply a constant and hence differentiating f with respect to t just then boils down to differentiating a constant times e^{-4t}. You try:

Exercise 4.1. Find the derivatives $\partial f/\partial t$ and $\partial f/\partial x$ at (t, x) of the following functions:

1. $f(t, x) = e^{-4t}\sin(2x)$ (we found $\partial f/\partial t$ above)
2. $f(t, x) = x^2 - t^2$
3. $f(t, x) = 4(t - x)^2$
4. $f(t, x) = g(t)h(x)$ where g and h are differentiable functions of one variable.

In the remainder of this subsection, we discuss higher order partial derivatives. Recall that, for a differentiable function g of one variable, its derivative g' is another function and we can talk about its derivative. That is, we can talk about $g''(x) = (g'(x))' = d(g')/dx$. Similarly, we can talk about higher-order partial derivatives when they exist. Let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ be differentiable. When it exists, we define the *second partial derivative* of f with respect to t by

$$\frac{\partial^2 f}{\partial t^2}(x, t) = \frac{\partial}{\partial t}\frac{\partial f}{\partial t}(t, x) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial t}(t + h, x) - \frac{\partial f}{\partial t}(t, x)}{h}.$$

Similarly, the *second partial derivative of f with respect to x* (when it exists) is defined by

$$\frac{\partial^2 f}{\partial x^2}(x,t) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(t,x) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(t,x+h) - \frac{\partial f}{\partial x}(t,x)}{h}.$$

You can also define mixed partial derivatives $\partial^2 f/\partial t\partial x$ and $\partial^2 f/\partial x\partial t$ and even higher order partial derivatives like $\partial^3 f/\partial x^3$. It turns out that these things are all of great use. We won't consider any more partial derivatives here because we don't have need for them. Don't worry, you'll get to study them in Calc 3.

Exercise 4.2. Find the second order partial derivatives $\partial^2 f/\partial t^2$ and $\partial^2 f/\partial x^2$ of all the f's in Exercise 4.1. For fun, compute the mixed partial derivatives $\partial^2 f/\partial x\partial t$ and $\partial^2 f/\partial t\partial x$ for the same functions. Notice anything?

4.2 The heat equation

Consider a rod of length L made of a thermally homogeneous material. Suppose that at time $t=0$, the initial temperature of the rod is known and is given by $u(0,x) = u_0(x)$ for all $0 \le x \le L$. Think of the function $u_0 : [0,L] \to \mathbb{R}$ as some function that is known experimentally at $t=0$. Let's assume that, for $t>0$, the ends of the rod are attached to something (a thermal bath) that keeps their temperature fixed at 0 for all $t>0$. That is, $u(t,0) = u(t,L) = 0$ for all $t>0$. This set-up is depicted in Figure 8.

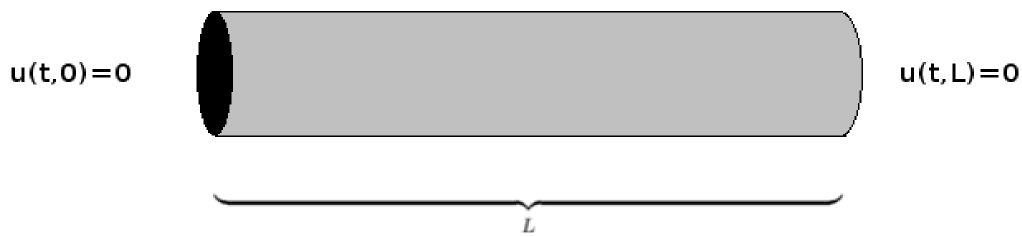


Figure 8: A rod of length L

The big question in the study of heat conduction is:

Question 4.3. *If we know the thermal properties of the rod, can we find the temperature of the rod, $u(t,x)$, for all $t>0$ and all $0 \le x \le L$?*

This question was one of the main motivations in Fourier's treatise, *Théorie analytique de la chaleur* (Analytic theory of heat). It is known, both experimentally and theoretically (see [9] for a clear derivation from first principles), that the temperature in the rod satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for $t>0$ and $0<x<L$, where $c>0$ is a constant that depends on the rod's thermal conductivity, density and specific heat. The value of c can be determined experimentally. So given our set-up, Question 4.3 can be restated as follows:

Problem 4.4. *Let $u_0 : [0,L] \to \mathbb{R}$ and $c>0$. Find a function $u : [0,\infty)\times[0,L] \to \mathbb{R}$ that satisfies*

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{17}$$

for $t>0$ and $0<x<L$,

$$u(t,0) = u(t,L) = 0 \tag{18}$$

for all $t > 0$ *and*

$$u(0,x) = u_0(x) \tag{19}$$

for all $0 \leq x \leq L$.

The equation (17) is called the *heat equation* or *diffusion equation*. It is a partial differential equation (PDE) in space and time. You can devote your entire life to studying this equation and equations like it! PDE's are usually paired with conditions like (18) and (19); they are called *boundary conditions* and *initial conditions* respectively. The rest of this section is devoted to solving Problem 4.4 and we will use Fourier series to do it!

Step 1. (Separation of variables)

Partial differential equations are notoriously difficult to solve. The known techniques for producing solutions are often heavily dependent on the domain of interest. In our case, this is the domain $[0,\infty) \times [0,L]$. We will seek solutions by the method of separation of variables. The basic idea is to assume that a solution of (17) is of the form

$$f(t,x) = g(t)h(x) \tag{20}$$

for $(t,x) \in [0,\infty) \times [0,L]$, where $g : [0,\infty) \to \mathbb{R}$ is continuously differentiable, denoted $g \in C^1[0,\infty)$, and $h : [0,L] \to \mathbb{R}$ is twice continuously differentiable, denoted $h \in C^2[0,L]$. In this way, we will reduce our PDE to a pair of ODE's which are easier to solve. Let's insert f into (17) and see what happens. We have,

$$h(x)g'(t) = \frac{\partial}{\partial t} g(t)h(x) = c^2 \frac{\partial^2}{\partial x^2} g(t)h(x) = c^2 g(t)h''(x)$$

for $0 \leq t$ and $0 \leq x \leq L$; here we have written

$$\frac{\partial g}{\partial t}(t) = \frac{dg}{dt}(t) = g'(t) \quad \text{and} \quad \frac{\partial^2 h}{\partial^2 x}(x) = \frac{d^2 h}{dx^2}(x) = h''(x).$$

Consequently,

$$\frac{g'(t)}{c^2 g(t)} = \frac{h''(x)}{h(x)}$$

for $(t,x) \in [0,\infty) \times [0,L]$. One now makes the observation that, because x and t have nothing, a priori, to do with each other (this takes into account that the domain is a rectangle), both sides of the above equation must be equal to a constant, say $-m^2$ (see Remark 4.5 below). This gives us two ODE's:

$$g'(t) = -m^2 c^2 g(t) \tag{21}$$

for $t \geq 0$ and

$$h''(x) = -m^2 h(x) \tag{22}$$

for $0 \leq x \leq L$. Solving (21) is simple, its solution is

$$g(t) = g(0)\exp(-(cm)^2 t)$$

for $t \geq 0$.

Remark 4.5. If $-m^2$ had been replaced by a positive constant, for a fixed x with $h(x) \neq 0$, the solution $f(t,x) \to \pm\infty$ as $t \to \infty$; this can't be reconciled with empirical observations of temperature, i.e., there is no reason to expect that temperature become infinitely hot as time progresses. Such constants and their corresponding solutions are rejected on physical grounds.

