Most of us love an extreme example. In their recent “bouquet of discontinuous functions” [1], Drago, Lamberti, and Tony omit our favorite flower: a function with a dense graph. A natural occurrence appears when we encounter additive homomorphisms of the real numbers. A typical exercise asks us to prove that such a homomorphism \( f \) satisfies \( f(s) = rs \), for some real number \( r \) and all rational numbers \( s \) ([6] p. 51). It is fun to point out to students that for continuous \( f \), the equation is good for all real numbers \( s \), but there exist discontinuous examples, and they are so wild that their graphs are dense in the plane ([3], [5] p. 200)!

Recall that a subset \( D \) of \( \mathbb{R} \) (or \( \mathbb{Q} \)) is dense when every infinite interval intersects \( D \). A function \( f : \mathbb{R} \to \mathbb{R} \) has a dense graph when every infinite rectangle intersects the graph of \( f \), or equivalently, when, for every infinite interval \( I \), the image \( f(I) \) is dense. This note describes new elementary examples of such functions. Start with an enumeration \( \{r_1, r_2, r_3, \ldots\} \) of the positive rational numbers \( \mathbb{Q}^+ \), then define the function \( f : \mathbb{Q}^+ \to \mathbb{Q}^+ \) that maps the rational \( m/n \) (written in reduced form) to \( r_m \). We prove that \( f \) has a graph dense in the first quadrant. We encourage the reader to play with various specific examples of such \( f \), for example when the enumeration is the classic one

\[
\left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \ldots \right\},
\]

or one of the interesting enumerations in [2]. We found a completely elementary proof (though quite intricate) that the graph of \( f \) is dense in the special case of the classic enumeration. For the general proof presented below, we rely on the prime number theorem, a calculus problem, and a short lemma.

For each \( x \in \mathbb{Q}^+ \), let \( \pi(x) \) denote the number of primes \( p \) less than or equal to \( x \). The prime number theorem gives a handle on the rate of growth of \( \pi \). A proof can be found in [4].
**Prime number theorem** The function $\pi(x)$ behaves asymptotically like $x/\ln x$, i.e.

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln x} = 1$$

**A good calculus problem** Assume that $a < b$. We have

$$\lim_{x \to \infty} \left[ \frac{bx}{\ln(bx)} - \frac{ax}{\ln(ax)} \right] = \infty.$$ A comparison with $x/\ln x$ does the trick:

$$\frac{\frac{bx}{\ln(bx)} - \frac{ax}{\ln(ax)}}{\frac{x}{\ln x}} = \frac{b}{\ln b + 1} - \frac{a}{\ln a + 1} \to b - a.$$ 

**A short lemma** Assume $a,b \in \mathbb{Q}^+$ with $a < b$. There exists $N \in \mathbb{N}$ such that, for all $m > N$, there exist $p_m$ relatively prime to $m$ with $p_m \in (am,bm)$. Choose $N_1$ so that $aN_1 > 1$, then find $q \geq 1$ so that $(aN_1)^q > N_1$. The calculus problem and the prime number theorem let us find $N \geq N_1$ with

$$q < \pi(bm) - \pi(am)$$

for all $m \geq N$. If $m > N$, then $a^q m^{q-1} > a^q N_1^{q-1} > 1$, so $(am)^q > m$. Our choice of $N$ ensures that the interval $(am,bm)$ contains at least $q$ primes $(p_i)_{i=1}^q$, and the inequality

$$m < (am)^q < \Pi_{i=1}^q p_i$$

shows that there exists $p_m \in \{p_1, \ldots, p_q\}$ such that $p_m$ does not divide $m$.  

**Proof of density** Assume $\sigma : \mathbb{N} \to \mathbb{Q}^+$ is an enumeration of the positive rational numbers, and define $f : \mathbb{Q}^+ \to \mathbb{Q}^+$, for reduced $m/n \in \mathbb{Q}^+$, by

$$f\left(\frac{m}{n}\right) = \sigma_m.$$ 

To show the graph of $f$ dense in the first quadrant, let an interval $(c,d)$ in $\mathbb{Q}^+$ be given. We need to show that $f(c,d)$ is a dense subset of $\mathbb{Q}^+$. Let $a$ and $b$ be the reciprocals of $d$ and $c$ (respectively), choose $N$ as in the short lemma, and note that $f(c,d)$ contains the dense set

$$\{ \sigma_m : m > N \},$$

a consequence of

$$c = \frac{1}{b} < \frac{m}{p_m} < \frac{1}{a} = d$$

(for all $m > N$). Thus $f(c,d)$ is itself dense in $\mathbb{Q}^+$.  

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References