Now let's solve (22). Its solution is of the form

$$h(x) = A\cos(mx) + B\sin(mx)$$

for $0 \leq x \leq L$ (you will study this equation when you take a course on ordinary differential equations). We note that the boundary condition (18) (combined with (20)) forces $h(0) = 0 = h(L)$; these are boundary conditions for h. One one hand,

$$0 = h(0) = A\cos(m0) + B\sin(m0) = A(1) + B(0) = A$$

and therefore $h(x) = B\sin(mx)$. On the other hand,

$$0 = h(L) = B\sin(mL),$$

but since we seek non-trivial solutions, i.e., solutions for which $h(x) \neq 0$ for all x and hence $B \neq 0$, we are forced to conclude that $mL = k\pi$ for some integer $k \in \mathbb{N}^+ = \{1, 2, 3, \dots\}$ because only then, $h(L) = B\sin(mL) = B\sin(k\pi) = 0$. Putting everything back together, we obtain

$$f(t, x) = g(0)Be^{-(cm)^2 t}\sin(mx) = g(0)B\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right)$$

for $(t, x) = [0, \infty) \times [0, L]$. But, in fact, this method gave us a collection of solutions for $k \in \mathbb{N}^+$, namely

$$f_k(t, x) = b_k\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right)$$

for $(t, x) = [0, \infty) \times [0, L]$ and $k \in \mathbb{N}^+$; here b_k are constants to be determined. Looking back, Figure 7 shows the solution for $k = 2$ where, $b_k = 1$, $c = 1$ and $L = \pi$.

Exercise 4.6. Verify directly that, for all $k \in \mathbb{N}^+$, $f_k(t, x)$ is a solution to (17) and satisfies (18).

Step 2. (The principle of superposition)

Let's begin by making an observation about (17):

Proposition 4.7. *(The principle of superposition) If $u(t, x)$ and $v(t, x)$ are solutions to* (17) *for $(t, x) \in [0, \infty) \times [0, L]$, then $(u + v)(t, x)$ is also a solution to* (17) *for $(t, x) \in [0, \infty) \times [0, L]$.*

Proof. This simply follows from the fact that the derivative of a sum is the sum of derivatives. Observe that

$$\frac{\partial}{\partial t}(u + v)(t, x) = \frac{\partial u}{\partial t}(t, x) + \frac{\partial v}{\partial t}(t, x) = c^2\frac{\partial^2 u}{\partial x^2}(t, x) + c^2\frac{\partial^2 v}{\partial x^2}(t, x) = c^2\frac{\partial^2}{\partial x^2}(u + v)(t, x)$$

for all $(t, x) \in [0, \infty) \times [0, L]$. □

Using the proposition and our knowledge that the functions $\{f_k(t, x)\}_{k=1}^{\infty}$ all satisfy (17), we can conclude that the (partial) sums

$$S_n(t, x) = \sum_{k=1}^{n} f_k(t, x) = \sum_{k=1}^{n} b_k\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right)$$

for $n \in \mathbb{N}^+$ all satisfy (17). What's more, the partial sums $S_n(t, x)$ also satisfy the boundary conditions (18)–this is easy to see (you should check it). So now we have a collection of partial sums, each satisfying (17). It seems reasonable to ask about the limit of these partial sums. Specifically, we'd like to ask if

$$S(t, x) = \sum_{k=1}^{\infty} b_k\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right), \tag{23}$$

when it converges, also satisfies (17)? In general, this is quite a delicate question because verification of (17) requires you to differentiate the series term-by-term. This question is reminiscent of those considered in Subsection 2.3. Fortunately, for the specific series above, this isn't a problem if the coefficients $\{b_k\}$ are uniformly bounded. This is due to fact that $\exp(-k^2c^2\pi^2/L^2t)$ decays fantastically fast for any fixed $t > 0$ as $k \to \infty$. We state it as a fact, the specific details can be found in [8].

Fact 4.8. *Suppose that, for some $M > 0$, $|b_k| \leq M$ for all $k \in \mathbb{N}^+$. Then, for any $t > 0$, the series* (23) *converges uniformly for all $0 \leq x \leq L$. Moreover, it can be differentiated term-by-term.*

We shall henceforth make the assumption that $|b_k| \leq M$ for all $k \in \mathbb{N}^+$ for some $M > 0$. In this case, the series (23) need not converge for $t = 0$ but, for any $t > 0$, it converges uniformly in x as the fact guarantees. Employing Fact 4.8,

$$\begin{aligned}\frac{\partial S}{\partial t}(t,x) &= \sum_{k=1}^{\infty} b_k \frac{\partial}{\partial t}\left(\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right)\right)\\ &= \sum_{k=1}^{\infty} b_k c^2 \frac{\partial^2}{\partial x^2}\left(\sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right)\right)\\ &= c^2\frac{\partial^2 S}{\partial x^2}(t,x)\end{aligned}$$

for $t > 0$ and $0 \leq x \leq L$. Therefore, $S(t,x)$ satisfies (17)! It is clear that $S(t,x)$ also satisfies the boundary conditions (18).

Remark 4.9. The astute reader will note that I've been a bit careless concerning differentiability and the satisfaction of (17) on the boundary of $[0,\infty) \times [0,L]$, $\{(t,x) : t = 0 \text{ or } x = 0, L\}$. You should take differentiability at the boundary to mean the existence of the appropriate left or right derivative. In (17), we only required differentiability and the satisfaction of (17) on the interior of $[0,\infty) \times [0,L]$, $(0,\infty) \times (0,L)$. However, by the method of separation of variables, we produced the collection of functions $\{f_k(t,x)\}_{k=1}^{\infty}$ which were differentiable and satisfied (17) on the entire domain $[0,\infty) \times [0,L]$. You should note that Fact 4.8 does not guarantee this beloved differentiability when $t = 0$, even though all of the partial sums had it. This is simply due to the fact that things can go bad when you take limits. All is not lost, for we only really need differentiability on the interior, in particular, for $t > 0$ as this was all that was required in (17) in the original problem.

Step 3. (Putting things together)

So far, we have produced a very general solution,

$$S(t,x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right),$$

to (17) for $t > 0$ and $0 \leq x \leq L$ provided $|b_k| \leq M$ for $k \in \mathbb{N}^+$, where $M > 0$. This solution also satisfied the boundary conditions (18). What we haven't yet dealt with is the initial condition (19), at $t = 0$. Observe that

$$S(0,x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}0\right) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)$$

for $0 \leq x \leq L$. Behold, this is simply a trigonometric series! What's more, it naturally appeared by solving (17). Since we have yet to choose the constants $\{b_k\}$ and, to solve Problem (4.4), we need our desired solution to satisfy $u(0,x) = u_0$, it seems only reasonable to represent u_0 by its half-range Fourier series. So let's do it: set

$$b_k = \frac{2}{L}\int_0^L \sin\left(\frac{\pi k x}{L}\right) u_0(x)\,dx \tag{24}$$

for $k \in \mathbb{N}^+$ and

$$u(t,x) = S(t,x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)\exp\left(-\frac{k^2c^2\pi^2}{L^2}t\right) \tag{25}$$

for $t \geq 0$ and $0 \leq x \leq L$. We then have the following theorem:

Theorem 4.10. *Suppose that $u_0 : [0, L] \to \mathbb{R}$ is Riemann-integrable and let $c > 0$. Define the constants b_k by* (24) *for $k \in \mathbb{N}^+$. If, for any reason whatsoever, u_0 is representable by its half-range expansion, i.e.,*

$$u_0(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi k x}{L}\right)$$

for $0 \le x \le L$, then $u(t, x)$, defined by (25)*, is a solution to Problem 4.4.*

Proof. Given the hypotheses, it is obvious that $u(0, x) = u_0(x)$ for $0 \le x \le L$ and so (19) is satisfied. To ensure that $u(t, x)$ satisfies (17) and (18), we must simply check that the coefficients $\{b_k\}$ are uniformly bounded. For any $k \in \mathbb{N}^+$, the hypothesis that u_0 is Riemann-integrable guarantees that

$$|b_k| \le \frac{2}{L} \int_0^L \left| \sin\left(\frac{\pi k x}{L}\right) u_0(x) \right| \, dx \le \frac{2}{L} \int_0^L |u_0(x)| \, dx = M < \infty$$

which is exactly what we needed to show. □

The previous theorem's hypothesis that the initial temperature distribution u_0 is representable by its (half-range) Fourier series is mysterious and somewhat unsatisfying. As we've discussed before, one can't simply look at u_0 and know whether or not this condition is true. This bothered Fourier! He wanted to know exactly which functions could be represented by their Fourier series (although he didn't call them by this name) because it was for these functions that Problem 4.4 could be solved.

Exercise 4.11. Find a condition on u_0 which ensures that its odd extension to $(-L, L]$ has a periodic extension to $\mathbb{R}$ which satisfies the hypotheses of Corollary 3.10 (or more precisely, your answer to Exercise 3.11). Using the condition and the result of Corollary 3.10, state a theorem analogous to Theorem 4.10 which achieves the same conclusion but has your condition as a hypothesis. This is a very difficult exercise!

5 Convergence revisited

In this section, we return to talking about convergence of Fourier series. In contrast to our study of uniform convergence of Fourier series (which was fruitful but limited) in Section 2 although this time, we will focus on pointwise convergence (in contrast to our study of the uniform convergence of Fourier series in Section 2). We will also (and finally) discuss the Gibbs phenomenon.

Before we begin to develop the machinery necessary to prove two very nice results about the pointwise convergence of Fourier series, Theorems 5.19 and 5.21, we should ask the inevitable questions: Why work so hard? Why is the pointwise convergence of Fourier series a non-trivial question? As the two theorems below show, things sometimes go very very wrong.

Theorem 5.1 (Du Bois-Reymond 1876). *There exists a continuous function whose Fourier series diverges at a point.*

Theorem 5.2 (Kolmogorov 1926). *There exists a function (Lebesgue integrable but not Riemann integrable) whose Fourier series diverges at every point.*

In light of the above theorems, we see that the pointwise convergence of Fourier series must be non-trivial and so we embark on a quest to find some function which are representable by their Fourier series. I'll remind you that this was Fourier's quest too. We begin our study in Section 5.1 where we introduce some notions from Linear algebra and proof two very important results: Bessel's inequality and the Riemann-Lebesgue lemma. In Section 5.1, we will also discuss (but not prove) Parseval's theorem and use it to solve Basel's problem; this is the problem that made Leonhard Euler instantly famous. In section 5.2, we will study the Dirichlet kernel and learn about using integration to approximate functions. This section is quite technical but the results therein will allow us to prove some very strong results. For instance, we will show that any function which is 2π-periodic and differentiable (everywhere) is equal to its Fourier series; this result is seen to improve upon our results from Section 2. In the final subsection, Subsection 5.3, we study the Gibb's phenomenon.

We return to discussing functions of period 2π. As we know, functions of arbitrary periods can be handled by a rescaling. We will adopt the following notation throughout:

Notation 5.3. We denote by $\mathcal{R}$, the set of 2π-periodic Riemann-integrable functions $f:\mathbb{R}\to\mathbb{R}$. In symbols,

$$\mathcal{R}=\{f:\mathbb{R}\to\mathbb{R} : f \text{ is } 2\pi\text{-periodic and Riemann integrable}\}.$$

We recall that, by definition, each $f\in\mathcal{R}$ is bounded.

5.1 Orthonormality and the Riemann-Lebesgue Lemma

The present subsection focuses on things you will later see in a course on linear algebra. I shall only present here the very essentials needed to discuss the convergence of Fourier series. Linear algebra is the study of things called vectors (like $(1,0,0)\in\mathbb{R}^3$) and special functions which map vectors to vectors, called linear operators. You should be sure to take as many classes in Linear algebra (and functional analysis) as you can. It is a truly beautiful, fascinating and far-reaching subject. In our setting, the "vectors" are simply 2π-periodic Riemann integrable functions (this is the set $\mathcal{R}$).

We first discuss a type of product of vectors which generalizes the dot product. For any two $f,g\in\mathcal{R}$, we define their *inner product* by

$$\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx;$$

note that this is just a number. The *norm*, or 2-norm, of f is defined to be

$$\|f\|_2=\left(\langle f,f,\rangle\right)^{1/2}=\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx\right)^{1/2}.$$

Exercise 5.4. Compute the following:

- $\langle \cos x, \sin x\rangle$
- $\langle \cos x, \cos 2x\rangle$
- $\|\cos x\|_2$
- $\|\sin x\|_2$

Proposition 5.5. *Let $f, g, h \in \mathcal{R}$ let α be a constant. The following properties hold:*

1. $$\langle f, g\rangle = \langle g, f\rangle$$
2. $$\langle \alpha f, g\rangle = \alpha\langle f, g\rangle = \langle f, \alpha g\rangle$$
3. $$\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$$
4. $$\langle f, h+g\rangle = \langle f, h\rangle + \langle f, g\rangle$$
5. $$\langle f, f\rangle \geq 0$$

Note that the last property ensures that $\|f\|_2$ is sensible.

Exercise 5.6. Prove the previous proposition. All it requires are facts you already know about the Riemann integral; just keep the factor of $1/\pi$ out in front.

Definition 5.7 (Orthonormal collection)**.** *Let $\{\phi_1, \phi_2, \dots\}$ be a collection of Riemann integrable 2π-periodic functions. We say that this collection is orthonormal if*

$$\langle \phi_i, \phi_j\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi_i(x)\phi_j(x)\,dx = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

The following proposition is an immediate consequence of Fact 3.3.

Proposition 5.8. *The collection $\{\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$ is orthonormal.*

Given any orthonormal collection $\{\phi_1, \phi_2, \dots\}$ and Riemann integrable 2π-periodic function f, we define the *generalized Fourier coefficients* of f by

$$c_k = \langle f, \phi_k\rangle$$

for $k = 1, 2, \dots$. Our next theorem is paramount.

Theorem 5.9 (Bessel's inequality)**.** *Let $\{\phi_1, \phi_2, \dots\}$ be an orthonormal collection, let $f \in \mathcal{R}$ and c_k be the generalized Fourier coefficients of f as defined above. Then*

$$\sum_{k=1}^{\infty} c_k^2 \leq \|f\|_2^2.$$

The inequality above is called Bessel's inequality. You should note that the theorem, in addition to asserting the validity of Bessel's inequality, also ensures the convergence of the series $\sum c_k^2$.

Proof. Define, for each natural number n,

$$S_n(x) = \sum_{k=1}^{n} c_k \phi_k(x).$$

Using Proposition 5.5, we observe that

$$\begin{aligned} 0 \quad \leq \quad & \langle f - S_n, f - S_n \rangle = \langle f, f \rangle - \langle f, S_n \rangle - \langle S_n, f \rangle + \langle S_n, S_n \rangle \\ & = \|f\|_2^2 - 2\langle f, S_n \rangle + \langle S_n, S_n \rangle. \end{aligned} \tag{26}$$

Now,

$$\langle f, S_n \rangle = \sum_{k=1}^{n} \langle f, c_k \phi_k \rangle = \sum_{k=1}^{n} c_k \langle f, \phi_k \rangle = \sum_{k=1}^{n} c_k^2 \tag{27}$$

where we have used the definition of the generalized Fourier coefficients and Proposition 5.5. Observe also that, for any k,

$$\langle S_n, \phi_k \rangle = \sum_{j=1}^{n} \langle c_j \phi_j, \phi_k \rangle = \sum_{j=1}^{n} c_j \langle \phi_j, \phi_k \rangle = c_k,$$

where we have used the fact that the collection $\{\phi_1, \phi_2, \dots\}$ is orthonormal. It follows that

$$\langle S_n, S_n \rangle = \sum_{k=1}^{n} \langle S_n, c_k \phi_k \rangle = \sum_{k=1}^{n} c_k \langle S_n, \phi_k \rangle = \sum_{k=1}^{n} c_k^2. \tag{28}$$

Combining (26),(27) and (28), we find that

$$0 \leq \|f\|_2^2 - \sum_{k=1}^{n} c_k^2$$

or equivalently,

$$\sum_{k=1}^{n} c_k^2 \leq \|f\|_2^2$$

for any n. Now, upon noting that f is Riemann integrable (which is by definition bounded) and so $\|f\|_2 < \infty$, we find the the series $\sum c_k^2$, being a series of positive numbers, converges and moreover

$$\sum_{k=1}^{\infty} c_k^2 = \lim_{n \to \infty} \sum_{k=1}^{n} c_k^2 \leq \|f\|_2^2$$

as desired. $\square$

Let's write down what the above theorem says about the Fourier coefficients $a_0, a_1, b_1, \dots$ of Section 2 for a Riemann integrable function f. Observe that

$$c_1 = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} f(x)\, dx = \sqrt{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \sqrt{2} a_0$$

and so

$$c_1^2 = 2a_0^2.$$

That's nice. Let's try it for other $c's$. We have

$$c_2 = \langle f, \cos x \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx = a_1$$

and

$$c_3 = \langle f, \sin x\rangle \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = b_1.$$

In general, we find that

$$2a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \sum_{k=1}^{\infty} c_k.$$

Thus, Theorem 5.9 hands us the following corollary:

Corollary 5.10. *Let $f \in \mathcal{R}$ and $a_0, a_1, b_2, a_2, b_2, \dots$ be the Fourier coefficients of f. Then*

$$2a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \|f\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx.$$

Here is another corollary:

Corollary 5.11 (The Riemann-Lebesgue lemma)**.** *For any $f \in \mathcal{R}$,*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$$

and

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0.$$

Proof. In view of the previous corollary, for any $f \in \mathcal{R}$, $\sum a_k^2 < \infty$ and $\sum b_k^2 < \infty$. Consequently, $a_n^2 \to 0$ and $b_n^2 \to 0$ as $n \to \infty$. This is a fact you should recall from the treatment of numerical series in your calculus class. It follows then that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \lim_{n\to\infty}\pi\langle f, \cos nx\rangle = \lim_{n\to\infty}\pi a_n = 0$$

and

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \lim_{n\to\infty}\pi\langle f, \sin nx\rangle = \lim_{n\to\infty}\pi b_n = 0.$$

□

The above result is the last one we will need to talk more precisely about pointwise convergence of Fourier series (which we do in the next section). Before we move on, let's briefly discuss a very important counterpart to Corollary 5.10; this is Parseval's theorem.

Theorem 5.12 (Parseval's Theorem)**.** *For any $f \in \mathcal{R}$, let $a_0, a_1, b_2, a_2, b_2, \dots$ be the Fourier coefficients of f. Then*

$$2a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx. \tag{29}$$

Parseval's theorem is a major result (and it's true for more general functions than simply Riemann integrable functions). It shows that Bessel's inequality is an equality. It's proof isn't actually that complicated; however, we don't have time in this seminar to go in this direction. For a clear proof of Parseval's theorem (which is at the level of these notes), see Chapter 9 of [6].

Example 5.13 (The Basel problem). You know very well that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges (if you don't, go back and do Exercise 2.5). The Basel problem, posed by Pietro Mengoli and solved nearly a century later by Leonhard Euler in 1734 [2], asks: What is the exact value of this series? Euler showed that the value of the series is exactly $\pi^2/6$. Euler's proof involved the consideration of the power series of $\sin(x)$. It turns out that, using Fourier series, we can give a very simple proof by using Parseval's theorem.

Theorem 5.14 (Euler 1734)**.**

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Proof. Let $f(x)=x$ for $-\pi<x\leq\pi$ and extend it periodically to $\mathbb{R}$. Let's compute the Fourier coefficients. We have

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x\,dx=0$$

because x is an odd function. Similarly,

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos(kx)\,dx=0.$$

We now find the coefficients b_k for $k=1,2,\dots$ using integration by parts. We have

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(kx)\,dx=\frac{-1}{k\pi}\int_{-\pi}^{\pi}x(-k\sin(kx))\,dx.$$

We recognize that $-k\sin(kx)=\frac{d}{dx}\cos(kx)$ and therefore

$$\begin{aligned}
b_k &= \frac{-1}{k\pi}\int_{-\pi}^{\pi}x\frac{d}{dx}\cos(kx)\,dx=\frac{-1}{k\pi}\left(x\cos(kx)\Big|_{x=-\pi}^{x=\pi}-\int_{-\pi}^{\pi}(\frac{d}{dx}x)\cos(kx)\,dx\right)\\
&= -\frac{\pi\cos(k\pi)-(-\pi)\cos(k(-\pi))}{k\pi}+\frac{1}{k\pi}\int_{-\pi}^{\pi}1\cdot\cos(kx)\,dx\\
&= -\frac{2\cos(k\pi)}{k}+0=\frac{2(-1)^{k+1}}{k},
\end{aligned}$$

where we have used the fact that the integral of $\cos(kx)$ from $-\pi$ to π is 0 for every k. Thus,

$$b_k^2=\frac{(2(-1)^{k+1})^2}{k^2}=\frac{4}{k^2}$$

for $k\in\mathbb{N}$. This will allow us to compute the left side of (29). To compute the right hand side is simple. We have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x)^2\,dx=\frac{1}{\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi}=\frac{\pi^3-(-\pi)^3}{3\pi}=\frac{2\pi^2}{3}.$$

So, invoking Parseval's theorem and inserting our results into (29), we have

$$\frac{2\pi^2}{3}=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=2a_0^2+\sum_{k=1}^{\infty}(a_k^2+b_k^2)=\sum_{k=1}^{\infty}\frac{4}{k^2}.$$

Multiplying both sides of this equality by $\frac{1}{4}$ gives

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{2\pi^2}{4\cdot 3}=\frac{\pi^2}{6}$$

as desired. $\square$

5.2 The Dirichlet kernel

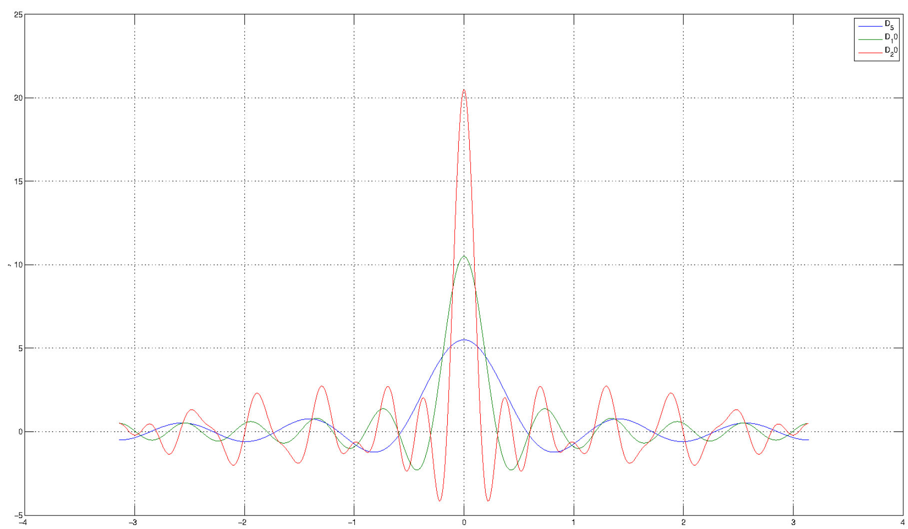
The name of the game in mathematical analysis is approximation. This has been the name of the game for us too–we've been trying to approximate functions by trigonometric polynomials. We begin our discussion here by taking a special function D_n, called a kernel, and integrating it against $f \in \mathcal{R}$ to produce a new function. More precisely, for any $f \in \mathcal{R}$, we consider the function of x defined by

$$x \mapsto \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x-u) f(u)\, du.$$

If we choose D_n to be sufficiently nice, we will often find that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x-u) f(u)\, du = f(x).$$

In this way, we are approximating f. This idea turns out to generalize in a pretty big way and is often called *approximating the identity*. In this subsection, we will define a specific kernel D_n and show that it was really something we've been working with all along.



Fix a natural number n and define

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx)$$

for $x \in \mathbb{R}$. The function $D_n(x)$ is called the *Dirichlet kernel*. In Figure 5.2, we illustrate the graphs of D_n for

$n = 5, 10, 20$. Let's make an extremely useful observation about D_n: For $f \in \mathcal{R}$, we observe that

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x-u)f(u)\,du &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(k(x-u))\right)f(u)\,du \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(ku)\cos(kx)+\sin(ku)\sin(kx)\right)f(u)\,du \\
&= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du\right)+\sum_{k=1}^{n}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ku)f(u)\,du\right)\cos(kx) \\
&\qquad +\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(ku)f(u)\,du\right)\sin(kx) \\
&= a_0+\sum_{k=1}^{n} a_k\cos(kx)+b_k\sin(kx) \\
&= S_n(x)
\end{aligned}$$

where we have used the trigonometric identity $\cos(b-a) = \cos(a)\cos(b) + \sin(a)\sin(b)$ and the linearity of the integral. This is quite a curious formula and it will prove to be extremely useful to us. It says that the n-th Fourier partial sum for f is equivalently defined by integrating f against the Dirichlet kernel, i.e.,

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x-u)f(u)\,du \tag{30}$$

for $x \in \mathbb{R}$. Of course, we want to understand when the $\lim_{n\to\infty} S_n(x) = f(x)$ (this is when the Fourier series converges for f converges to f at x) and for this, for each $n = 1, 2, \dots$, we define the *remainder* by

$$R_n(x) = S_n(x) - f(x)$$

for each $x \in \mathbb{R}$. Our next proposition records some important facts about S_n and R_n.

Proposition 5.15. *For any $f \in \mathcal{R}$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$,*

1. $$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x-u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)\,du = 1,$$

2. $$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)f(x+u)\,du$$

   and

3. $$R_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)(f(x+u)-f(x))\,du$$

Exercise 5.16. Proof the proposition using (30). We started the proof in class, all you need to do is change variables and use the periodicity of f and D_n.

We want to use the above proposition to show that, under certain conditions, the remainder converges to 0 as $n \to \infty$. To this end, we focus on properties of the Dirichlet kernel. Let's play around a little bit: Observe that whenever $\sin(x/2)\neq 0$ (which happens whenever $x/2\pi$ is not an integer),

$$\begin{aligned}
\frac{\sin(x(1+1/2))}{2\sin(x/2)} &= \frac{\sin(x+x/2)}{2\sin(x/2)} = \frac{\sin x\cos x/2+\cos x\sin x/2}{2\sin x/2}\\
&= \frac{\sin(2(x/2))\cos x/2+\cos x\sin x/2}{2\sin x/2} = \frac{2\sin(x/2)\cos(x/2)\cos(x/2)+\cos x\sin x/2}{2\sin x/2}\\
&= \cos^2(x/2)+\frac{1}{2}\cos x = \frac{1+\cos x}{2}+\frac{1}{2}\cos x = \frac{1}{2}+\cos x\\
&= D_1(x).
\end{aligned}$$

where we have simply used the angle-sum trigonometric identities. Consequently, we observe that

$$D_1(x)=\begin{cases}\frac{\sin(x(1+1/2))}{2\sin x/2} & \text{if } x\neq 2\pi m \text{ for } m\in\mathbb{N}\\ 1+1/2 & \text{when } x=2\pi m \text{ for } m\in\mathbb{N}.\end{cases}$$

As it turns out, this generalizes:

Proposition 5.17. *For each n,*

$$D_n(x)=\begin{cases}\frac{\sin(x(n+1/2))}{2\sin x/2} & \textit{if } x\neq 2\pi m \textit{ for } m\in\mathbb{N}\\ n+1/2 & \textit{when } x=2\pi m \textit{ for } m\in\mathbb{N}\end{cases}$$

Exercise 5.18. Prove the proposition. Your proof should use induction in some way. Notice that we have started the induction process by checking the $n=1$ step.

So we've now developed a lot of machinery in this subsection. Luckily, it is precisely the machinery needed to prove a great result about the convergence of Fourier series. This result is a statement about pointwise convergence and, in this sense, it is stronger than our previous theorem, Theorem 3.7. As such, it also is another confirmation of Claim 1.6.

Theorem 5.19. *Let $f\in\mathcal{R}$. If f is differentiable at x_0, then the Fourier series of f converges at x_0 and*

$$f(x_0)=\lim_{n\to\infty}\left(a_0+\sum_{k=1}^n a_k\cos(kx)+b_k\sin(kx)\right)$$

i.e., the Fourier series of f, $\tilde{f}$, is equal to f at x_0.

Proof. We have

$$R_n(x_0)+f(x_0)=S_n(x_0)=a_0+\sum_{k=1}^n(a_k\cos(kx_0)+b_k\sin(kx_0)).$$

So, if we can show that $R_n(x_0)\to 0$ as $n\to 0$, using the fact that the the limit of the sum is the sum of the limits, we can, at once, conclude that Fourier series for f converges and is equal to f. This is our goal. Let's observe that, in view of Propositions 5.15 and 5.17,

$$\begin{aligned}
R_n(x_0) &= \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)(f(x_0+u)-f(x_0))\,du\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(u(n+1/2))}{2\sin(u/2)}(f(x_0+u)-f(x_0))\,du\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(nu)\cos(u/2)+\cos(nu)\sin(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0))\,du\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(nu)\cos(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0))\,du+\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(nu)\sin(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0))\,du\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0))\sin(nu)\,du+\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x_0+u)-f(x_0))\cos(nu)\,du. \qquad (31)
\end{aligned}$$

Now, as $u \mapsto f(x_0+u) - f(x_0)$ is 2π-periodic and Riemann integrable, the Riemann-Lebesgue lemma guarantees that, for the second integral above,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x_0+u)-f(x_0))\cos(nu)\,du = 0.$$

We want to make the same conclusion concerning the first integral in (31) and so it remains to show that

$$g(u) = \frac{\cos(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0))$$

is Riemann integrable (you should check that it is certainly 2π-periodic). We note that g is composed of Riemann integrable functions (because we have assumed that $f \in \mathcal{R}$) and so the only thing that stands in our way is the possibility of things going bad when we divide by 0. More precisely, we are worried that $g(u)$ blows up as $\sin(u/2) \to 0$. We note that, on the interval $[-\pi,\pi]$, $\sin(u/2) \to 0$ only when $u \to 0$ and so we need to examine $g(u)$ near 0. Observe that

$$\lim_{u\to 0} g(u) = \lim_{u\to 0} \frac{\cos(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0)) = \lim_{u\to 0} \cos(u/2)\frac{\sin(u/2)}{u/2}\frac{f(x_0+u)-f(x_0)}{u}.$$

But, because you know that $\sin\theta/\theta \to 1$ as $\theta \to 0$ and we have assumed that f is differentiable at x_0,

$$\lim_{u\to 0} g(u) = \lim_{u\to 0}\frac{f(x_0+u)-f(x_0)}{u} = f'(x_o)$$

and so nothing bad happens to $g(u)$ near 0 because the difference quotients of f at x_0 are necessarily bounded. Therefore g must be Riemann integrable, because it is bounded everywhere and continuous at every point except possibly 0. Thus, we can apply the Riemann-Lebesgue lemma to g from which we conclude that

$$\lim_{n\to\infty} R_n(x_0) = \lim_{n\to 0}\frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin(nu)\,du + \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x_0+u)-f(x_0))\cos(nu)\,du = 0$$

as desired. □

Let's note that we immediately obtain the next corollary (it improves upon Theorem 3.7).

Corollary 5.20. *If f is 2π-periodic and continuously differentiable, then f is equal to its Fourier series.*

Let's show something stronger and, perhaps, more interesting.

Theorem 5.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and piecewise differentiable, i.e., it is continuously differentiable on $[-\pi,\pi]$ except possibly at a finite number of points where f and f' have (at worst) removable or jump discontinuities. For any $x_0 \in \mathbb{R}$, define*

$$f(x_0^+) = \lim_{x\to x_0^+} f(x) \quad \text{and} \quad f(x_0^-) = \lim_{x\to x_0^-} f(x).$$

Then, at each $x_0 \in \mathbb{R}$, the Fourier series for f converges and

$$\frac{f(x_0^+)+f(x_0^-)}{2} = \lim_{n\to\infty} S_n(x_0) = a_0 + \sum_{k=1}^{\infty} a_k\cos(kx_0) + b_k\sin(kx_0).$$

The theorem says that, if f is 2π-periodic and piecewise differentiable then, at all points x_0, the Fourier series for f converges to the average value of the left and right had limits at x_0. It f happens to be continuous at x_0, then this average is simply $f(x_0)$. Perhaps this is best illustrated by an example:

Example 5.22. The sawtooth function from Example 1.5 is a typical 2π-periodic and piecewise differentiable function. You should go back and look at that example now. We observe directly that is continuously differentiable everywhere on the interval $[-\pi,\pi]$ except at the endpoints. In fact, it is continuously differentiable at every point which is not of the form πm for any odd $m\in\mathbb{Z}$. If x_0 is not of the form πm for an odd integer m, then it is continuous there and in particular

$$\frac{f(x_0^+)+f(x_0^-)}{2}=\frac{f(x_0)+f(x_0)}{2}=f(x_0).$$

Thus the theorem guarantees that, at such an x_0,

$$f(x_0)=a_0+\sum_{k=1}^\infty a_k\cos(kx_0)+b_k\sin(kx_0);$$

we however already knew this. This was our conclusion from Theorem 5.19. What's more interesting is what happens at the breakpoints $x_0=\pi m$ where m is an odd integer. By direct calculation using the sawtooth function, we find that $f(x_0^-)=-\pi$ and $f(x_0^+)=\pi$ (look at Figure 2 to confirm this). Consequently, the theorem shows that

$$0=\frac{\pi+-\pi}{2}=a_0+\sum_{k=1}^\infty a_k\cos(kx_0)+b_k\sin(kx_0).$$

This is precisely what's happening in Figure 2, the trigonometric polynomials are all 0 at $0,\pi,3\pi$, etc.

We now prove the theorem.

Proof of Theorem 5.21. As in the proof of the last theorem, our aim is to show that

$$\lim_{n\to\infty}\left(S_n(x_0)-\frac{f(x_0^+)+f(x_0^-)}{2}\right)=0.$$

To this end, upon making similar calculations to those done in the previous proof we find that

$$\begin{aligned}\left(S_n(x_0)-\frac{f(x_0^+)+f(x_0^-)}{2}\right) &= \frac{1}{\pi}\int_0^\pi \frac{f(x_0+u)-f(x_0^+)}{2\sin(u/2)}\sin(u(n+1/2))\,du\\ &\quad+\frac{1}{\pi}\int_{-\pi}^0 \frac{f(x_0+u)-f(x_0^-)}{2\sin(u/2)}\sin(u(n+1/2))\,du. \end{aligned}\tag{32}$$

Now, we consider the integrals one at a time. For the first integral, we expand the integrand using our trigonometric identities to find that

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi \frac{f(x_0+u)-f(x_0^+)}{2\sin(u/2)}\sin(u(n+1/2))\,du &= \frac{1}{\pi}\int_0^\pi \frac{\cos(u/2)}{2\sin(u/2)}(f(x_0+u)-f(x_0^+))\sin(nu)\,du\\ &\quad+\frac{1}{2\pi}\int_0^\pi (f(x_0+u)-f(x_0^+))\cos(nu)\,du\end{aligned}$$

An appeal to the Riemann-Lebesgue lemma shows that the second term in the above equation converges to 0 as $n\to\infty$. As for the first term, we essentially repeat the argument we gave for the proof of Theorem 5.19 while noting that we are only considering positive u's and so the corresponding left hand difference quotients for f at x_0 are bounded by our hypothesis. Hence, as we concluded in the proof of the previous theorem, the Riemann-Lebesgue lemma applies to the first term too and we have

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi \frac{f(x_0+u)-f(x_0^+)}{2\sin(u/2)}\sin(u(n+1/2))\,du=0.\tag{33}$$

By the same argument, using right hand difference quotients, we conclude that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^0 \frac{f(x_0+u)-f(x_0^-)}{2\sin(u/2)}\sin(u(n+1/2))\,du=0\tag{34}$$

and thus, by combining (32), (33) and (34),

$$\lim_{n\to\infty}\left(S_n(x_0)-\frac{f(x_0^+)+f(x_0^-)}{2}\right)=0$$

as desired. □

Theorem 5.21 is the strongest result about the convergence of Fourier series we will prove in this seminar. I hope that you find the result satisfying, it encompasses most of the functions that you know about and can write down. The Nobel prize-winning physicist, Richard Feynman, was quite happy with this results (and ones like it) when made the following statement: "The mathematicians have shown, for a wide class of functions, in fact for all that are of interest to physicists, that if we can do the integrals we will get back $f(t)$." He made this statement in the 1960's in his famous lecture series at Caltech, just before Lennart Carleson completely solved the problem and determined the exact class of functions representable by their Fourier series [4, 7]. Carleson's result, now known as Carleson's theorem, was a long standing conjecture known as Luzin's conjecture. In its full form, Carleson's theorem is the final and definitive result that addresses the claim made on the first day of our seminar, Claim 1.6. For your cultural benefit, I will state it here; I'll first need to make a definition.

Definition 5.23. *For any interval $I=[a,b]\subseteq\mathbb{R}$, we define*

$$\ell(I)=b-a$$

to be the length of I. Now, for any subset E of $\mathbb{R}$, we say that E is a set of measure zero (or a null set) if, for every $\epsilon>0$, there is an infinite collection of intervals $\{I_n\}$ such that

$$E\subseteq\bigcup_{n=1}^{\infty}I_n$$

and

$$\sum_{n=1}^{\infty}\ell(I_n)<\epsilon.$$

You should think of a set of measure zero as an extremely small set. For instance, any finite (or countably infinite) collection of points is of measure zero. Using this notion of small sets we can state Carleson's theorem in the context of Riemann integrable functions; the general result is formulated using the Lebesgue integral [1]. Here it is:

Theorem 5.24 (Carleson 1966)**.** *For any 2π-Riemann integrable function f, there exists a set E of measure zero such that*

$$f(x)=a_0+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx$$

for all $x\notin E$.

5.3 The Gibb's phenomenon

In this final subsection, we will study the Gibb's phenomenon. The Gibb's phenomenon, named after J. Willard Gibb's (yes, the free energy Gibbs), describes the pointwise convergence of Fourier series of a function with a jump discontinuity. We have already discussed this briefly in the seminar before. Instead of working in the general setting, we will study the Gibb's phenomenon as it occurs when we consider the Fourier series of the sawtooth function. Focusing on this specific case will allow us to very precisely see what's going on. If you are worried about the general case, I'll refer you to a very nice discussion by T. W. Körner in which he describes how to extend the results pertaining to this example to the general class of piecewise differentiable functions in $\mathcal{R}$ [7].

So let's return to our favorite example, the sawtooth function f, defined by

$$f(x) = x$$

for all $-\pi < x \leq \pi$ and extended periodically to $\mathbb{R}$. You should note that our solution of the Basel problem in Subsection 5.1 made use of the sawtooth function and its Fourier series. In particular, we've already computed the Fourier coefficients for f. We found that $a_k = 0$ for $k = 0, 1, 2, \dots$ and

$$b_k = \frac{2(-1)^{k+1}}{k}$$

for $k = 1, 2, \dots$. Thus, the Fourier partial sums for f are

$$S_n(x) = \sum_{k=1}^{n} \frac{2(-1)^{k+1}}{k} \sin(kx)$$

for $n \in \mathbb{N}$ and the Fourier series for f is

$$\tilde{f}(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx).$$

Figure 2 in Section 1 depicts the graphs of S_n for $n = 1, 2, 3, 4, 5, 40$ and the graph of f. Let's momentarily focus on the the graph of S_n for $n = 40$; this is illustrated in Figure 9.

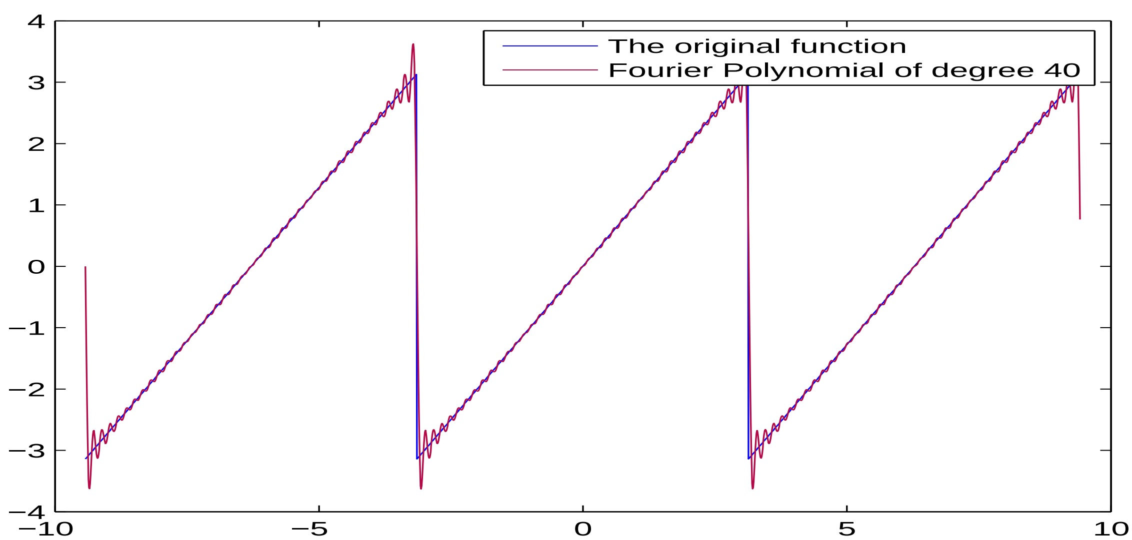


Figure 9: f and S_{40}

As we discussed in the last subsection, Theorem 5.21 guarantees that

$$\lim_{n\to\infty} S_n(x) = f(x)$$

for all $x \neq m\pi$ where $m \in \mathbb{Z}$ is odd. We also showed that, at any $x = m\pi$ where $m \in \mathbb{Z}$ is odd, $S_n(x) \to 0$ as $n \to \infty$. These two things should appear to be somewhat clear by looking at Figure 9. There is however one thing that should bother you: near (but not at) the breakpoints, $\{\dots,-3\pi,-\pi,\pi,3\pi,\dots\}$, the graph of S_{40} seems to "overshoot" (or "undershoot") the graph of f. These are the spikes you see close to the discontinuity in f. This behavior is called the *Gibb's phenomenon*. Let's study this behavior more closely at, say, $x=\pi$; Figure 10 shows the graphs of $S_n(x)$ for $n=25,26,\dots,50$.

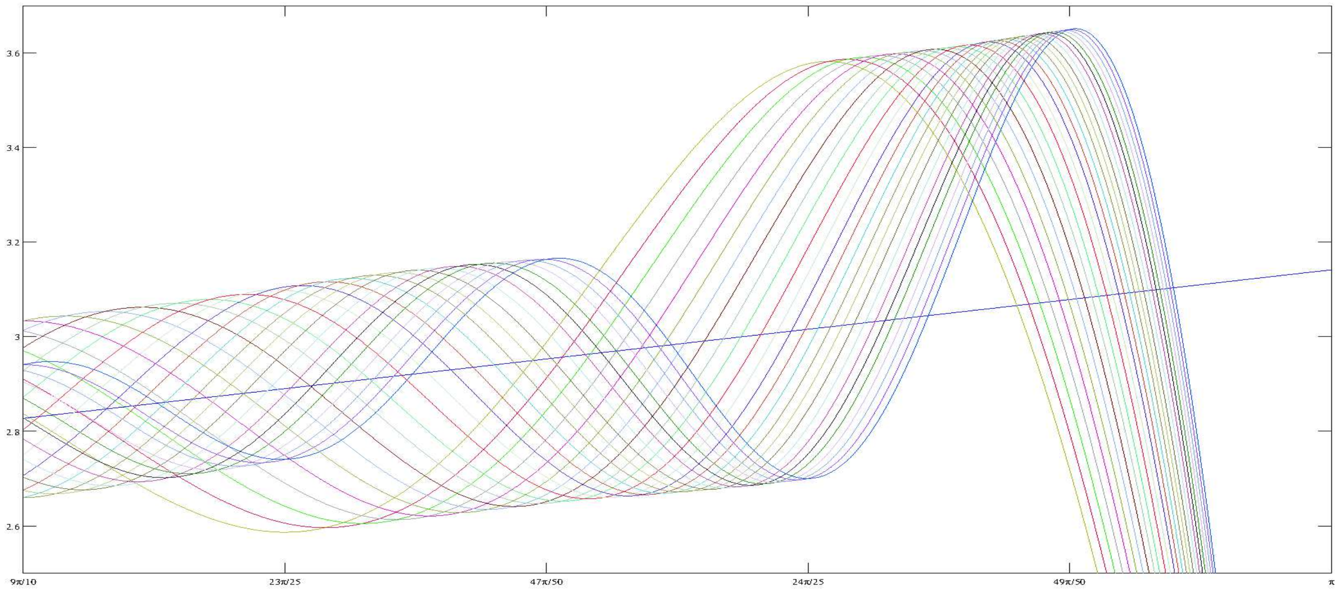


Figure 10: The graphs of $S_n(x)$ for $n=25,26,\dots 50$ and $f(x)$ for $9\pi/10\le x\le\pi$.

Upon studying Figure 10 closely, we see that the overshoot is moving right as n increases. You might say: Theorem 5.21 guarantees that $S_n(x)\to f(x)=x$ for all $-\pi<x<\pi$ but, upon looking at the figure, $S_n(x)$ isn't converging to $f(x)$ for x very close to π. So where did we go wrong? The answer is that we haven't gone wrong at all, the apparent discrepancy can be understood by recognizing that pointwise convergence is weaker than convergence in the graph–this is the difference between pointwise convergence and uniform convergence. Remember, that for pointwise convergence, we first select x and ϵ and find a natural number $N=N(\epsilon,x)$ for which

$$|S_n(x)-f(x)|<\epsilon$$

for all $n\ge N$. In the case at hand, we can understand this notion as follows: If I select an $x<\pi$, but as close to π as I want, since the overshoot in the Fourier polynomials are moving to the right, I simply have to wait until they have moved so far right that they've passed x–this will determine N. After this, the Fourier polynomials evaluated at x will get much much closer to $f(x)$. Okay, so now you understand how we still get pointwise convergence. Let's now try to understand the overshoot.

Using the same numbers I've used to make the graphs in Figure 10, I can quantify this overshoot. For each $n\in\mathbb{N}$, denote by

$$M_n=\max_{9\pi/10\le x\pi} S_n(x),$$

the maximum of the function $S_n(x)$ near π. We also denote by x_n the unique x near π for which

$$f(x_n)=M_n.$$

The following table shows M_n, M_n/π, x_n and $\pi-\pi/n$ to four decimal places for $n=25,30,\dots,50$.

n	M_n	M_n/π	x_n	$\pi - \pi/n$
25	3.5822	1.403	3.0204	3.0159
30	3.6020	1.1465	3.0404	3.0369
35	3.6172	1.1514	3.0544	3.0518
40	3.6321	1.1561	3.0654	3.0631
45	3.6433	1.1597	3.0734	3.0718
50	3.6516	1.1624	3.0804	3.0788

Upon looking at the table, we see that, as n increases the x_n's are close to $\pi - \pi/n$ and the ratio M_n/π grows toward $1.17\dots$. So, by following the x at which $S_n(x)$ is maximized, the ratio M_n/π approaches some number A as $n \to \infty$ (note that π is half the gap of the discontinuity of f at π); this describes the overshoot. The following theorem formalizes it:

Theorem 5.25. *Let f, S_n be as above. Then*

$$\lim_{n\to\infty} S_n(\pi - \pi/n) = \pi A$$

where

$$A = 1.178979744447216727\dots$$

Thus, the $S_n(\pi - \pi/n)$ converges to A (called the Gibb's constant) times half of the gap of the jump discontinuity.

Proof. Using our trigonometric identities, we find that

$$\begin{aligned} S_n(\pi - \pi/n) &= \sum_{k=1}^{n} \frac{2(-1)^{k+1}}{k} \sin(k\pi - k\pi/n) \\ &= \sum_{k=1}^{n} \frac{2(-1)^{k+1}}{k} (\sin(kx)\cos(k\pi/n) - \sin(k\pi/n)\cos(k\pi) \\ &= \sum_{k=1}^{n} \frac{2(-1)^{k+1}}{k} (0 - \sin(k\pi/n)\cos(k\pi) \\ &= \sum_{k=1}^{n} \frac{2(-1)^{k+1}}{k} ((-1)^{k+1}\sin(k\pi/n) \\ &= 2\sum_{k=1}^{n} \frac{\sin k\pi/n}{k\pi/n}\frac{\pi}{n}. \end{aligned}$$

You should recognize that

$$\sum_{k=1}^{n} \frac{\sin k\pi/n}{k\pi/n}\frac{\pi}{n}$$

is a (right) Riemann sum for the integral

$$\int_0^\pi \frac{\sin(x)}{x}\,dx$$

and because $\sin x/x$ is Riemann integrable on $(0, \pi]$, we immediately conclude that

$$\lim_{n\to\infty} S_n(\pi - \pi/n) = \lim_{n\to\infty} 2\sum_{k=1}^{n} \frac{\sin k\pi/n}{k\pi/n}\frac{\pi}{n} = 2\int_0^\pi \frac{\sin x}{x}\,dx = \pi A$$

where

$$A = \frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx.$$

It remains to compute A. Luckily, you now know some facts about Taylor series or, in this case, Maclaurin series. We note that, for all $x \neq 0$,

$$\frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\cdots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots .$$

Using the Ratio test, you find that this series converges uniformly on $(0,1]$ and hence, by Proposition 2.18,

$$\begin{aligned} A &= \frac{2}{\pi}\int_0^\pi (1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots)\,dx = \frac{2}{\pi}\left(\int_0^\pi 1\,dx - \frac{1}{3!}\int_0^\pi x^2\,dx + \frac{1}{5!}\int_0^\pi x^4\,dx\right) \\ &= \frac{2}{\pi}\left(\pi - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} + \cdots\right) = 2 - \frac{2\pi^2}{3\cdot 3!} + \frac{2\pi^4}{5\cdot 5!}\cdots \\ &= 1.178979744447216727\ldots \end{aligned}$$

which can be checked to any desired accuracy using the remainder part of Taylor's theorem. □

References

[1] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. **116** (1966) 135-157.

[2] W. Dunham, Euler, The Master of Us All. The Mathematical Association of America, 1999.

[3] H. Dym and H. P. McKean, Fourier Series and Integrals. Academic Press, New York, New York, 1972.

[4] R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures on Physics. Addison-Wesley Pub. Co., 1965.

[5] J. Franks, A (Terse) Introduction to Lebesgue Integration. Student Mathematical Library, American Mathematical Society (v 48), Providence, RI, 2009.

[6] J. Kirkwood, An Introduction to Analysis. Waveland Press, Inc., Long Grove, IL, 2002.

[7] T.W. Körner, Fourier Analysis. Cambridge University Press, Cambridge, England, 1988.

[8] W. Rudin, Principles of Mathematical Analysis, 3rd ed. McGray-Hill, New York, New York, 1976.

[9] F. W. Sears, Mechanics Wave Motion and Heat. Addison-Wesley Publishing, Reading, Massachusetts, 1958.